PH 451/551 QM

Lecture #5

①

Harmonic oscillator: wave functions, position representation

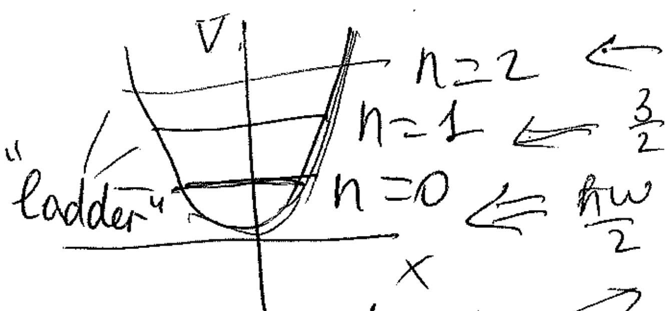
Recall: ladder operators $\Rightarrow a, a^+$

a.k.a annihilation or lowering (a)

creation or raising (a^+)

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$



$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

$n = 0, 1, 2, \ldots$

$a \to$ go down the "ladder" $\to$ lower energy by $\hbar\omega$

$a^+ \to$ go up $\to$ increase energy by $\hbar\omega$

- If we start at some state $|n\rangle$ & keep applying the annihilation operator $a \Rightarrow$ What is the lowest we can go?

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^2|n\rangle = a\,a|n\rangle = a\sqrt{n}\,|n-1\rangle =$

$= \sqrt{n}\sqrt{n-1}\,|n-2\rangle$

$\vdots$

Recall the number operator $\hat{N}$:

$$\hat{N}|n\rangle = n|n\rangle$$

$\times \langle n|$: $\langle n|\underbrace{\hat{N}|n\rangle}_{\substack{\| \\ n|n\rangle}} = n\underbrace{\langle n|n\rangle}_{\substack{\| \\ 1}}$

So $n = \langle n|\hat{N}|n\rangle \underset{\hat{N}=a^{\dagger}a}{=} \langle n|a^{\dagger}a|n\rangle =$

$= (\langle n|a^{\dagger})(a|n\rangle) = \underbrace{\|a|n\rangle\|^2}_{\text{norm of the vector}} \geq 0 \Rightarrow$

$\boxed{n \geq 0}$

If we apply a to $|n\rangle$, repeatedly, must stop at $n=0$ so that $\underbrace{a|0\rangle = 0}$

$\Updownarrow$

$E_0 = \frac{\hbar\omega}{2}$

ground state energy

- What if we start at $|0\rangle$ & apply $a^{\dagger}$ many times?

Start from $|0\rangle$: $a^{+}|0\rangle = |1\rangle$

$a^{+}|1\rangle = \sqrt{2}\,|2\rangle$

$a^{+}|2\rangle = \sqrt{3}\,|3\rangle$

$\vdots$

$$|3\rangle = \frac{1}{\sqrt{3}} a^{+}|2\rangle =$$

$$= \frac{1}{\sqrt{3}} a^{+}\left(\frac{1}{\sqrt{2}} a^{+}|1\rangle\right) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} \underbrace{a^{+}a^{+}a^{+}}_{\text{"}(a^{+})^{3}}|0\rangle$$

Generalize:

$$\boxed{|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle} \Leftarrow$$ can construct any eigenstate using creation operators applied to $|0\rangle$!

- How do we use states $|n\rangle$ to generate wave functions (in the position space)?

$\Downarrow$

start from the ground state

$$a|0\rangle = 0$$

recall that $\langle x|0\rangle = \varphi_0(x)$ ← ground-state wave function

So $a\,\varphi_0(x) = 0$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) \Rightarrow$$ since we are working in the position space

$$\Downarrow \qquad\qquad \Downarrow$$

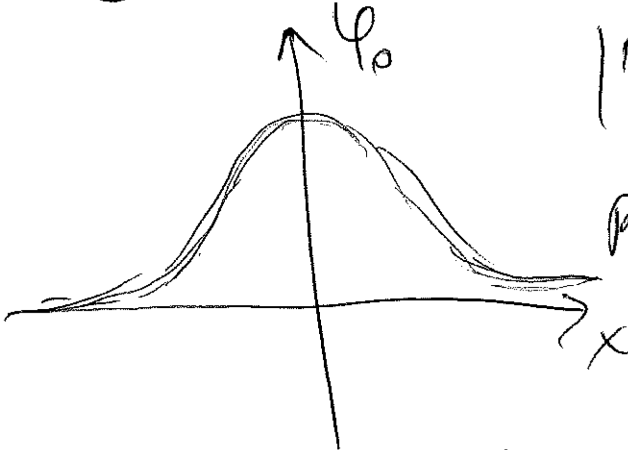
$$\hat{X} \doteq x \;;\; \hat{P} \doteq -i\hbar\frac{d}{dx}$$ need to express everything in terms of x!

$$\left(x + \frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\right)\right)\varphi_0(x) = 0$$

$$\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\varphi_0(x) = 0 \Rightarrow$$ differential equation for the ground-state wave function

$$\Downarrow$$

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$ $\leftarrow$ Gaussian!

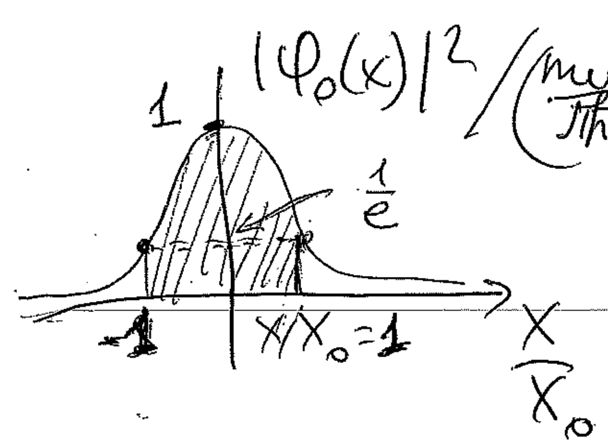


$$|\varphi_0(x)|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega x^2}{\hbar}} =$$

prob. density

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\left(\frac{x}{x_0}\right)^2}$$

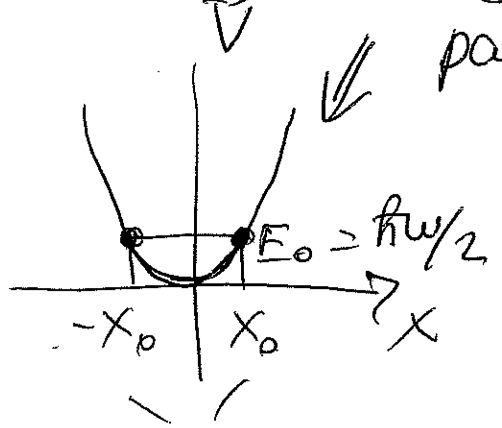
characteristic scale for wave function decay $\rightarrow$ $x_0 = \sqrt{\frac{\hbar}{m\omega}}$



$$\int_{-x_0}^{x_0} |\varphi_0(x)|^2 dx$$

probability to find the particle from $-x_0$ to x_0 $\Rightarrow$ clearly < 1!

Compare to a classical particle with $E_0 = \frac{\hbar\omega}{2} = \frac{1}{2} m\omega^2 x_0^2$



$$\Downarrow$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

turning points

$\Downarrow$ classical particle motion at this energy is completely confined between $-x_0$ & x_0!

Can we construct wave functions $\varphi_n(x)$ for any n? $\Rightarrow$ use $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \Rightarrow$

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\right]^n \varphi_0(x)$$

Ex. $\varphi_1(x) = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot$

$\cdot\, e^{-m\omega x^2/2\hbar} = \sqrt{\frac{m\omega}{2\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \left(x - \cancel{\frac{\hbar}{m\omega}} \cdot\right.$

$\left.\cdot\, \cancel{\left(-\frac{m\omega}{2\hbar}\right)\cdot 2x}\right) e^{-m\omega x^2/2\hbar} = \sqrt{\frac{m\omega}{2\hbar}} \cdot 2x \cdot$

$\cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

More generally: $\varphi_n(x) = \text{polynomial} \cdot e^{-m\omega x^2/2\hbar}$

Simplified notation: $\xi = \frac{x}{x_0} = \sqrt{\frac{\cancel{\hbar} m\omega}{\hbar}}\, x$

dimensionless

$$\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$$

$$\varphi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}$$

Hermite polynomials

$H_0(\xi) = 1$

$H_1(\xi) = 2\xi$

$H_2(\xi) = 4\xi^2 - 2$

$H_3(\xi) = 8\xi^3 - 12\xi$

$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$