Math 481/581 –Final

Name: ______________________________

Score

1.	**(25)**	
2.	**(25)**	
3.	**(15)**	
4.	**(25)**	
5.	**(10)**	
Total		

1. (25 %) Linear System
2. (25 %) Dynamical System
3. (15 %) Regular Perturbation Problem
4. (25 %) Singular Perturbation Problem
5. (10 %) Neatness

- Please write neatly: it's worth 10% of your grade (as you can see above, I mean it).
- Turn in this cover page with your exam and don't forget to put your name on it.

1. Find the solution to

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= A\mathbf{x}, \quad t > 0 \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned}$$

   where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

2. Investigate the linearized asymptotic stability of the equations

$$\begin{aligned} \dot{y}_1 &= y_1(3 - y_1 - y_2) \\ \dot{y}_2 &= y_2(y_1 - 1) \end{aligned}$$

   - Find critical points.
   - Investigate the stability of the critical points.
   - For each of the critical points, plot the local structure of perturbations.

3. Let

$$f(x; \epsilon) = \int_x^\infty e^{-t} t^\epsilon dt.$$

   Investigate the limit,

$$\lim_{x \to \infty} f(x; \epsilon), \tag{1}$$

   as a function of ϵ. To do this,

   - obtain a regular perturbation series, using repeated integration by parts on $f(x; \epsilon)$. Obtain the first 3 terms of the series.
   - Infer the form of the n^{th} term in the series.
   - Estimate the limit indicated in Equation (1) by applying the limit to its approximation.
   - How does that limit behave, as a function of the parameter ϵ, for $\epsilon \approx 0$? You might find it convenient to define $\epsilon = \lambda - 1$.

4. Consider the problem

$$
\begin{aligned}
\epsilon \frac{d^2y}{dx^2} - \frac{dy}{dx} &= 2x, \quad 0 < x < 1, \\
y(0) = a, y(1) = b.
\end{aligned}
$$

- Solve the equation exactly.
- Using the exact solution, argue about whether a boundary layer is present.
- Argue, based upon the differential equation whether a boundary layer forms and where it is to be found.
- Find an inner and (a matched) outer approximate solution to the differential equation.
- Find the uniform approximation that uses the inner and outer approximations.
- Plot approximate solution, with $a = -1, b = 1$. Superimposed, plot the uniform approximation on top of the exact solution. Take $\epsilon = 0.1$, and $\epsilon = 0.5$ (present 2 plots).