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MTH 581
11/4/16

Homework 4

Problem 1

Verify that $\int_0^\varepsilon e^{-x^2}\,dx = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0^+$.
Solution: To verify the result first note that

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \mathcal{O}(x^8).$$

Furthermore with this approximation we have

$$\begin{aligned}
\int_0^\varepsilon e^{-x^2}\,dx &\approx \int_0^\varepsilon 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \mathcal{O}(x^8)\,dx \\
&= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \mathcal{O}(x^9)\Big|_0^\varepsilon \\
&= \varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{10} - \frac{\varepsilon^7}{42} + \mathcal{O}(\varepsilon^9).
\end{aligned}$$

With these two approximations we have the following result by the applying the definition of $\mathcal{O}$,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{\left|\int_0^\varepsilon e^{-x^2}\,dx\right|}{|\varepsilon|} &\approx \lim_{\varepsilon\to 0^+} \left|\frac{\varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{10} - \frac{\varepsilon^7}{42} + \mathcal{O}(\varepsilon^9)}{\varepsilon}\right| \\
&= \lim_{\varepsilon\to 0^+} \left|1 - \frac{\varepsilon^2}{3} + \frac{\varepsilon^4}{10} - \frac{\varepsilon^6}{42} + \mathcal{O}(\varepsilon^8)\right| \\
&= 1
\end{aligned}$$

Hence we have $\int_0^\varepsilon e^{-x^2}\,dx = \mathcal{O}(\varepsilon)$.

Another way of looking at the problem is keeping it in summation notation to suppress the approximations. That is we have

$$e^{-x^2} \approx \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}.$$

Then making the same substitution into the integral we have

$$\begin{aligned}
\int_0^\varepsilon e^{-x^2}\,dx &\approx \int_0^\varepsilon \sum_{k=0}^\infty \frac{(-x^2)^k}{k!}\,dx \\
&= \sum_{k=0}^\infty \int_0^\varepsilon \frac{(-1)^k(x^{2k})}{k!}\,dx \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!}\,\frac{x^{2k+1}}{2k+1}\bigg|_0^\varepsilon \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!}\,\frac{\varepsilon^{2k+1}}{2k+1}
\end{aligned}$$

With these two approximations we have the following result by the applying the definition of $\mathcal{O}$,

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\left|\frac{\int_0^\varepsilon e^{-x^2}}{\varepsilon}\right| &= \lim_{\varepsilon\to 0^+}\left|\frac{\sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{\varepsilon^{2k+1}}{2k+1}}{\varepsilon}\right| \\
&= \lim_{\varepsilon\to 0^+}\left|\sum_{k=0}^\infty \frac{(-1)^k\varepsilon^{2k}}{2k+1}\right| \\
&= 1
\end{aligned}$$

Hence we have $\int_0^\varepsilon e^{-x^2}\,dx = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0^+$.

Problem 2

Verify that $e^{-\varepsilon} = o(1)$ as $\varepsilon \to \infty$.

Solution: From the last problem we saw that either expanding it out or keeping it in summation notation lead to the same result, the only difference is that summation notation tends to be more condensed. Note that $e^{-\varepsilon} = \frac{1}{e^\varepsilon}$ we can consider

$$e^\varepsilon \approx \sum_{k=0}^\infty \frac{\varepsilon^k}{k!}.$$

Thus by the definition of o we want to verify the following

$$\begin{aligned}
\lim_{\varepsilon\to\infty}\frac{e^{-k}}{1} &= \lim_{\varepsilon\to\infty}\frac{1}{e^k} \\
&= \lim_{\varepsilon\to\infty}\frac{1}{\sum_{k=0}^\infty \frac{\varepsilon^k}{k!}}
\end{aligned}$$

Here it is obvious that the denominator is a summation of positive terms that are increasing larger, and as $\varepsilon \to \infty$ we have it getting even larger faster. Thus it is clear to see

$$\lim_{\varepsilon\to\infty}\frac{e^{-k}}{1} = 0$$

, hence $e^{-\varepsilon} = o(1)$ as $\varepsilon \to \infty$.

Problem 3

Use Poincaré-Lindstedt Method to find a 2-term approximation to the solution of

$$\ddot{u} + u = \varepsilon(1 - u^2)\dot{u}$$

Solution: With the Poincaré-Lindstedt Method we introduce a new time scale $\tau = \omega t$ in which we can note the differential operator will be

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \omega\frac{d}{d\tau} \implies \frac{d^2}{dt^2} = \omega^2\frac{d^2}{d\omega^2}.$$

Making this substitution into the original system we have

$$\omega^2 u'' + \varepsilon(u^2 - 1)\omega u' + u = 0.$$

Here we can note that u is now a function of τ. Now we can make the following approximation of u and ω with $\varepsilon \ll 1$:

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3) \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \mathcal{O}(\varepsilon^3) \end{aligned}$$

Note that the base harmonic oscillator frequency is 1 in the original ODE, which implies that $\omega_0 = 1$ making this substitution and the substitution with the approximations into the system we have the following

$$\begin{aligned} 0 &= \left(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \mathcal{O}(\varepsilon^3)\right)^2\left(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \mathcal{O}(\varepsilon^3)\right) \\ &\quad + \varepsilon\left(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \mathcal{O}(\varepsilon^3)\right)\left(\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)\right)^2 - 1\right)\left(u_0' + \varepsilon u_1' + \varepsilon^2 u_2' + \mathcal{O}(\varepsilon^3)\right) \\ &\quad + \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)\right) \\ &= (u_0'' + u_0) + \varepsilon\left(u_1'' + (u_0^2 - 1)u_0' + u_1 + 2\omega_1 u_0''\right) \\ &\quad + \varepsilon^2\left(u_2'' + u_2 + (\omega_1^2 + 2\omega_2)u_0'' + 2\omega_1 u_1'' + (u_0^2 - 1)(u_1' + \omega_1 u_0'\right) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

Thus we can see for ε^0 order terms we have $u_0'' + u_0 = 0$. Since it is an order 2 with constant coefficients we can use the corresponding characteristic function $\lambda^2 + 1 = 0$ to find the solution. From the characteristic function we have $\lambda = \pm i$, thus

$$u_0 = A_0\cos(\tau) + B_0\sin(\tau)$$

Since we are not given any initial conditions we cannot solve for A_0 and B_0. So we have to use the hint given in class in which we have the following trigonometric identity:

$$u_0 = A_0\cos(\tau) + B_0\sin(\tau) = D_0\cos(\tau + \phi)$$

with $D_0 = \sqrt{A_0^2 + B_0^2}$ and $\tan(\phi) = \frac{B_0}{A_0}$. Now that we have a solution for u_0 in which we can solve for the next term u_1 in which we need to consider u_0' and u_0'. Note that

$$u_0 = D_0\cos(\tau + \phi) \implies u_0' = -D_0\sin(\tau + \phi) \implies u_0'' = -D_0\cos(\tau + \phi)$$

Consider the ε^1 order terms.

$$u_1'' + (u_0^2 - 1)u_0' + u_1 + 2\omega_1 u_0'' = 0.$$

Plugging in what we have already solved for and breaking it down using trigonometry identities gives us

$$\begin{aligned} u_1'' + u_1 &= (D_0^2 \cos^2(\tau + \phi) - 1)D_0 \sin(\tau + \phi) + 2\omega_1 D_0 \cos(\tau + \phi) \\ &= 2\omega_1 D_0 \cos(\tau + \phi) - D_0 \sin(\tau + \phi) + D_0^3 \cos^2(\tau + \phi)\sin(\tau + \phi) \\ &= 2\omega_1 D_0 \cos(\tau + \phi) - D_0 \sin(\tau + \phi) + \frac{D_0^3}{4}\sin(\tau + \phi) + \frac{D_0^3}{4}\sin(3(\tau + \phi)) \\ &= 2\omega_1 D_0 \cos(\tau + \phi) + D_0\left(\frac{D_0^2}{4} - 1\right)\sin(\tau + \phi) + \frac{D_0^3}{4}\sin(3(\tau + \phi)) \end{aligned}$$

To get rid of the secular terms we need to have $\omega_1 = 0$ and we can let $D_0 = 2$ in which we get

$$u_1'' + u_1 = 2\sin(3(\tau + \phi)).$$

Here we know that the homogeneous solution will be $u_1^H = A_1 \cos(\tau) + B_1 \sin(\tau)$. For the particular solution we can guess that the solution will look like $u_1^P = C_1 \cos(3\tau) + D_1 \sin(3\tau)$. Taking the derivatives accordingly and plugging them into the ODE corresponding to u_1 we have

$$\begin{aligned} 2\sin(3(\tau + \phi)) &= u_1'' + u_1 \\ &= -9C_1 \cos(3(\tau + \phi)) - 9D_1 \sin(3(\tau + \phi)) + C_1 \cos(3(\tau + \phi)) + D_1 \sin(3(\tau + \phi)) \\ &= -8C_1 \cos(3(\tau + \phi)) - 8D_1 \sin(3(\tau + \phi)) \end{aligned}$$

Comparing the left and right side of the equation above we get that $C_1 = 0$ and $D_1 = -\frac{1}{4}$. Thus we have

$$u_1 = A_1 \cos(\tau) + B_1 \sin(\tau) - \frac{1}{4}\sin(3(\tau + \phi)).$$

From this we can conclude that our approximate solution is

$$\begin{cases} u = -2\cos(\tau + \phi) + \varepsilon\left(A_1 \cos(\tau) + B_1 \sin(\tau) - \frac{1}{4}\sin(3(\tau + \phi))\right) + \mathcal{O}(\varepsilon^2) \\ \omega = 1 + 0 + \mathcal{O}(\varepsilon^2) \end{cases}$$

Problem 4

Use a regular perturbation series to find a 2-term approximation to the solution of

$$\ddot{u} + u = \varepsilon(1 - u^2)\dot{u}.$$

Solution: For this problem we will follow somewhat of the same procedure as Problem 5, the difference being we will only perturb our solution u and leave the same time scale t. So for $\varepsilon \ll 1$ we can approximation u by

$$u \approx u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3).$$

Plugging this into ODE, taking the appropriate derivative we have

$$\begin{aligned}
0 &= \left(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \mathcal{O}(\varepsilon^3)\right) \\
&\quad + \varepsilon \left(\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)\right)^2 - 1 \right) \left(u_0' + \varepsilon u_1' + \varepsilon^2 u_2' + \mathcal{O}(\varepsilon^3)\right) \\
&\quad + \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \mathcal{O}(\varepsilon^3)\right) \\
&= (u_0'' + u_0) + \varepsilon \left(u_1'' + u_0'(u_0^2 - 1) + u_1\right) + \varepsilon^2 \left(u_2'' + u_2 + u_1'(u_0^2 - 1) + 2u_0 u_0' u_1\right) + \mathcal{O}(\varepsilon^3)
\end{aligned}$$

Now we can consider them case by case in which we first look at ε^0 and note we have

$$u_0'' + u_0 = 0 \implies A_0 \cos(t) + B_0 \sin(t) \implies u_0 = D_0 \cos(t + \phi)$$

where we use the same trigonometric identity from the previous problem in which $D_0 = \sqrt{A_0^2 + B_0^2}$ and $\tan(\phi) = \frac{B_0}{A_0}$.

Now to consider the ε^1 case we have, by substituting the known value of u_0

$$\begin{aligned}
u_1'' + u_1 &= (D_0^2 \cos^2(\tau + \phi) - 1) D_0 \sin(\tau + \phi) \\
&= -D_0 \sin(\tau + \phi) + D_0^3 \cos^2(\tau + \phi) \sin(\tau + \phi) \\
&= -D_0 \sin(\tau + \phi) + \frac{D_0^3}{4} \sin(\tau + \phi) + \frac{D_0^3}{4} \sin(3(\tau + \phi)) \\
&= D_0 \left(\frac{D_0^2}{4} - 1\right) \sin(\tau + \phi) + \frac{D_0^3}{4} \sin(3(\tau + \phi))
\end{aligned}$$

Using Maple to solve this problem (due to tall the constants floating around) we have the general solution to be

$$u_1 = \left(\frac{D_0^3}{4} - D_0\right) \left(-\frac{t}{2} \cos(t)\right) - \frac{D_0^3}{32} \sin(3t) + A_1 \cos(t) + B_1 \sin(t).$$

So a 2-term approximation for $u(t)$ is given by

$$u(t) = D_0 \cos(t) + \varepsilon \left(\left(\frac{D_0^3}{4} - D_0\right) \left(-\frac{t}{2} \cos(t)\right) - \frac{D_0^3}{32} \sin(3t) + A_1 \cos(t) + B_1 \sin(t) \right) + \mathcal{O}(\varepsilon^2)$$

Problem 5

Consider the following system:

$$\begin{cases} \frac{dy}{dt} = 1 + (1 + \varepsilon) y^2 & t > 0 \quad \varepsilon \ll 1 \\ y(0) = 1 \end{cases}$$

(a) Solve exactly.

(b) Find 2-term approximation via regular series expansion.

(c) Find the first two terms of series expansion of exact solution found in (a) and compare to the approximation found in (b).

Solution:

(a) Considering the system given in the problem we can note that is a first order equation that is separable. Thus by separation of variables within the ODE we have

$$\frac{dy}{dt} = 1 + (1+\varepsilon)y^2 \implies \frac{1}{1+(1+\varepsilon)y}\,dy = dt$$

Integrating the right hand side is trivial so we concentrate on the left hand side of the last equation above. That is we are considering

$$\int \frac{1}{1+(1+\varepsilon)y}\,dy.$$

Now through a u-substitution, let $u = \sqrt{1+\varepsilon}y$ in which we can note that $du = \sqrt{1+\varepsilon}\,dy$. Hence we have

$$\begin{aligned}\int \frac{1}{1+(1+\varepsilon)y}\,dy &= \frac{1}{\sqrt{1+\varepsilon}}\int \frac{1}{u^2+1}\,du \\ &= \frac{\tan^{-1}(u)}{\sqrt{1+\varepsilon}} \\ &= \frac{\tan^{-1}(\sqrt{1+\varepsilon}y)}{\sqrt{1+\varepsilon}}\end{aligned}$$

Putting this back into the separation of variables equation and taking the integral of the right hand side (in which we are combining the integrating constant on the right hand side) we have

$$\frac{\tan^{-1}(\sqrt{1+\varepsilon}y)}{\sqrt{1+\varepsilon}} = t + C.$$

Solving for y we have

$$y = \frac{\tan\left((t+C)\sqrt{1+\varepsilon}\right)}{\sqrt{1+\varepsilon}}.$$

Using the initial condition $y(0) = 1$ we have

$$1 = y(0) = \frac{\tan(C\sqrt{1+\varepsilon})}{\sqrt{1+\varepsilon}} \implies C = \frac{\tan^{-1}(\sqrt{1+\varepsilon})}{\sqrt{1+\varepsilon}}.$$

(b) Following the same procedure as done in previous problems we make the assumption for $\varepsilon \ll 1$ we have

$$y \approx y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2) \implies y_0(0) = 1 \text{ and } y_i(0) = 0 \text{ for } i > 0.$$

Plugging this into the original ODE we have

$$\begin{aligned}0 &= y' - 1 - y^2 - \varepsilon y \\ &= \left(y_0' + \varepsilon y_1' + \mathcal{O}(\varepsilon^2)\right) - 1 - \left(y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)\right)^2 - \varepsilon\left(y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)\right) \\ &= \left(y_0' - y_0^2 - 1\right) + \varepsilon\left(y_1' - 2y_0y_1 - y_0^2\right) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

For the ε^0 case we have $y_0' - y_0^2 - 1 = 0$ with $y_0(0) = 1$. We can solve this problem through separation of variables and then use the initial condition to solve for the integration constant that is:

$$y_0' - y_0^2 - 1 = 0 \implies \int \frac{dy_0}{y_0^2+1} = \int dt \implies y_0 = \tan(t+C).$$

Using the initial condition we have $C = \frac{\pi}{4}$ and thus $y_0 = \tan(t + \frac{\pi}{4})$.

Now for ε^1 case we have

$$0 = y_1' - 2y_0y_1 - y_0^2 = y_1' - 2y_1 \tan\left(t + \frac{\pi}{4}\right) - \tan^2\left(t + \frac{\pi}{4}\right).$$

Because of the complexity of the ODE we will turn to software to solve this problem, that is using Maple we have that

$$y_1 = \left(\frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right)\right)\tan\left(t + \frac{\pi}{4}\right)^2 + \tan\left(t + \frac{\pi}{4}\right)^2 C + \frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right) + C$$

Using maple to isolate C in order to use the initial condition we have

$$C = \frac{-y(t) + \left(\frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right)\right)\tan\left(t + \frac{\pi}{4}\right)^2 + \frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right)}{-\tan\left(t + \frac{\pi}{4}\right)^2 - 1}$$

Using hat $y_1(0) = 0$ we have $C = \frac{1}{4}$. Thus we can finally write down our 2 term approximation to be

$$y = \tan(t + \frac{\pi}{4}) + \varepsilon\left(\left(\frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right)\right)\tan\left(t + \frac{\pi}{4}\right)^2 + \frac{1}{4}\tan\left(t + \frac{\pi}{4}\right)^2 + \frac{t}{2} - \frac{1}{4}\sin\left(2t + \frac{\pi}{2}\right) + \frac{1}{4}\right) + \mathcal{O}(\varepsilon^2)$$

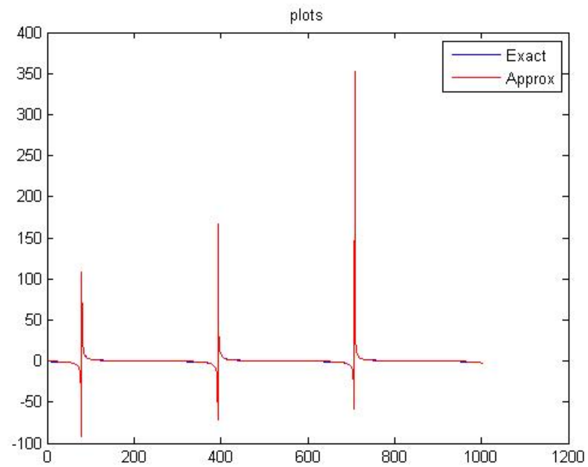
(c) Using Maple again to find the Taylor series of the exact solution from part (a) we have

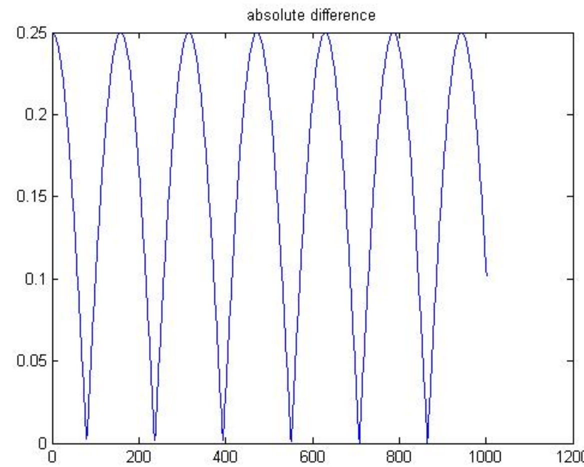
$$y_{exact} = \tan\left(t + \frac{\pi}{4}\right) + \varepsilon\left(-\frac{1}{2}t - \tan\left(t + \frac{\pi}{4}\right) + \frac{t}{2} + \frac{1}{4} + \frac{1}{2}\tan^2\left(t + \frac{\pi}{4}\right)t + \frac{1}{4}\tan^2\left(t + \frac{\pi}{4}\right)\right) + \mathcal{O}(\varepsilon^2)$$

Comparing this to what we found in part (b) we can note that the firs term in the sequence of the series agree. For the second term, by setting them equal to each other and canceling out corresponding terms we have the different terms being

Exact 2nd term	Approximation 2nd term
$-\frac{1}{2}\tan\left(t + \frac{\pi}{4}\right)$	$-\frac{1}{4}\cos(2t)\tan^2\left(t + \frac{\pi}{4}\right)$

To see if these two things are even close approximations of one another (since they are the only different terms) we can plot them with each other and there absolute difference. Using MATLAB we can generate the following plots (code is readily available upon request).





To make some comments about these graphs, we can note that periodic behavior do to it being tan. With this in mind we would expect our approximation to be worse at the vertical asymptotes, which is shown in the correlation of the two figures. Where we have these asymptotes we have the absolute difference being growing large and then going back down towards the center of the periods.

Problem 6

The goal of this problem is to use perturbation methods to find the roots of

$$(*) \qquad x^3 - 4.001x + 0.002 = 0.$$

One method is to write $(*)$ as

$$(**) \qquad x^3 - (4+\varepsilon)x + 2\varepsilon = 0.$$

where ε is the perturbation.

(a) Compute the approximations to the roots of $(*)$ to $\mathcal{O}(\varepsilon^2)$.

(b) Evaluate the estimate using $\varepsilon = 0.001$.

(c) To compare your estimates to the "exact" answer, factor $(*)$ as

$$(x-2)(x^2 + ax + b) = 0$$

i.e. one of the roots is exactly 2. The other 2 will be the roots of $x^2 + ax + b$.

Solution:

(a) We first assume that $x \approx x_0 + \varepsilon x_1 + \mathcal{O}(\varepsilon^2)$, from this we have the direct implication that $x^3 \approx x_0^3 + 3\varepsilon x_0^2 x_1 + \mathcal{O}(\varepsilon^2)$. By substituting these approximations into (**) and expanding out we have the following:

$$\begin{aligned} 0 &= x^3 - (4+\varepsilon)x + 2\varepsilon \\ &\approx x_0^3 + 3\varepsilon x_0^2 x_1 + \mathcal{O}(\varepsilon^2) - (4+\varepsilon)(x_0 + \varepsilon x_1 + \mathcal{O}(\varepsilon^2)) + 2\varepsilon \\ &= x_0^3 + 3\varepsilon x_0^2 x_1 - 4x_0 - 4\varepsilon x_1 - \varepsilon x_0 + 2\varepsilon + \mathcal{O}(\varepsilon^2) \\ &= \left(x_0^3 - 4x_0\right) + \varepsilon\left(3x_0^2 x_1 - 4x_1 + 2\right) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Now we can look at the orders of ε and find the values of x_0 and x_1 and makes the equations true. Note that we are not considering any order of ε greater or equal to 2, by the designation of the problem.

For the ε^0 case we have:

$$x_0^2 - 4x_0 = 0 \implies x_0(x_0^2 - 4) = 0 \implies x_0 = \{0, 2, -2\}$$

Since $x_0 = \{0, 2, -2\}$ and the ε^1 order terms $(3x_0^2 x_1 - 4x_1 + 2)$ are dependent on x_0, for the ε^1 we have 3 sub-cases. So consider the following cases.

For ε^1 with $x_0 = 0$ we have

$$-4x_1 + 2 = 0 \implies x_1 = \frac{1}{2}$$

For ε^1 with $x_0 = 2$ we have

$$8x_1 + 2 = 0 \implies x_1 = -\frac{1}{4}$$

For ε^1 with $x_0 = -$ we have

$$8x_1 + 2 = 0 \implies x_1 = -\frac{1}{4}$$

Now since we found 3 different values for x_0 we have that there will be 3 different approximations for x. Thus, by plugging in the values of x_0 and the corresponding x_1 into our assumption of x we have

$$\begin{aligned} \tilde{x}_1 &= \frac{\varepsilon}{2} \\ \tilde{x}_2 &= 2 - \frac{\varepsilon}{4} \\ \tilde{x}_3 &= -2 - \frac{\varepsilon}{4} \end{aligned}$$

(b) To evaluate our approximations for the roots with $\varepsilon = 0.001$ we simply substitute this value into the equations from a). Hence we have

$$\begin{aligned} \tilde{x}_1 &= 5\text{E} - 04 \\ \tilde{x}_2 &= 1.99975 \\ \tilde{x}_3 &= -2.00025 \end{aligned}$$

(c) To solve for the unknown a, b, and c we can simply multiply out the expression given to us and compare it to the original equation. That is we can consider

$$\begin{aligned} x^3 - 4.001x + 0.002 &= (x-2)(x^2 + ax + b) \\ &= x^3 + (a-2)x^2 + (b-2a)x + (-2b) \end{aligned}$$

Thus we have the system of equations and solutions to be

$$\begin{cases} a - 2 = 0 \\ b - 2a = -4.001 \\ -2b = .002 \end{cases} \implies \begin{cases} a = 2 \\ b = -0.001 \end{cases}$$

So we have

$$x^3 - 4.001x + 0.002 = (x-2)(x^2 + 2x - 0.001)$$

Now solving for the roots the right hand side we have

$$\begin{aligned} x_1 &= -1 + \frac{\sqrt{4.004}}{2} \\ x_2 &= 2 \\ x_3 &= -1 - \frac{\sqrt{4.004}}{2} \end{aligned}$$

Now to compare them in a numerical fashion we have

$$\begin{aligned} |\tilde{x_1} - x_1| &= 1.24937\text{E} - 07 \\ |\tilde{x_2} - x_2| &= 2.49875\text{E} - 04 \\ |\tilde{x_3} - x_3| &= 2.50000\text{E} - 04 \end{aligned}$$

Here we can note that our approximation is a good approximation for basic root finding, especially considering the low computational cost, however for real application we would like to have a smaller error in which we can just consider more higher order terms.