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Homework 1

Problem 1

In the blast wave problem discussed in class, assume instead

$$g(t, r, \rho, e, p) = 0$$

where P is the ambient pressure.

(a) How many dimensionless groups can be formed?

(b) Using these groups, does it still follow that r varies like the two-fifths power of t?

Solution: **(a)** Considering the dimensions of the variables in our g function we have the following:

$$t = [T] \quad r = [L] \quad \rho = [ML^{-3}] \quad e = [ML^2T^{-2}] \quad P = [M^{-2}T^{-2}L^{-1}].$$

Defining the same variables from class we can let m denote the number of variables of g and n denote the number of unique basic units. That is to say from above we can note that $m = 5$ and $n = 3$. Forming our matrix of variable to basic units we have the following matrix

$$A = \begin{array}{c} \\ T \\ L \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} t & r & \rho & e & P \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & -3 & 2 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Using MATLAB (code is not included due to it being a calculation but ready upon request) we can calculate the rank of$A$ to be 3. Applying *Buckingham's* π *Theorem* we have that there will be $m - \text{rank}(A)$ many dimensionless groupings. Since $m = 5$ and rank(A)=3 we know that there will be 2 dimensionless groupings.

Note: since this is the first problem it will be worked out completely in the sense of superscripts as well, then we will note the correlation to the matrix A in which the correlation will be used to determine other system of equations derived throughout the homework.

Since we want to system to be dimensionless we apply superscripts to the variables to simplify the problem into a system of equations, then solve. That is we have the following steps:

$$\begin{aligned} 1 &= [t^{\alpha_1} r^{\alpha_2} \rho^{\alpha_3} e^{\alpha_4} P^{\alpha_5}] \\ &= \left[T^{\alpha_1} L^{\alpha_2} (ML^{-3})^{\alpha_3} (ML^2T^{-2})^{\alpha_4} (MT^{-2}L^{-1})^{\alpha_5}\right] \\ &= \left[T^{\alpha_1 - 2\alpha_4 - 2\alpha_5} L^{\alpha_2 - 3\alpha_3 + 2\alpha_4 - \alpha_5} M^{\alpha_3 + \alpha_4 + \alpha_5}\right] \end{aligned}$$

Because the system needs to be dimensionless and the last line corresponds to our basic dimensions we need the superscripts to be equal to zero for every term, that is to say we have the following system of equations

$$\begin{cases} \alpha_1 - 2\alpha_4 - 2\alpha_5 = 0 \\ \alpha_2 - 3\alpha_3 + 2\alpha_4 - \alpha_5 = 0 \\ \alpha_3 + \alpha_4 + \alpha_5 = 0 \end{cases}$$

To see the correspondence between the system and the matrix A we define a vector $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]^T$. Each α_i corresponds to the i'th column variable, for example $\rho = \alpha_3$. With this defined vector, the system is derived by the following equation

$$A\alpha = 0.$$

With this in mind, every problem after this will use this process, instead of doing the calculations for the system through the basic dimension terms and their superscripts.

Back to the problem at hand, we are interested in time t and radius r, with this in mind we are interested in α_1 and α_2 the corresponding superscripts. Through basic algebra we get the following simplified system of equations

$$\begin{cases} \alpha_1 = 2\alpha_4 + 2\alpha_5 \\ \alpha_2 = -5\alpha_4 - 2\alpha_5 \\ \alpha_3 = -\alpha_4 - \alpha_5 \end{cases}$$

Being interested in t and r we can consider the following two cases.
Case 1 Let $\alpha_1 = 1$ and $\alpha_2 = 0$. This gives the following system

$$\begin{cases} 1 = 2\alpha_4 + 2\alpha_5 \\ 0 = -5\alpha_4 - 2\alpha_5 \\ \alpha_3 = -\alpha_4 - \alpha_5 \end{cases}$$

Solving through MATLAB (again the code is not included for being a simple calculation, available upon request) we get that

$$\alpha_3 = 0$$
$$\alpha_4 = -\frac{1}{3}$$
$$\alpha_5 = \frac{1}{3}$$

Thus we get the following dimensionless grouping $\pi_1 = re^{-\frac{1}{3}}P^{\frac{1}{3}}$.
Case 2 Let $\alpha_1 = 0$ and $\alpha_2 = 1$. This gives the following system

$$\begin{cases} 0 = 2\alpha_4 + 2\alpha_5 \\ 1 = -5\alpha_4 - 2\alpha_5 \\ \alpha_3 = -\alpha_4 - \alpha_5 \end{cases}$$

Solving through MATLAB (code is not included, available upon request) we get that

$$\alpha_3 = -\frac{1}{2}$$
$$\alpha_4 = -\frac{1}{3}$$
$$\alpha_5 = \frac{5}{6}$$

Thus we get the following dimensionless grouping $\pi_2 = t\rho^{-\frac{1}{2}}e^{-\frac{1}{3}}P^{\frac{5}{6}}$.

(b) Now to see if the problem still follow that r varies like the two-fifth power of t, we use these values for our superscripts pertaining to the variables. That is we let $\alpha_1 = -\frac{2}{5}$ and $\alpha_2 = 1$. From this substitution we get the following system

$$\begin{cases} -\frac{2}{5} = 2\alpha_4 + 2\alpha_5 \\ 1 = -5\alpha_4 - 2\alpha_5 \\ \alpha_3 = -\alpha_4 - \alpha_5 \end{cases}$$

Solving this system in MATLAB we get that

$$\alpha_3 = \frac{1}{5}$$
$$\alpha_4 = -\frac{1}{5}$$
$$\alpha_5 = 0$$

Thus we do have a dimensionless grouping that follows the set values, $\pi = t^{-\frac{2}{5}} r \rho^{\frac{1}{5}} e^{-\frac{1}{5}}$. Now to find a function of r we can choose π_1 to be the other dimensionless grouping for the following step. So we have a function $F(\pi, \pi_1) = 0$, in which we can make the assumption that $\pi = h(\pi_1)$ for some function h. In other words we have the following implication by solving for r

$$\pi = t^{-\frac{2}{5}} r \rho^{\frac{1}{5}} e^{-\frac{1}{5}} \implies r = t^{\frac{2}{5}} \rho^{\frac{1}{5}} e^{-\frac{1}{5}} h(\pi) = t^{\frac{2}{5}} \rho^{\frac{1}{5}} e^{-\frac{1}{5}} h\left(re^{-\frac{1}{3}} P^{\frac{1}{3}}\right)$$

Hence we still have that relationship, however with a function h that is still dependent on r.

Problem 2

A physical system is described by a law $f(E, P, A) = 0$ where E is energy, P is pressure, and A is area. Show that

$$\frac{PA^{\frac{3}{2}}}{E} = \text{constant} \qquad \text{or} \qquad E = \kappa P A^{\frac{3}{2}}$$

where κ is a constant.

Solution: First we take note of the variables and their dimensions,

$$E = [ML^2T^{-2}] \quad A = [L^2] \quad P = [MT^{-2}L^{-1}].$$

From here we can construct the following matrix:

$$A = \begin{matrix} & E & A & P \\ T \\ L \\ M \end{matrix} \begin{pmatrix} -2 & 0 & -2 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Taking note we have that the number of variables is $m = 3$ and calculating the rank of A being 2, using the *Buckingham's π Theorem* we have $3 - 2 = 1$ dimensionless groups. Defining $\alpha = [\alpha_1, \alpha_2, \alpha_3]^T$ with respect to the variables E,A, and P (the columns of the matrix) we calculate $A\alpha = 0$ to determine our system of equations. This quick calculation derives the following

$$\begin{cases} -2\alpha_1 - 2\alpha_3 = 0 \\ 2\alpha_1 + 2\alpha_2 - \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \end{cases}$$

Solving the system algebraically we can find that there is a dependence between α_1 and α_3, that is we have $\alpha_1 = -\alpha_3$. With this dependence we can we can let $\alpha_1 = -1$ which would imply that $\alpha_3 = 1$, solving for α_2 we have that $\alpha_2 = \frac{\alpha_3}{2} - \alpha_1 = \frac{1}{2} + 1 = \frac{3}{2}$. Thus we have that $E^{-1}A^{\frac{3}{2}}P$ will be dimensionless, or in other words

$$\frac{PA^{\frac{3}{2}}}{E} = \text{constant}$$

Problem 3

We want to find the power P that must be applied to keep a ship of length L moving at constant speed V. Assume P depends on the water density ρ, acceleration of gravity g, the viscosity of water ν (units of length squared/time) as well as L and V. Show that

$$\frac{P}{\rho L^2 V^3} = f(Fr, Re)$$

where $Fr = \frac{V}{\sqrt{Lg}}$ and $Re = \frac{VL}{\nu}$.

Solution: First we take note of the variables and their dimensions,

$$P = [ML^2T^{-3}] \quad \nu = [L^2T^{-1}] \quad \rho = [ML^{-3}] \quad V = [LT^{-1}] \quad g = [LT^{-2}] \quad l = [L]$$

Constructing the matrix as done in previous problems we have

$$A = \begin{array}{c} \\ T \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} P & \nu & \rho & V & g & l \end{array} \\ \begin{pmatrix} -3 & -1 & 0 & -1 & -2 & 0 \\ 2 & 2 & -3 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

To apply *Buckingham's π Theorem* , we note that that the number of variables $m = 6$ and computed rank of A is 3. Thus there are 3 dimensionless groupings. Here we can define $\alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6]^T$ to correspond to P, ν, ρ, V, g, and l. Then to get the system of equation we calculate $A\alpha = 0$ and get

$$\begin{cases} -3\alpha_1 - \alpha_2 - \alpha_4 - 2\alpha_5 = 0 \\ 2\alpha_1 + 2\alpha_2 - 3\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0 \\ \alpha_1 + \alpha_3 = 0 \end{cases}$$

It is clear that we have an over-determined system of equations so we look at the question at hand to give us some insight to the problem, that is to say we consider the following cases.
Case 1 We have that $Fr = \frac{V}{\sqrt{Lg}} = Vl^{-\frac{1}{2}}g^{-\frac{1}{2}}$, this would imply that $\alpha_4 = 1$, $\alpha_5 = -\frac{1}{2}$, and $\alpha_6 = -\frac{1}{2}$. With this in mind we can use these values to turn our over-determined system to a solvable system of equation through substitution. The resulting system is

$$\begin{cases} -3\alpha_1 - \alpha_2 = 0 \\ 2\alpha_1 + 2\alpha_2 - 3\alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \end{cases}$$

Solving the system we get that $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. That is to say that with $\alpha_4 = 1$, $\alpha_5 = -\frac{1}{2}$, and $\alpha_6 = -\frac{1}{2}$ we have a dimensionless grouping. Hence $\pi_1 = Fr$.

Case 2 We also have that $Re = \frac{Vl}{\nu} = Vl\nu^{-1}$, this would imply that $\alpha_2 = -1$, $\alpha_4 = 1$, and $\alpha_6 = 1$. Substituting this into our over-determined system we get the new system to be

$$\begin{cases} -3\alpha_1 - 2\alpha_5 = 0 \\ 2\alpha_1 - 3\alpha_3 + \alpha_5 = 0 \\ \alpha_1 + \alpha_3 = 0 \end{cases}$$

Solving the system we get that $\alpha_1 = 0$, $\alpha_3 = 0$, and $\alpha_5 = 0$. That is to say that with $\alpha_2 = -1$, $\alpha_4 = 1$, and $\alpha_6 = 1$ we have a dimensionless grouping. Hence $\pi_2 = Re$.
Case 3 Lastly we have from the question $\frac{P}{\rho l^2 V^3} = P\rho^{-1}l^{-2}V^{-3}$, this would imply that $\alpha_1 = 1$, $\alpha_3 = -1$, $\alpha_4 = -3$, and $\alpha_6 = -2$. With these values we can substitute it into the original system to get a solvable one, which turns out to be

$$\begin{cases} -\alpha_2 - 2\alpha_5 = 0 \\ 2\alpha_3 + \alpha_5 = 0 \\ 0 = 0 \end{cases}$$

Solving the system confirms that $\alpha_2 = 0$ and $\alpha_5 = 0$, which implies that with $\alpha_1 = 1$, $\alpha_3 = -1$, $\alpha_4 = -3$, and $\alpha_6 = -2$ we have a dimensionless system. Hence $\pi_3 = \frac{P}{\rho l^2 V^3}$.

With these 3 cases we have that $F(\pi_1, \pi_2, \pi_3) = 0$ in which we can make the assumption that $\pi_1 = f(\pi_2, \pi_3)$ for some function of f. That is to say we have shown, with the substitutions of π_i for $i = 1 : 3$, the following

$$\frac{P}{\rho L^2 V^3} = f(Fr, Re)$$

Problem 4

The *Lennerd-Jones* potential is defined by

$$V(r) = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$

where σ and ε are constants. This is the potential for many molecular systems, separated by a distance r. In general it is known that the viscosity μ (mass/(length and time)) of a gas depends only on the temperature and molecular properties. Show that it is not possible, however, for

$$\mu = f(m, \varepsilon, \sigma, \tau)$$

for some function f, where m is the molecular mass, τ is the temperature. On the other hand, by including *Boltzmann's constant* κ (1.38×10^{-6}erg/K°) show that

$$\frac{\mu\sigma^2}{\sqrt{m\varepsilon}} = f\left(\frac{\kappa\tau}{\varepsilon}\right).$$

Solution: To show that it is not possible for the viscosity of a gas depends only on the temperature and molecular properties for $\mu = f(m, \varepsilon, \sigma, \tau)$ we first consider the dimensions of the variables for f. That is we have the following

$$\sigma = [L] \qquad \varepsilon = [ML^2T^{-2}] \qquad m = [M] \qquad \tau = [K]$$

Now we also know that $\mu = [ML^{-1}T^{-1}]$, so for $\mu = f(m, \varepsilon, \sigma, \tau)$ we have that $f = [ML^{-1}T^{-1}]$ as well. Thus we can see if the relationship described is not possible by solving for the superscripts as done so for *Problem 1.* that is we have the following

$$\begin{aligned} ML^{-1}T^{-1} &= [m^{\alpha_1}, \varepsilon^{\alpha_2}, \sigma^{\alpha_3}, \tau^{\alpha_4}] \\ &= M^{\alpha_1}\left(ML^2T^{-2}\right)^{\alpha_2} L^{\alpha_3} K^{\alpha_4} \\ &= M^{\alpha_1+\alpha_3} L^{2\alpha_2+\alpha_3} T^{-2\alpha_2} K_4^{\alpha} \end{aligned}$$

Thus we have the following system of equations by comparing the left and right hand side of the equality:

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ 2\alpha_2 + \alpha_3 = -1 \\ -2\alpha_2 = -1 \\ \alpha_4 = 0 \end{cases}$$

Solving the system leads to the following values: $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = 0$, and $\alpha_4 = 0$. Hence we see that the superscript dealing with the temperature τ is $\alpha_4 = 0$, thus the relationship described in the problem does not exist for $\mu = f(m, \varepsilon, \sigma, \tau)$.

Now we are asked to add the constant $\kappa = [KL^2T^{-1}K^{-1}]$, so that it is claimed that

$$\frac{\mu\sigma^2}{\sqrt{m\varepsilon}} = f\left(\frac{\kappa\tau}{\varepsilon}\right).$$

For briefness of discussion we will label the left hand side of the equality to be the LH, and the right hand side of the equality to be RH. Constructing the matrix as done so in the previous problems we have

$$A = \begin{matrix} & m & \varepsilon & \sigma & \tau & \kappa & \mu \\ T & 0 & -2 & 0 & 0 & -2 & -1 \\ L & 0 & 2 & 1 & 0 & 2 & -1 \\ M & 1 & 1 & 0 & 0 & 1 & 1 \\ K & 0 & 0 & 0 & 1 & -1 & 0 \end{matrix}$$

Constructing the α vector to correspond with the variables we have the following system of equations

$$\begin{cases} -2\alpha_2 - 2\alpha_5 - \alpha_6 = 0 \\ 2\alpha_2 + \alpha_3 + 2\alpha_5 - \alpha_6 = 0 \\ \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 = 0 \\ \alpha_4 - \alpha_5 = 0 \end{cases}$$

For the LH we can see that $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$, $\alpha_3 = 2$, and $\alpha_6 = 1$. Plugging these values into the system we get the following system of equations

$$\begin{cases} -2\alpha_5 = 0 \\ 2\alpha_5 = 0 \\ \alpha_5 = 0 \\ \alpha_4 - \alpha_5 = 0 \end{cases}$$

Here it is obvious that $\alpha_5 = 0$ which directly implies that $\alpha_4 = 0$, thus we the α values that the LH side holds the relationships responsible for a dimensionless grouping. For the RH we can see that

$\alpha_2 = -1$, $\alpha_4 = 1$, and $\alpha_5 = 1$. Plugging these values into the system we get the following system of equations

$$\begin{cases} \alpha_6 = 0 \\ \alpha_3 - \alpha_6 = 0 \\ \alpha_1 + \alpha_6 = 0 \\ 0 = 0 \end{cases}$$

Since we can see that $\alpha_6 = 0$ which directly implies that $\alpha_1 = 0$ and $\alpha_2 = 0$, so the properties of a dimensionless group holds. Now considering *Buckingham's π Theorem* we also can note that the number of variables is $m = 6$ where the rank of A is $n = 4$ implies that there are 2 dimensionless groupings, in which we have in our LH and RH. So we have that $F(\pi_1, \pi_2) = 0$ with $\pi_1 = \frac{\mu\sigma^2}{\sqrt{m\varepsilon}}$ and $\pi_1 = \frac{\kappa\tau}{\varepsilon}$. Thus letting $\pi_1 = f(\pi_2)$ and solving for μ we have the following

$$\mu = \frac{\sqrt{m\varepsilon}}{\sigma^2} f\left(\frac{\kappa\tau}{\varepsilon}\right)$$

which shows the desired description of μ.

Problem 5

Consider the heat equation problem on a 1D rod:

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$$

where $u = u(x,t)$ is temperature, $0 \leq x \leq L$ is the position coordinate, $0 < t$ is time, κ is thermal diffusivity, κ is in dimension L^2T^{-1}. The boundary conditions are $u(x = 0, t) = T_\circ$ for $t > 0$. $T_\circ$ is a constant known temperature. The rod is initially at $u(x, 0) = 0$ for $0 < x < l$.

(a) Find the scaling that turns the problem into

$$\begin{aligned} \frac{\partial \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{t}} - \frac{\partial^2 \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{x}^2} &= 0 \qquad \tilde{t} > 0 \text{ and } 0 < \tilde{x} < 1 \\ \tilde{u}(\tilde{x}, 0) &= 0 \qquad 0 < \tilde{x} < 1 \\ \tilde{u}(0,\tilde{t}) = \tilde{u}(1,\tilde{t}) &= 1 \qquad \tilde{t} > 0 \end{aligned}$$

(b) What can you say about the typical rate of change of u by your choice of time scale?

Solution: **(a)** To find the scaling to transform the problem into the desired result we first consider the following dimensions

$$\kappa = [L^2t^{-1}] \qquad x = [L] \qquad t = [T] \qquad u = [K] \qquad T_\circ = [K] \qquad l = [L]$$

To write the variables in a dimensionless form we can consider the following

$$\begin{aligned} \tilde{u} = T_\circ^{-1}u &\implies u = \tilde{u}T_\circ \\ \tilde{t} = t\kappa l^{-2} &\implies t = \tilde{t}\kappa^{-1}l^2 \\ \tilde{x} = xl^{-1} &\implies x = \tilde{x}l \end{aligned}$$

Since we will wish to substitute these dimensionless terms into the PDE, we will also consider the partial differential operators under this transformation by using the chain rule as such:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} = \kappa l^{-2}\frac{\partial}{\partial \tilde{t}}$$
$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x}\right) = \frac{\partial}{\partial x}\left(l^{-1}\frac{\partial}{\partial \tilde{x}}\right) = l^{-2}\frac{\partial^2}{\partial \tilde{x}^2}$$

Thus by considering each term of the original PDE separately we see the following result:

$$\frac{\partial u}{\partial t} = \kappa l^{-2}\frac{\partial u}{\partial \tilde{t}} = \kappa l^{-2} T_\circ \frac{\partial \tilde{u}}{\partial \tilde{t}}$$
$$\frac{\partial^2 u}{\partial x^2} = l^{-2}\frac{\partial^2 u}{\partial \tilde{x}^2} = l^{-2} T_\circ \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}$$

Making the substitution into the PDE we have the following implication:

$$\kappa l^{-2} T_\circ \frac{\partial \tilde{u}}{\partial \tilde{t}} - \kappa l^{-2} T_\circ \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0 \implies \frac{\partial \tilde{u}}{\partial \tilde{t}} - \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = 0$$

Now considering the domain, initial conditions and boundary conditions of the problem and the dimensionless terms we have the following list of implications with substitutions:

$$0 < x < l \implies 0 < \tilde{x} l < l \implies 0 < \tilde{x} < 1$$
$$0 < t \implies 0 < \tilde{t}\kappa^{-1} l^2 \implies 0 < \tilde{t}$$
$$u(x,0) = 0 \implies \tilde{u}(x,0)T_\circ = 0 \implies \tilde{u}(x,0) = 0$$
$$u(x,t) = T_\circ \implies \tilde{u}(0,t)T_\circ = T_\circ \implies \tilde{u}(0,t) = 1$$
$$u(l,t) = T_\circ \implies \tilde{u}(1,t)T_\circ = T_\circ \implies \tilde{u}(1,t) = 1$$

Combining all this information into a more ordinary looking format we have

$$\frac{\partial \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{t}} - \frac{\partial^2 \tilde{u}(\tilde{x},\tilde{t})}{\partial \tilde{x}^2} = 0 \qquad \tilde{t} > 0 \text{ and } 0 < \tilde{x} < 1$$
$$\tilde{u}(\tilde{x},0) = 0 \qquad 0 < \tilde{x} < 1$$
$$\tilde{u}(0,\tilde{t}) = \tilde{u}(1,\tilde{t}) = 1 \qquad \tilde{t} > 0$$

(b) A typical change of rate in u pertaining to our choice of time scale is simply to look at $\frac{\partial \tilde{u}}{\partial \tilde{t}}$, that is we are looking at the term that was calculated for part **(a)**

$$\frac{\partial u}{\partial t} = \kappa l^{-2} T_\circ \frac{\partial \tilde{u}}{\partial \tilde{t}}$$

Solving for $\frac{\partial \tilde{u}}{\partial \tilde{t}}$ we have that

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = l^2 \kappa^{-1} T_\circ^{-1} \frac{\partial u}{\partial t}$$

Considering what we see above we can determine that one change in dimensionless quantities represents a change in $l^2\kappa^{-1}$ per unit of temperature. Also in taking note of what the variables stand for it also gives us a time scale that relates the length of the bar to the diffusivity.