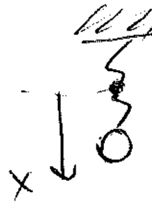
Lecture #4

Harmonic oscillator: number representation

Recall classical mechanics:



$\vec{F} = -Kx \Rightarrow F = -\frac{dV}{dx} \Rightarrow V = \frac{Kx^2}{2}$

↑ restoring force (Hooke's law) $\Rightarrow m\frac{d^2x}{dt^2} = -Kx \Rightarrow$

$\ddot{x} + \omega^2 x = 0$, $\omega = \sqrt{\frac{K}{m}}$

$x(t) = A\cos(\omega t + \varphi)$

↑ oscillatory motion

Now we want to study QM harmonic oscillator!

Why? → very important model for a variety of physical systems!

- molecular vibrations ↑ drive physical processes & chemical reactions
- lattice vibrations in solids (phonons) ↑ determine electronic properties
- EM field (photons) ↑ spontaneous emission; light-matter interactions
- → spin waves (magnons) ↑ information processing & computing

(2)

So, how do we approach the problem?

Typically start with classical mechanics

Lagrange formalism $\Rightarrow$ equations of the motion $\Leftarrow \mathcal{L}$

Hamiltonian mechanics

classical Hamiltonian

$$\sum_i p_i \dot{q}_i - \mathcal{L} = \mathcal{H} = \frac{p^2}{2m} + V(x)$$

p_i - generalized momentum

q_i - generalized coordinate

Transition to QM: take classical $\mathcal{H}$ & replace variables with operators

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$$

H. O.

$$\frac{k\hat{x}^2}{2} = \frac{m\omega^2\hat{x}^2}{2}$$

What do we want to know? $\Rightarrow$ energy!

$\hat{H}|n\rangle = E|n\rangle$ abstract

$\hat{H}\,\varphi_E(x) = E\,\varphi_E(x)$ position space

$\hat{H}\,\varphi_p(p) = E\,\varphi_p(p)$ momentum space

What do we want to solve?

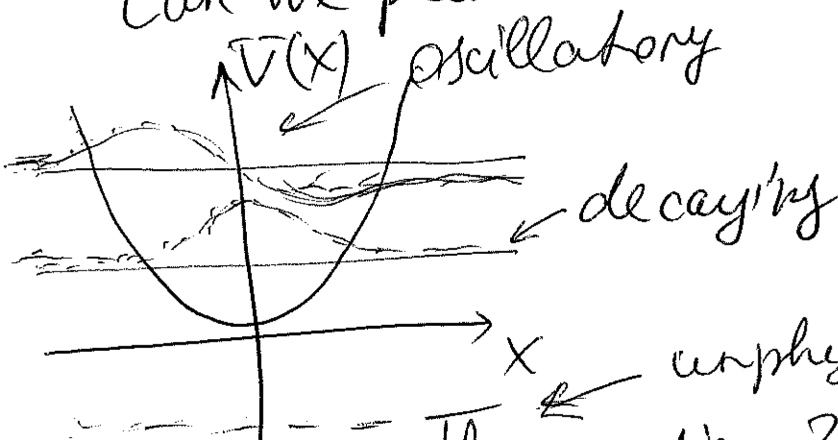
Examine position space $\Rightarrow$ ③

$$\hat{X} \doteq x \; ; \; \hat{P} \doteq -i\hbar \frac{d}{dx} \Rightarrow$$

need to solve

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_E}{dx^2} + \frac{m\omega^2 x^2}{2}\varphi_E = E\varphi_E(x)$$

Can we predict some behaviors without solving?



discrete spectrum

Solving the equation requires ansatz for a decay (behavior at $x \to \pm\infty$) & series solution.

like for Ⓗ-atom

Ch. 8

How about momentum space? $\Rightarrow$

$$\hat{P} \doteq p, \quad \hat{X} \doteq i\hbar \frac{d}{dp}$$

$$\left(\frac{p^2}{2m} - \frac{m\omega^2}{2}\hbar^2\frac{d^2}{dp^2}\right)\varphi_p(p) = E\varphi_p(p)$$

Is this better? looks like the equation for $\varphi_E(x)$...

Introduce a new basis / method!

$\Downarrow$

Operator method ; number representation

First introduce new operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + i\frac{\hat{P}}{m\omega}\right)$$ ← lowering (annihilation) operator

$$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X}^{+} - i\frac{\hat{P}^{+}}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - i\frac{\hat{P}}{m\omega}\right)$$

↑ raising (creation) operator

$$\left.\begin{aligned}\hat{X}^{+} &= \hat{X}\\ \hat{P}^{+} &= \hat{P}\end{aligned}\right\}$$ Hermitian operators

$$(a^{+})^{+} = a \; ; \; (a)^{+} = a^{+}$$ ← Hermitian conjugates of each other

How are these useful?

Consider $a^{+}a = \hat{N}$ ← number operator

Let's examine the relationship between $\hat{N}$ & $\hat{H}$ $\Rightarrow$

Note: $a \Rightarrow u + iv$
$a^{+} \Rightarrow u - iv$

$$a^{+}a \Rightarrow (u - iv)(u + iv) = u^2 - ivu + iuv + v^2$$

=

$$= u^2 + v^2 + i\,\underbrace{[u, v]}$$

$\uparrow$ commutator!

So $a^+ a = \frac{m\omega}{2\hbar}\left(\hat{X}^2 + \frac{\hat{P}^2}{m^2\omega^2} + i\left[\frac{\hat{P}}{m\omega}, \hat{X}\right]\right)$

$$= \frac{\hat{P}^2}{2\hbar m\omega} + \frac{m\omega}{2\hbar}\hat{X}^2 + \underbrace{\frac{-i}{2\hbar}\cdot(-i\hbar)}_{= -\frac{1}{2}} = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

$\uparrow$ $[\hat{X}, \hat{P}] = i\hbar$

Then, $\boxed{\hat{H} = \hbar\omega(\underbrace{a^+ a}_{= \hat{N}} + \tfrac{1}{2})}$

Note: $[\hat{H}, \hat{N}] = 0 \Rightarrow$ share eigenstates!

So if we have $\hat{N}|n\rangle = n|n\rangle$

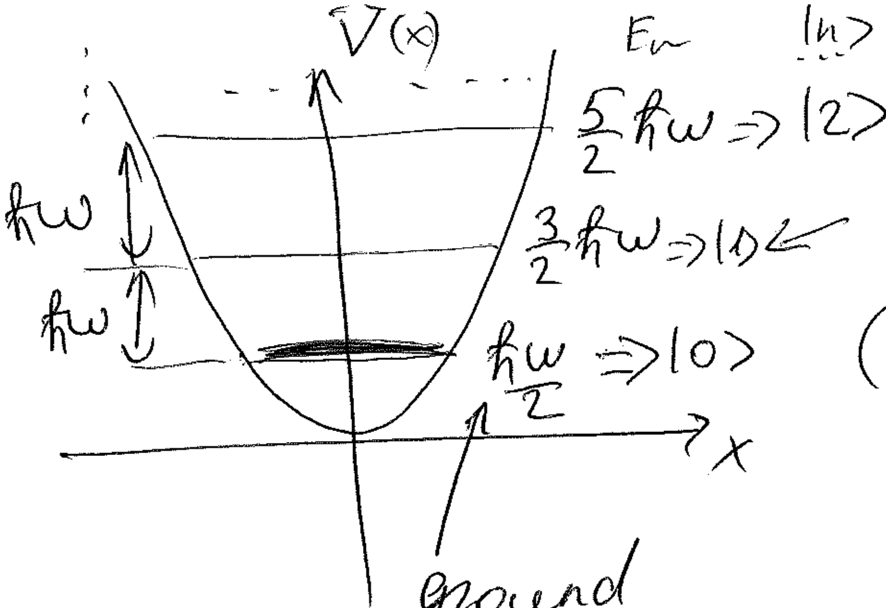
$\Downarrow$ eigenstate eigenvalues of the number operator $\langle n|m\rangle = \delta_{nm}$

$\Downarrow$

$$\hat{H}|n\rangle = \hbar\omega(\hat{N} + \tfrac{1}{2})|n\rangle =$$

$$= \underbrace{\hbar\omega(n + \tfrac{1}{2})}_{= E_n}|n\rangle \qquad n = 0, 1, 2, \ldots$$

$\leftarrow$ energies of H.O.



discrete energies of quantum H. O.

(note: classical limit $\hbar \to 0 \Rightarrow$ energy becomes continuous)

ground state

zero-point energy

$\hat{N} \leftarrow$ dimensionless quantity

Ex. Photons as quantum H. O.

quanta of EM wave $\leftarrow$ energy $\sim |E|^2$ or $|B|^2$

E- or B-fields

$n \leftarrow$ # of photons in the field

eigenvalues of $\hat{N}$

$|n\rangle \leftarrow$ state of the system with n photons

$|n\rangle \Rightarrow |n-1\rangle$

absorption event $\leftarrow$ "annihilation" of a photon

$\Downarrow$

$a|n\rangle \Rightarrow |n-1\rangle$

$|n\rangle \Rightarrow |n+1\rangle$

emission event

$\searrow$ creation of a photon $\Rightarrow a^\dagger|n\rangle \Rightarrow |n+1\rangle$

$n = 0 \Rightarrow$

$|0\rangle$ - vacuum state $\Leftarrow$ observable via light-matter interaction!

Is $a^\dagger a = a a^\dagger$? $\Leftrightarrow [a, a^\dagger] = 0$?

$$\underbrace{a a^\dagger}_{= (u+iv)\cdot(u-iv)} - \underbrace{a^\dagger a}_{= (u-iv)(u+iv) = u^2+v^2+i[u,v]} = 2i\,[u,v] = -2i \cdot \frac{1}{m\omega}\underbrace{[\hat X, \hat P]}_{= i\hbar} \cdot \frac{m\omega}{2\hbar} = 1$$

$$(u+iv)\cdot(u-iv) = u^2+v^2+i[v,u] = u^2+v^2-i[u,v]$$

So $[a, a^\dagger] = 1 \Rightarrow a a^\dagger = \underbrace{\hat N}_{= a^\dagger a} + 1$

What is the significance of a & $a^\dagger$? $\Rightarrow$

Consider $[\hat N, a] = [a^\dagger a, a] =$

commutator algebra $\rightarrow$ $[\hat A \hat B, \hat C] = \hat A[\hat B, \hat C] + [\hat A, \hat C]\hat B$

$$= a^\dagger \underbrace{[a, a]}_{= 0} + \underbrace{[a^\dagger, a]}_{= -1} a = -a$$

Use this: $\hat N a = -a + a \hat N \Rightarrow$

$$\underbrace{\hat N a|n\rangle}_{= |?\rangle} = \underbrace{-a|n\rangle}_{= |?\rangle} + a\underbrace{\hat N |n\rangle}_{= n|n\rangle}$$

$$\hat N|?\rangle = (n-1)|?\rangle \Rightarrow |?\rangle = c|n-1\rangle$$

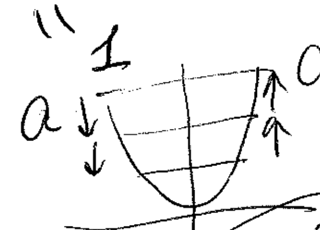
So $|?\rangle = a|n\rangle = C|n-1\rangle$

- action of ~~lowering~~ lowering operator a on $|n\rangle$ lowers the state to $|n-1\rangle$

- C ← ~~have to make sure that~~ ~~normalised~~

use $\langle ? | ? \rangle = |C|^2 \underbrace{\langle n-1|n-1\rangle}$

$$\underbrace{\langle n|a^+}_{\langle ?|}\underbrace{a|n\rangle}_{|?\rangle} = \langle n|\hat{N}|n\rangle = n$$



ladder operators

So $|C|^2 = n \Rightarrow C = \sqrt{n}$

Altogether: $\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$

Similarly: $[\hat{N}, a^+] = [a^+a, a^+] = a^+[a,a^+] + [a^+,a^+]a = a^+ \Rightarrow \hat{N}a^+|n\rangle = a^+|n\rangle + a^+\underbrace{N|n\rangle}_{n|n\rangle} = (n+1)\underbrace{a^+|n\rangle}_{|?\rangle}$

$\hat{N}|?\rangle = (n+1)|?\rangle$

$a^+|n\rangle = \tilde{C}|n+1\rangle$ → a^+ raising $|n\rangle$ to $|n+1\rangle$

$\langle n|\underbrace{aa^+}_{\hat{N}+1}|n\rangle = n+1 = |\tilde{C}|^2 \Rightarrow \boxed{a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$