PH 457/557   Lecture #7

QM

Time dependence in the harmonic oscillator

Classical oscillator: $x(t) \sim \overset{\sin}{\cos}(\omega t+\varphi)$

Quantum oscillator: ?

$\Downarrow$ examine time dependence!

Consider a general state $|\Psi(0)\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$

where $|n\rangle$ is an eigenstate of $\hat{H}$, so $\hat{H}|n\rangle = E_n|n\rangle$

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

Then,

$$|\Psi(t)\rangle = \sum_n C_n e^{-\frac{i}{\hbar}E_n t}|n\rangle =$$

$$= \sum_n C_n e^{-i(n+\frac{1}{2})\omega t}|n\rangle = \underbrace{e^{-\frac{i\omega t}{2}}}_{\uparrow} \cdot \sum_n C_n e^{-in\omega t}|n\rangle$$

global phase factor doesn't change the physical state

What is the probability to measure energy E_n at some time t? $\Rightarrow P(E_n) = |C_n e^{-in\omega t}|^2 =$

$= |C_n|^2 \leftarrow$ is not time dependent! $\Rightarrow$

⇒ energy eigenstates are stationary states

$$|n\rangle|_t = |n\rangle_0 \, e^{-\frac{i}{\hbar}E_n t}$$

For example,

$$\underbrace{{}_t\langle n|\hat{X}|n\rangle_t}_{\text{expectation value of position with respect to stationary states}} = {}_0\langle n| e^{\frac{i}{\hbar}E_n t}\hat{X}\, e^{-\frac{i}{\hbar}E_n t}|n\rangle_0$$

$$= {}_0\langle n|\underbrace{\hat{X}}_{\sqrt{\frac{\hbar}{2m\omega}}(a^{+}+a)}|n\rangle_0 =$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle n|a^{+}+a|n\rangle = 0$$

⇒ expectation value of X is 0 in the stationary state regardless of time

How do we get a time-dependent $\langle X\rangle(t)$?

⇓

Consider a superposition:

e.g. $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \Rightarrow$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-\frac{3}{2}i\omega t}|1\rangle =$$

$$= \frac{1}{\sqrt{2}}e^{-i\omega t/2}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

Now, $\langle \Psi(t)|\hat{X}|\Psi(t)\rangle =$

$= |1\rangle$ $= 0$

$$= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\Big[\left(\langle 0| + e^{i\omega t}\langle 1|\right)\left(a^\dagger + a\right)\left(|0\rangle + e^{-i\omega t}|1\rangle\right)\Big] = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big[\left(\langle 0| + e^{i\omega t}\langle 1|\right)\cdot\left(|1\rangle + e^{-i\omega t}\left(\sqrt{2}|2\rangle + |0\rangle\right)\right)\Big] =$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\underbrace{\langle 0|0\rangle}_{=1} + e^{i\omega t}\underbrace{\langle 1|1\rangle}_{=1}\right) = \underbrace{\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t}$$

↑ Only 1st harmonic like class.!

Note: if $|\Psi(0)\rangle = c_m|m\rangle + c_n|n\rangle$ with $m \neq n \pm 1 \Rightarrow \langle \Psi(t)|\hat{X}|\Psi(t)\rangle = 0$

What else is time dependent?

⇓

E.g. Spatial probability density ⇒ for superposition states

$$P(x,t) = |\langle x|\Psi(t)\rangle|^2 = \leftarrow \text{e.g. } \Psi(t) = \frac{1}{\sqrt{2}}e^{-i\omega t/2}\left(|0\rangle + e^{-i\omega t}|1\rangle\right) \text{ like above}$$

$$= \frac{1}{2}\left|\varphi_0(x) + e^{-i\omega t}\varphi_1(x)\right|^2 =$$

$$= \frac{1}{2}\left(\varphi_0^2(x) + \varphi_1^2(x) + 2\varphi_0(x)\varphi_1(x)\cos\omega t\right)$$

Note: If $|\psi(0)\rangle = C_m|m\rangle + C_n|n\rangle$

with $m \neq n \pm 1 \Rightarrow \rho(x,t)$ would have higher harmonics!

⇙

e.g. if $m-n = 2 \Rightarrow \cos 2\omega t$

$- - - - 3 \quad \cos 3\omega t$

$. \; - - -$

} not like a classical oscillator

Also: only a particular superposition (coherent state) has a behavior similar to a classical oscillator! (wave packet that moves in the H.O potential without distortion)

⇑ Roy Glauber ⇐ David Wineland

Roy Glauber ⇑ Nobel Prize in Physics 2005

Roy Glauber ↑ realized these in optics ⇓ quantum optics

David Wineland ⇓ Nobel Prize in Physics 2012 ⇙ realized these in laser-trapping of ions