SINGULAR PERTURBATION METHODS

HW

(I) Consider
$$\begin{cases} \ddot{u} + \omega_0^2 u = \varepsilon \dot{u}^2 u \\ u(0) = A \\ \dot{u}(0) = 0 \end{cases} \qquad u = u(t)$$

Find the leading order and first correction approximation to u. Use Poincaré Lindstedt (optional: try regular perturbations)

(II) Consider the boundary layer problem

$$\text{BVP} \begin{cases} \varepsilon y'' + y' + y = 0 & 0 < x < 1 \\ y(0) = \alpha, \; y(1) = \beta & \alpha, \beta > 0 \end{cases}$$

(a) Find exact solution $y(x)$

(b) Show BVP is singular

(c) Find an inner & outer approximation to solution, to lowest order, to within constant(s)

(d) Find matching condition that determines the unknown constants in (c).

(e) Show that the inner & outer solutions approximate the exact solution in each of the regions.

(III) Consider the boundary layer problem

$$\text{BVP} \begin{cases} \varepsilon y'' - y' = 1 & 0 < x < 1 \\ y(0) = \alpha,\ y(1) = \beta & \alpha, \beta > 0 \end{cases}$$

(a) Find exact solution

(b) Plot exact solution

(c) Argue why we expect a boundary layer close to $x = 1$ (use plot for guidance, but use BVP for the argument).

(d) Find inner & outer approximate solutions, use the transformation $\xi = \frac{1-x}{\varepsilon^{\lambda}}$ in the boundary layer and show that $\lambda = 1$ balances BVP terms in the layer

(e) Use matching to fully determine constant(s) in the inner & outer solutions