HW1

(1) Determine the eigenvalues & normalized eigenfunctions to

$$(BVP)\begin{cases} y'' + \lambda^2 y = 0 \\ y(0) = 0 \quad , \quad y'(1) + y(1) = 0 \end{cases}$$

Also, show that BVP is Sturm-Liouville problem.

(2) We sketch out a proof that the eigenfunctions of the SL problem

$$(SL)\begin{cases} [p(x)y']' + q(x)y = \lambda r(x) y, \quad 0 < x < 1 \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$$

are real:

(a) Let λ be an e'value and ϕ a corresponding e'function. Show that $\bar{\phi}$ (its complex conjugate) is also an eigenfunction corresponding to the e'value λ.

(b) Show that ϕ and $\bar{\phi}$ are Linearly DEPENDENT.

Use the theorem on SL that proves that all eigenvalues of the SL are simple and they correspond, one to one, to individual eigenfunctions. Also, use the Wronskian of ϕ and $\bar{\phi}$ to prove that ϕ and $\bar{\phi}$ must be LINEARLY DEPENDENT.

(c) Following up on the result of (b), show that ϕ must be real (apart from a multiplicative constant).

(3) Let $\hat{\phi}_n = \sqrt{2}\sin n\pi x \qquad n=1,2,3\ldots$

(a) Show that

$$(\hat{\phi}_n, \hat{\phi}_m) = \delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

for $0 \leq x \leq 1$,

and thus $\{\hat{\phi}_n\}_{n=1,2\ldots}$ are orthogonal and normalized on $x \in [0,1]$.

(b) Find the explicit expression for c_i, $i=1,2\ldots$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \hat{\phi}_i(x) \qquad (*)$$

on $0 \leq x \leq 1$

when $f(x) = x$

$(*)$ is called an "eigenfunction expansion" of $f(x)$

To find c_i, multiply both sides of (*) by $\hat{\phi}_m(x)$ and integrate both sides with respect to x from $x=0$ to $x=1$. Here $m=1,2,3\ldots$

(4) The general solution to the general linear differential equation

$$Ly = f(x) \text{ and B.C.}$$

here, L is a differential operator is

$$y(x) = y_H(x) + y_p(x)$$

where $Ly_H = 0$ and $Ly_p = f(x)$

Use the results of (3) and the eigenfunction expansion

$$y_p(x) = \sum_{i=1}^{\infty} \alpha_i \phi_i(x)$$

to solve

$$\begin{cases} y'' + 2y = -x, & 0<x<1 \\ y(0) = y(1) = 0 \end{cases}$$

(5) Determine whether the problem

$$(1+x^2)y'' + 2xy' + y = \lambda(1+x^2)y$$

$$\left.\begin{aligned} y(0) &= y'(1) \\ y'(0) &= -2y(1) \end{aligned}\right\}$$

is self adjoint