HW3

① $\dot{\underset{\sim}{x}} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \underset{\sim}{x}$

Solve the above system. Note: while one of the eigenvalues is repeated, you will not have trouble find 2 associated vectors. Hint: Although there are repeated roots, you won't need to use the matrix exponential expansion.

② Solve $\dot{\underline{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underline{x}$; $\underline{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

③ Optional: show that the eigenvalues of $T^{-1}AT$ are the same as those for A

④ Optional: verify the Cayley Hamilton Theorem for

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

⑤ Compute e^{At}, given $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$

⑥ Solve $\dot{\underset{\sim}{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$ with $\underset{\sim}{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⑦ Optional: fill in the details for the problem done in class, namely

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 0 \\ 0 \\ e^t \cos 2t \end{pmatrix} ; \quad x(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

8 Solve $\dot{\underset{\sim}{x}} = \begin{pmatrix} 3 & -4 \\ -1 & -1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$

$$\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

HW3 Solutions

$\dot{x} = Ax \qquad A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

$\det(A - \lambda \mathbb{I}) = 0 \qquad p(\lambda) = -(\lambda+1)^2(\lambda-8) = 0$

so $\lambda = -1$ is repeated, λ is a simple root

For $\lambda = 8 \qquad (A - 8\mathbb{I})v = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \underset{\sim}{v} = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

For $\lambda = -1 \qquad \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \underset{\sim}{v} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underset{\sim}{v} = 0$

here we can propose $\underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \underset{\sim}{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

both are linearly independent of each other and $\underset{\sim}{v}_1$

$\therefore$ Not necessary to use the expansion procedure on e^{At}:

the solution is $\underset{\sim}{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} e^{-t}$

(2) $\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\det(A - \lambda \mathbb{I}) = 0 \quad \Rightarrow p(\lambda) = (\lambda - 3)(\lambda + 1) = 0$

$\lambda_1 = 3 \qquad \lambda_2 = -1$

for $\lambda_1 = 3 \qquad \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \underset{\sim}{v} = 0 \qquad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{x}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

for $\lambda_2 = -1$

$(A + 1)\underset{\sim}{v} = 0 \qquad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \underset{\sim}{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\underline{x}_2(t) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t}$

$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} \quad ; \quad x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow c_1 = 7/4 \quad c_2 = -1/4$

$\therefore \underset{\sim}{x}(t) = \frac{7}{4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} - \frac{1}{4} \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} = \frac{1}{4} \begin{pmatrix} 7e^{3t} + 4e^{-t} \\ 2e^{3t} - 2e^{-t} \end{pmatrix}$

(3) Show that eigenvalues of A are the same as those of

$T^{-1}AT$

$$\det(A-\lambda I) = \cancel{\det(TAT}\ \det(T^{-1}AT-\gamma I) = 0$$

$$\det(A-\lambda I) = \det[\underbrace{T T^{-1}}_{I} A \underbrace{T T^{-1}}_{I} - T\gamma I T^{-1}] = 0$$

$$\det(A-\lambda I) = \det[A - \gamma T I T^{-1}] = \det[A - \gamma I \overbrace{T T^{-1}}^{I}] = 0$$

$$\therefore \det(A-\lambda I) = \det(A-\gamma I) = 0$$

$$\therefore \lambda = \gamma$$

(5) Compute e^{At} given $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}$

first, find 3 lin indep vectors solving $\dot{x} = Ax$

since A is diagonal then e'values are diagonal entries:

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda_3 = 5$$

λ_1: $(A-1)v_1 = 0 \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad x_1 = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

λ_2: $(A-3)v_2 = 0 \quad \underline{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad x_2 = e^{3t} \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = (A-5)v_3 = 0 \quad v_3 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \quad x_3 = e^{5t} \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

$$\bar{X} = \begin{bmatrix} e^t & e^{3t}/2 & e^{5t}/2 \\ 0 & e^{3t} & e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

$$\bar{X}(0) = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \therefore [\bar{X}(0)]^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \Phi\, X(0)^{-1} = \begin{bmatrix} e^{t} & -\frac{1}{2}e^{t}+\frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t}+\frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t}+e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

$$= \exp\left[\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} t\right]$$

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(8) $\dot{x} = \begin{pmatrix} 3 & -4 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = x_H + x_p$

$\det \begin{vmatrix} 3-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = 1 \pm 2\sqrt{2}$

$(A - \lambda_1 I) v_1 = 0 \qquad v_1 = \begin{pmatrix} \frac{-4}{2\sqrt{2}-2} \\ 1 \end{pmatrix} \qquad \lambda_1 = 1+2\sqrt{2}$

$(A - \lambda_2 I) v_2 = 0 \qquad v_2 = \begin{pmatrix} \frac{+4}{2\sqrt{2}+2} \\ 1 \end{pmatrix}$

$x_H = c_1 \begin{pmatrix} \frac{-4}{2\sqrt{2}-2} \\ 1 \end{pmatrix} e^{(1+2\sqrt{2})t} + c_2 \begin{pmatrix} \frac{4}{2\sqrt{2}+2} \\ 1 \end{pmatrix} e^{(1-2\sqrt{2})t}$

$\Phi = \begin{pmatrix} \frac{-4}{2\sqrt{2}-2} e^{(1+2\sqrt{2})t} & \frac{4}{2\sqrt{2}+2} e^{(1-2\sqrt{2})t} \\ 1 e^{(1+2\sqrt{2})t} & 1 e^{(1-2\sqrt{2})t} \end{pmatrix} = \begin{pmatrix} \alpha e^{\beta t} & \gamma e^{\theta t} \\ e^{\beta t} & e^{\theta t} \end{pmatrix}$

$\Phi^{-1} = \begin{pmatrix} -\frac{1}{e^{\beta t}(\gamma-\alpha)} & \frac{\gamma}{e^{\beta t}(\gamma-\alpha)} \\ \frac{1}{e^{\theta t}(\gamma-\alpha)} & -\frac{\alpha}{e^{\theta t}(\gamma-\alpha)} \end{pmatrix}$

here $\alpha = -\frac{4}{2\sqrt{2}-2}$

$\beta = 1+2\sqrt{2}$

$\gamma = \frac{4}{2\sqrt{2}+2}$

$\theta = 1-2\sqrt{2}$

$X_p = \Sigma u$ substitute into

$\therefore \quad \dot{x} = \Delta x + f \qquad$ where $f =$

$$\dot{X}_p = \Delta x_p + f$$

$$\dot{\Sigma} u + \Sigma \dot{u} = \Delta \Sigma u + f$$

$$\Sigma \dot{u} = f \Rightarrow \dot{u} = \Sigma^{-1} f$$

$$u = \int_0^t \Sigma^{-1} f \, ds + u(0)$$

$$u = \int_0^t \begin{pmatrix} -\frac{e^{s-\beta s}}{\gamma - \alpha} + \frac{\gamma e^{s-\beta s}}{(\gamma - \alpha)} \\ \frac{e^{s} e^{-\theta s}}{(\gamma - \alpha)} - \frac{\alpha e^{s-\theta s}}{(\gamma - \alpha)} \end{pmatrix} ds + u(0)$$

$$u = \begin{pmatrix} -\frac{(\gamma - 1) e^{(1-\beta)t}}{(\beta - 1)(\gamma - \alpha)} \\ \frac{\alpha - 1}{(\theta - 1)} \frac{e^{(1-\theta)t}}{(\gamma - \alpha)} \end{pmatrix} + u(0)$$

$$\therefore\ X = \Phi\int\Phi^{-1}u + c_1 x_1 + c_2 x_2$$

$$= \frac{e^t}{(\beta-1)(\gamma-\alpha)(\theta-1)}\begin{bmatrix} \alpha\beta\gamma - \alpha\gamma\theta + \alpha\theta - \beta\gamma - \alpha + \gamma \\ -\alpha\beta + \gamma\theta + \alpha + \beta - \gamma - \theta \end{bmatrix} + c_1\begin{pmatrix}\alpha \\ 1\end{pmatrix}e^{\beta t} + c_2\begin{pmatrix}\gamma \\ 1\end{pmatrix}e^{\theta t}$$

$$x(0) = \begin{pmatrix}1 \\ 1\end{pmatrix} = \frac{1}{(\beta-1)(\gamma-\alpha)(\theta-1)}\begin{bmatrix} \end{bmatrix} + c_1\begin{pmatrix}\alpha \\ 1\end{pmatrix} + c_2\begin{pmatrix}\gamma \\ 1\end{pmatrix}$$

Substitute numerical values:

$$\begin{bmatrix} \end{bmatrix}\frac{1}{(\beta-1)(\gamma-\alpha)(\theta-1)} = \begin{pmatrix}\frac{1}{4} \\ 0.375\end{pmatrix}$$

$$\therefore \begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}\frac{1}{4} \\ 0.375\end{pmatrix} + c_1\begin{pmatrix}\alpha \\ 1\end{pmatrix} + c_2\begin{pmatrix}\gamma \\ 1\end{pmatrix}$$

$$\begin{aligned} \alpha c_1 + \gamma c_2 &= 3/4 \\ c_1 + c_2 &= 0.625 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= -0.0411 \\ c_2 &= 0.6661 \end{aligned}$$

$$\therefore\ x = e^t\begin{pmatrix}\frac{1}{4} \\ 0.375\end{pmatrix} - 0.0411\begin{pmatrix}\frac{-4}{2\sqrt{2}-2} \\ 1\end{pmatrix}e^{(1+2\sqrt{2})t} + 0.6661\begin{pmatrix}\frac{4}{2\sqrt{2}+2} \\ 1\end{pmatrix}e^{(1-2\sqrt{2})t}$$

(6) Solve $\dot{\underline{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underline{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix} \qquad \underline{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\underline{x} = \underline{x}_H + \underline{x}_p$$

Find $x_H = C_1 x_1 + C_2 x_2$

$$\det(A - \lambda I) = 0 \qquad p(\lambda) = \lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i$$

$$(A - \lambda I)\underline{v}_1 = (A - i)\underline{v} = 0 \qquad v_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$(A - \lambda I)v_2 = 0 = (A + i)\underline{v} = 0 \qquad v_2 = \cancel{\begin{pmatrix} 2-i \\ 1 \end{pmatrix}} \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

$$e^{it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = (\cos t + i\sin t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

$$\therefore \quad x_1 = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} \qquad x_2 = \begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix}$$

Next, find $x_p = \underline{X}\,\underline{u}$

$$X = \begin{pmatrix} 2\cos t - \sin t & 2\sin t - \cos t \\ \cos t & \sin t \end{pmatrix},$$

Substitute

$x_p = \Phi u$ into $\dot{x} = Ax + f$

$$\underline{\underline{\dot{\Phi} u}} + \Phi \dot{u} = \underline{\underline{A \Phi u}} + f$$

cancel

$$\Phi \dot{u} = f \Rightarrow \dot{u} = \Phi^{-1} f$$

$$\Phi^{-1} = \begin{bmatrix} \sin t & -2\sin t - \cos t \\ -\cos t & 2\cos t - \sin t \end{bmatrix} \frac{1}{2\cos^2 - 1}$$

$$\Phi^{-1} f = \Phi^{-1} \begin{pmatrix} \sin t \\ 0 \end{pmatrix} = \frac{1}{2\cos^2 t - 1} \begin{bmatrix} \sin^2 t \\ -\cos t \sin t \end{bmatrix}$$

$$u = \int_0^t \Phi^{-1}(s) f(s)\, ds + u(0) = \begin{bmatrix} \frac{1}{4}\ln|\tan t - 1| - \frac{1}{4}\ln(\tan t - 1) - \frac{1}{2}t \\ \frac{1}{4}\ln(2\cos^2 t - 1) \end{bmatrix}$$

$+ u(0)$

$$x_p = \Phi u = \Phi \int_0^t \Phi^{-1}(s) f(s)\, ds + \Phi u(0)$$

$$x(t) = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t - \cos t \\ \sin t \end{pmatrix} + \Phi\int_0^t \Phi^{-1} f(s)\,ds$$

note: the contribution $\Phi u(0)$ has been absorbed into the first two terms above.

Now, apply I.C.

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$c_1 = 0 \qquad c_2 = 0$$

$$\therefore\ x(t) = \Phi\int_0^t \Phi^{-1} f(s)\,ds \Rightarrow$$

$$x(t) = \begin{pmatrix} (2\cos t - \sin t)\left(\frac{1}{4}\ln(\tan t+1) - \frac{1}{4}\ln(\tan t - 1 - \frac{1}{2}t)\right) \\ + \frac{1}{4}(2\sin t - \cos t)\ln(2\cos^2 t - 1) \\ \\ \cos t\left(\frac{1}{4}\ln(\tan t+1) - \frac{1}{4}\ln(\tan t - 1) - \frac{1}{2}t\right. \\ + \frac{1}{4}\sin t \cdot \ln(2\cos^2 t - 1) \end{pmatrix}$$