Harmonic oscillator; measurements, uncertainty relations

Examples of measurements

Ex. 1 A quantum H. O. is in a state

$$|\psi\rangle = \sqrt{\tfrac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|3\rangle \quad \leftarrow \hat{H}|n\rangle = E_n|n\rangle$$

If energy is measured, what are outcomes / probabilities?

$$|\psi\rangle = \sum_n C_n |n\rangle \Rightarrow P(E_n) = |C_n|^2 \Rightarrow$$

outcomes: $E_0 = \frac{\hbar\omega}{2} \Rightarrow P_0 = \frac{2}{3}$

$E_3 = \hbar\omega \cdot \underbrace{(3+\tfrac{1}{2})}_{= 7/2} \Rightarrow P_3 = \frac{1}{3}$

Ex. 2 A quantum H. O. is in a state described by a wavefunction $\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega^2 x^2}{2\hbar}} \cdot \left(\frac{1}{\sqrt{2}} + \frac{2x}{\sqrt{2}}\sqrt{\frac{m\omega}{2\hbar}}\right)$. If energy is measured, what are the outcomes & probabilities?

Need to present

$$\psi(x) = \sum_n \varphi_n(x) = \frac{1}{\sqrt{2}}\varphi_0(x) + \frac{1}{\sqrt{2}}\varphi_1(x) \Rightarrow$$

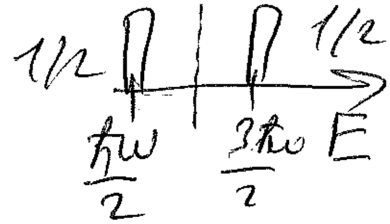
$\hat{H}\varphi_n(x) = E_n\varphi_n(x)$

$E_0 = \frac{\hbar\omega}{2} \Rightarrow P_0 = 1/2$

$E_1 = 3/2\,\hbar\omega \Rightarrow P_1 = 1/2$

What is the expectation value of energy? (2)

$$\langle E\rangle = \underbrace{P_0}_{1/2}\cdot\frac{\hbar\omega}{2} + \underbrace{P_1}_{1/2}\cdot\frac{3\hbar\omega}{2} = \frac{\hbar\omega}{4}+\frac{3\hbar\omega}{4} = \hbar\omega$$



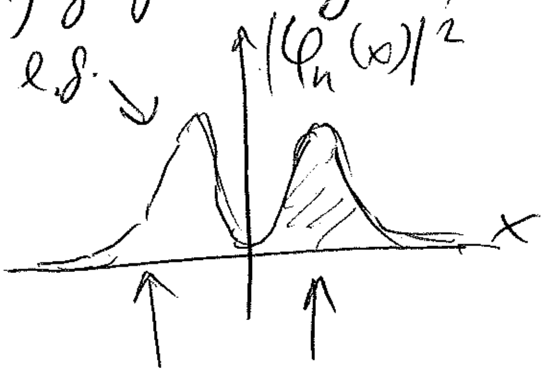
sanity check: $\frac{3}{2}\hbar\omega$, $\frac{\hbar\omega}{2}$ with equal probability $\Rightarrow$ average in the middle!

<u>Ex. 3</u> Find expectation value of X if a quantum H.O. is in some state $|n\rangle$ (eigenstate of $\hat{H}$, so $\hat{H}|n\rangle = E_n|n\rangle$)

X: position operator

1) graphically $\Rightarrow$ $\varphi_n(x)$ is either even or odd $\Rightarrow$ $|\varphi_n|^2$ is always even

since $V(X)=V(-X)$ for H.O.



same probability to find the particle at x & $-x$! $\Rightarrow \langle X\rangle = 0$

2) integral $\Rightarrow$ $\langle X\rangle = \int_{-\infty}^{+\infty}\varphi_n^*(x)\,X\,\varphi_n(x)\,dx =$

$$= \int_{-\infty}^{+\infty}\underbrace{\underbrace{|\varphi_n(x)|^2}_{\text{even}}\underbrace{X}_{\text{odd}}}_{\text{odd}}dx = 0$$

$\int \text{odd} = \text{even}$; $\text{even}\,\Big|_{-\infty}^{+\infty} = 0$

3) ket (abstract) (the best!)

recall: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + i\frac{\hat{P}}{m\omega}\right)$

$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - i\frac{\hat{P}}{m\omega}\right)$

$\Downarrow$

solve for $\hat{X}$ & $\hat{P}$ (in terms of a, a^+)

$\Downarrow$

$$\boxed{\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(a^+ + a\right)} \qquad \boxed{\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^+ - a\right)}$$

$\Downarrow$

$$\langle n|\hat{X}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a^+ + a|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \cdot$$

$$\cdot\left(\sqrt{n+1}\underbrace{\langle n|n+1\rangle}_{=0} + \sqrt{n}\underbrace{\langle n|n-1\rangle}_{=0}\right) = 0$$

$a|n\rangle = \sqrt{n}|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}|n+1\rangle$

(Note: similarly $\langle P\rangle \sim \langle n|a^+ - a|n\rangle = 0$)

What about $\langle X^2\rangle$? $\Rightarrow$ use abstract! $\Rightarrow$

$$\langle n|\hat{X}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|\underbrace{(a^+ + a)^2}_{= a^{+2} + a^2 + aa^+ + a^+a}|n\rangle =$$

$$= \frac{\hbar}{2m\omega}\langle n|\underbrace{aa^+}_{N+1} + \underbrace{a^+a}_{N}|n\rangle = \frac{\hbar}{2m\omega}(2n+1)$$

Uncertainty relations

Recall: $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$

$\Delta p = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$ $\Leftrightarrow$ standard deviation

In general, $\Delta A \cdot \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$

if $[\hat{A}, \hat{B}] \neq 0 \Rightarrow \hat{A}$ & $\hat{B}$ are incompatible observables

can't be simultaneously measured with high degree of accuracy

if $\Delta A \to 0$
$\Delta B \to \infty$
(and the other way around)
if $[\hat{A}, \hat{B}] \neq 0$

can achieve $\begin{cases} \Delta A \to 0 \\ \Delta B \to 0 \end{cases}$ for $[\hat{A}, \hat{B}] = 0$

but not for $[\hat{A}, \hat{B}] \neq 0$

If $\hat{A} = \hat{X}$, $\hat{B} = \hat{P} \Rightarrow \Delta X \Delta p \geq \frac{\hbar}{2}$

$[\hat{X}, \hat{P}] = i\hbar$

So how does this look like for quantum H.O.?

$\Delta X \Rightarrow \langle X\rangle = 0$, $\langle X^2\rangle = \frac{\hbar}{2m\omega}(2n+1)$

So $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{2n+1}$

$\Delta p \Rightarrow \langle p \rangle = 0, \quad \langle p^2 \rangle = \frac{m\hbar\omega}{2}(2n+1)$

HW#2

So $\Delta p = \sqrt{\frac{m\hbar\omega}{2}}\sqrt{2n+1}$

Then, $\Delta x \Delta p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\hbar\omega}{2}}\,(2n+1) =$

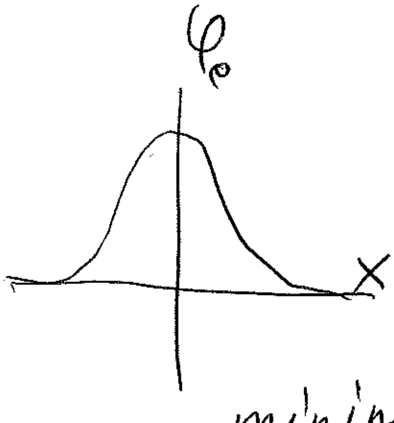
$= \frac{\hbar}{2}(2n+1)$

So, for H.O. the uncertainty relations are: $\frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}$

• Observation 1: ground-state $\Rightarrow n = 0$

get $\frac{\hbar}{2} = \frac{\hbar}{2}$ ← minimal uncertainty!

is this expected?



recall

$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

Ch. 6

minimal uncertainty wavepacket ⇐ Gaussian!!

- Observation 2: as $n \uparrow \Rightarrow$ the uncertainty $\uparrow$

harder to make precise measurements for excited-states