PH451/551
QM

Lecture #18

①

Additional properties of C.-G. coefficients

Recall: $$\underbrace{|JM\rangle}_{\text{coupled basis state}} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C^{j_1 j_2 J}_{m_1 m_2 M} \underbrace{|j_1 j_2 m_1 m_2\rangle}_{\text{uncoupled basis state}}$$

$C^{j_1 j_2 J}_{m_1 m_2 M}$ → Clebsch-Gordan coefficients

use for HW#6 problem 1

Additional properties (some were already discussed in Lecture #17):

- $|JM\rangle$ with $J = j_1 + j_2$, $M = J = j_1 + j_2$ } maximal possible J & M for given j_1 & j_2

⇙ stretched state

Convention: $$|\underbrace{j_1+j_2}_{J}\ \underbrace{j_1+j_2}_{M}\rangle = |j_1 j_2 \underbrace{j_1}_{m_1} \underbrace{j_2}_{m_2}\rangle$$

Note: we have seen this for $j_1 = j_2 = 1/2$ in hyperfine ⇒

$$|\underset{F M_F}{11}\rangle = |\underset{m_s\, m_I}{++}\rangle \equiv |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

- Recursion relations obtained by applying $J_\pm$ ← ladder operators

Ex.

$$J_-|JM\rangle = \hbar\sqrt{J(J+1)-M(M-1)}\;|J\;M-1\rangle$$

So, if we had $|JM\rangle = \sum_{m_1,m_2} C^{j_1 j_2 J}_{m_1 m_2 M}\,|j_1 j_2 m_1 m_2\rangle$

Then, apply J_- to both sides ⟹ LHS the equation above;

RHS: $\sum_{m_1,m_2} C^{j_1 j_2 J}_{m_1 m_2 M}\,\underbrace{J_-}_{= J_{1-}+J_{2-}}|j_1 j_2 m_1 m_2\rangle =$

$$= \sum_{m_1,m_2} C^{j_1 j_2 J}_{m_1 m_2 M}\left(J_{1-}|j_1 j_2 m_1 m_2\rangle + J_{2-}|j_1 j_2 m_1 m_2\rangle\right)$$

$$= \sum_{m_1,m_2} C^{j_1 j_2 J}_{m_1 m_2 M}\Big(\hbar\sqrt{j_1(j_1+1)-m_1(m_1-1)}\;|j_1 j_2\, m_1-1\, m_2\rangle +$$

$$+\,\hbar\sqrt{j_2(j_2+1)-m_2(m_2-1)}\;|j_1 j_2\, m_1\, m_2-1\rangle\Big)$$

Now we have a relationship between

$|J\,M-1\rangle$ & $|j_1 j_2\, m_1-1\; m_2\rangle$, $|j_1 j_2\, m_1\; m_2-1\rangle$

<u>Note:</u> If you do this $\Rightarrow J_-|J\,J\rangle$ to the stretched state (i.e. $M = J = j_1 + j_2$; $m_{1,2} = j_{1,2}$)

$\Downarrow$ get a useful relation

$$|\underbrace{j_1+j_2}_{J}, \underbrace{j_1+j_2-1}_{M}\rangle \underset{\substack{\uparrow \\ \text{show!}}}{=} \sqrt{\frac{j_1}{j_1+j_2}}\, |j_1 j_2 \overbrace{j_1-1}^{m_1} \overbrace{j_2}^{m_2}\rangle$$

$$+ \sqrt{\frac{j_2}{j_1+j_2}}\, |j_1 j_2 \underbrace{j_1}_{m_1} \underbrace{j_2-1}_{m_2}\rangle$$

(so if you don't have C.-G. tables, you can use this!)

By applying J_- you can generate any $|J = j_1 + j_2, M\rangle$ relationships with uncoupled basis vectors

$-J \le M \le J$

What if we need to change J?

E.g. $|\underbrace{j_1+j_2-1}_{J}, \underbrace{j_1+j_2-1}_{M}\rangle = ?$ $\Rightarrow$

Can use orthogonality of $|JM\rangle$ states, (4)

$$\langle J'M'|JM\rangle = \delta_{JJ'}\delta_{MM'}$$

First construct:

$$|\underbrace{j_1+j_2-1}_{J}, \underbrace{j_1+j_2-1}_{M}\rangle = a\,|j_1 j_2 m_1 m_2\rangle +$$

$$+ b\,|j_1 j_2 m_1' m_2'\rangle + \ldots \text{ (more?)}$$

note: $M = m_1 + m_2$ → how many combos of m_1 & m_2 give $m_1+m_2 = j_1+j_2-1$

$|m_{1/2}| \le j_{1,2}$

⇓

only two: $m_1 = j_1$, $m_2 = j_2 - 1$ & $m_1 = j_1 - 1$, $m_2 = j_2$

So:

$$|j_1+j_2-1, j_1+j_2-1\rangle =$$

$$= a\,|j_1 j_2\, j_1-1\, j_2\rangle +$$

$$+ b\,|j_1 j_2\, j_1\, j_2-1\rangle\,;$$

now use orthogonality $\langle j_1+j_2-1, j_1+j_2-1 | j_1+j_2, j_1+j_2\rangle = 0$

⇓

$$a\sqrt{\frac{j_1}{j_1+j_2}} + b\sqrt{\frac{j_2}{j_1+j_2}} = 0 \Rightarrow \frac{a}{b} = -\sqrt{\frac{j_2}{j_1}}\,;\; a^2+b^2=1$$

↑ normalized

$$b = \sqrt{\frac{j_1}{j_1+j_2}}\,;\; a = -\sqrt{\frac{j_2}{j_1+j_2}}$$