HW1 Solution

(1) $y'' + \lambda^2 y = 0$ ODE

$y(0) = 0 \qquad y'(1) + y(1) = 0$ B.C.

is an SL with $r = 1$ $q = 0$ $p = 1$

The solution of ODE is

$$y = A\cos\lambda x + B\sin\lambda x$$

apply B.C. $y(0) = 0 \Rightarrow A = 0$

$y = B\sin\lambda x \qquad \Rightarrow$ let $\phi_n(x) = \sin\lambda_n x$

$$y(1) + y'(1) = \sin\lambda_n + \lambda_n\cos\lambda_n = 0$$

or $\lambda_n = -\tan\lambda_n \quad n = 1, 2 \ldots$

The orthogonality condition is

$$\int_0^1 \phi_n(x)\phi_m(x)\,dx = N^2\delta_{nm} \quad \text{, so if } n = m$$

$$\int_0^1 \phi_n^2(x)\,dx = \left[\frac{x}{2} - \frac{\sin 2\lambda_n x}{4\lambda_n}\right]_0^1 = \frac{1 + \cos^2\lambda_n}{2}$$

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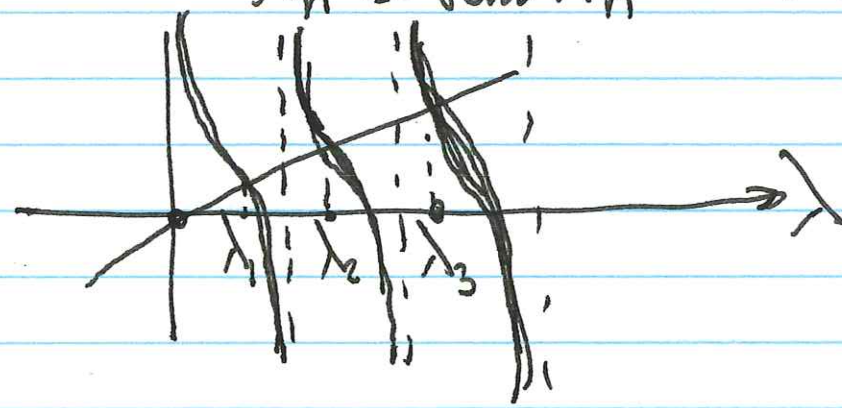
so let $\hat{\phi}_n = \dfrac{\sqrt{2}}{(1+\cos^2\lambda_n)^{1/2}} \sin\lambda_n x$

$n=1,2\ldots$

and λ_n are the roots of

$\lambda_n = -\tan\lambda_n \qquad n=1,2\ldots$



(2) In class we showed that λ is real

(a) Take $\mathcal{L}\phi = r\lambda\phi$; $\overline{\mathcal{L}\phi} = \overline{r\lambda\phi}$ or

$\mathcal{L}\bar{\phi} = r\lambda\bar{\phi}$ $\therefore$ $\bar{\phi}$ is an eigenfunction of S.L. w/ eigenvalue λ.

(b) Part (a) shows that λ is an eigenvalue of SL and that ϕ & $\bar{\phi}$ are both eigenfunctions of SL. But in class we also showed that there is a 1 to 1 correspondence between each λ and an eigenfunction. So we might have a contradiction, unless $\phi = \bar{\phi}$.

(i) Two functions are LI (linearly independent on $x=a$, to $x=b$ if

$$W(f,g) \neq 0 \text{ for some } a \leq x \leq b.$$

(ii) Two functions f, g are LI iff $c_1 = c_2 = 0$ are the only values taken by these constants to make

$$(*) \qquad c_1 f + c_2 g = 0 \text{ for sure } a \leq x \leq b.$$

If we write $c_1 \phi + c_2 \bar{\phi} = 0$ (consider ϕ sure arbitrary function)

$$c_1(\phi_r + i\phi_i) + c_2(\phi_r - i\phi_i) = 0$$

then $c_1 = c_2 = 0$. Unless EITHER ϕ_r or ϕ_i is zero.

We compute $W(\phi, \bar{\phi})(x) = (\phi \bar{\phi}_x - \phi_x \bar{\phi}) = 2i(\phi_{rx}\phi_i - \phi_r \phi_{ix})$

If ϕ solves SL and obeys B.C. then at $x=0$

$W(\phi, \bar{\phi})(0) = 0$. This is because at $x=0$

$$\phi_x(0) = -\frac{1}{A}\phi(0)$$

Some more detail:

(I) We first note that $\phi \equiv \phi_r + i\phi_i$ and $\bar{\phi} = \phi_r - i\phi_i$ are linearly independent (cannot write $\phi = c\bar{\phi}$, c constant). UNLESS either ϕ_r or ϕ_i is zero.

(II) $W(\phi, \bar{\phi}) = (\phi \bar{\phi}_x - \phi_x \bar{\phi}) = 2i(\phi_{rx}\phi_i - \phi_r \phi_{ix})$

I pick a convenient place to evaluate $W(x)$. For example, pick $x = 0$. There $\phi(0) + A\phi'(0) = 0$

$\therefore$ I can write $\phi_x(0) = -\frac{1}{A}\phi(0)$ $\therefore$

$$W(0) = \left(-\frac{1}{A}\phi_r\phi_i + \frac{1}{A}\phi_r\phi_i\right)(0) = 0$$

(III) The BVP is a linear second order equation with the conditions required to guarantee a unique solution and the general solution is $y = c_1 y_1(x) + c_2 y_2(x)$

~~(IV)~~

where $W(y_1, y_2) \neq 0$ (linearly independent).

This calculation suggests that ϕ and $\bar{\phi}$, both solutions to SL + B.C. are not LI.

Since $W(0)=0$ this implies that a solution of the form $y = c_1\phi + c_2\bar{\phi}$ has nontrivial c_1 & c_2 solutions to

$$\begin{cases} c_1\phi(0) + c_2\bar{\phi}(0) = 0 \\ c_1\phi'(0) + c_2\bar{\phi}'(0) = 0 \end{cases}$$

by the existence & uniqueness theorem of 2nd order ODE's

$$y(0) = y'(0) = 0 \;\therefore\; y(x) = 0 \;\forall\; 0<x<1.$$

In order for all inconsistencies to be removed and the results be correct, we have to have that $\phi = \phi_r$ (to within a multiplicative constant) : $\phi = \bar{\phi}$!

③ The eigenfunction expansion of x, for $0 \le x \le 1$ is

$$(\ddagger) \qquad x = \sum_{n=1}^{\infty} c_n \hat{\varphi}_n(x)$$

$$\hat{\varphi}_n(x) = \sqrt{2}\sin n\pi x \qquad n = 1, 2 \ldots$$

Multiply b.s. of $(\ddagger)$ by $\sqrt{2}\sin m\pi x$ & integrate:

$$\int_0^1 x\,\hat{\varphi}_m(x)\,dx = \int_0^1 \hat{\varphi}_m(x) \sum_{n=1}^{\infty} c_n \hat{\varphi}_n(x)\, dx$$

$$\sqrt{2}\int_0^1 x \sin m\pi x\, dx = \sum_{n=1}^{\infty} c_n \int_0^1 \hat{\varphi}_m(x)\hat{\varphi}_n(x)\, dx = c_m$$

integrate by parts to obtain:

$$c_m = -\sqrt{2}\,\frac{1}{m\pi}\left[\cos m\pi + \frac{1}{m\pi}\overset{0}{\sin m\pi}\right] = +\frac{\sqrt{2}}{m\pi}(-1)^{m+1}$$

$$\therefore \; x = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\pi}(-1)^{n+1} \sin n\pi x$$

(4) Solve $y''+2y=-x \qquad 0<x<1$

$$y(0)=y(1)=0$$

$y=y_H+y_p$ but $y_H=0$

$$\therefore y=y_p=\sum_{n=1}^{\infty} a_n \sin n\pi x$$ a_n's unknown.

Substituting into ODE,

$$\frac{d^2}{dx^2}\sum_{n=1}^{\infty} a_n \sin n\pi x + 2\sum_{n=1}^{\infty} a_n \sin n\pi x = -\sum_{n=1}^{\infty}\frac{\sqrt{2}}{n\pi}\overset{(-1)^{n+1}}{\vphantom{x}}\sin n\pi x$$

but $\frac{d^2}{dx^2}\sin n\pi x = -n^2\pi^2 \sin n\pi x$

$$\therefore \left(2-n^2\pi^2\right)a_n = \frac{\sqrt{2}}{n\pi}(-1)^{n+1} \qquad n=1,2\ldots$$

$$\therefore a_n = \frac{\sqrt{2}}{(2-n^2\pi^2)}\frac{1}{n\pi}(-1)^{n+1} \qquad n=1,2\ldots$$

then $$y=\sum_{n=1}^{\infty}\frac{\sqrt{2}}{(2-n^2\pi^2)}\frac{1}{n\pi}(-1)^{n+1}\sin n\pi x$$

(5) Check that

$$\begin{cases} (1+x^2)\,y'' + 2x\,y' + y = \lambda(1+x^2)\,y & \text{for } 0<x<1 \\ y(0) - y'(1) = 0 \quad \text{and} \quad y'(0) + y(1) = 0 \end{cases}$$

is self adjoint.

The ODE is of the form of a S.L:

the SL is $\mathcal{L}y = \alpha W y$ plus $\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$

$$\mathcal{L}y = -(p(x)\,y')' + q(x)\,y$$

$$(\star)\ \left[\begin{array}{l} p(x) > 0 \text{ for } 0<x<1, \quad p, q \in C[0,1] \\ \alpha \text{ is a constant} \\ W(x) > 0 \text{ for } 0 \le x \le 1 \end{array}\right.$$

so let $\alpha = -\lambda \quad p(x) = 1+x^2 \quad q = -1$

$W(x) = 1+x^2$

which clearly satisfy $(\star)$ properties

Next, we confirm that $(Lu,v) = (Lv,u)$

Let $L = (1+x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx} + 1$

or $L = \frac{d}{dx}\left[(1+x^2)\frac{d}{dx}\right] + 1$

$$\int_0^1 (Lu)v\,dx = \int_0^1 \frac{d}{dx}\left[(1+x^2)u'\right]v\,dx + \int_0^1 uv\,dx$$

integrate 1st term by parts:

$$\begin{cases} ds = v & w = \left[(1+x^2)u'\right] \\ ds = \frac{dv}{dx}dx & dw = \frac{d}{dx}\left[(1+x^2)u'\right]dx \end{cases}$$

$$\int_0^1 (Lu)v\,dx = v(1+x^2)u'\Big|_0^1 - \int_0^1 \left[(1+x^2)u'\right]\frac{dv}{dx}dx + \int_0^1 uv\,dx$$

integrate 2nd term by parts:

$$s = (1+x^2)\frac{dv}{dx} \qquad w = u$$

$$ds = \frac{d}{dx}\left[(1+x^2)\frac{dv}{dx}\right]dx \qquad dw = \frac{du}{dx}dx$$

$$\int_0^1 (Lu)v\,dx = v(1+x^2)u'\Big|_0^1 - (1+x^2)\frac{dv}{dx}u\Big|_0^1$$
$$+ \int_0^1 \frac{d}{dx}\left[(1+x^2)v'\right]u\,dx + \int_0^1 uv\,dx$$

$$\int_0^1 Luv\,dx = \underbrace{(1+x^2)\left[vu' - v'u\right]\Big|_0^1}_{} + \int_0^1 (Lv)u\,dx$$

if ↗ is zero then

$$\int_0^1 (Lu)v\,dx = \int (Lv)u\,dx$$ self-adjoint.

so $v(1)\underline{\underline{u'(1)}} - v(0)\underline{\underline{u'(0)}} - v'\underline{\underline{(1)}}u(1) + v'(0)u(0)$

the B.C. $y'(1) = y(0)$ and $y'(0) = -y(1)$

so $v(1)\underline{\underline{u(0)}} + v(0)\underline{\underline{u(1)}} - v\underline{\underline{(0)}}u(1) - \underline{\underline{v(1)}}u(0) = 0$

so it checks out.