PH 451/551
QM

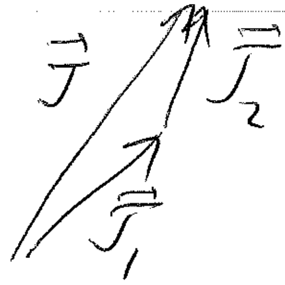
Lecture #17

(1)

Addition of generalized angular momenta

Need: a general rule of adding $\vec{J}_1$ & $\vec{J}_2$

$\vec{J} = \vec{J}_1 + \vec{J}_2 \Rightarrow$



$\Rightarrow |\vec{J}_{max}| = |\vec{J}_1| + |\vec{J}_2|$

$\vec{J}_1 \uparrow\uparrow \vec{J}_2$

$\uparrow \vec{J}_2$, $\uparrow \vec{J}_1$, $\uparrow \vec{J}_{max}$

$\vec{J}_1 \uparrow\downarrow \vec{J}_2$ $\quad J_{min} = |J_1 - J_2|$

Classically: $|\vec{J}|$ varies from $\left||\vec{J}_1| - |\vec{J}_2|\right|$ to $|\vec{J}_1| + |\vec{J}_2|$

QM: in quantized steps; $J = j_1 + j_2, j_1 + j_2 - 1, \ldots |j_1 - j_2|$

J: quantum number correspond. to $\vec{J}$

$j_{1,2}$: quantum number corresponding to $\vec{J}_{1,2}$

$-j_{1,2} \le m_{j_{1,2}} \le j_{1,2}$

$M = m_1 + m_2 \rightarrow M: -J \le M \le J$

$-J, -J+1, -J+2, \ldots, J$

<u>Ex.</u> $j_1 = 2$; $j_2 = 3 \rightarrow J = ?$

$J = \underbrace{2+3}_{=5}, 4, 3, 2, 1$

$|2-3| = 1$

Note that $J \geq 0$!

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$$\{\vec{J}_1^2, \vec{J}_2^2, J_{1z}, J_{2z}\} \Rightarrow \underbrace{|j_1 j_2 m_1 m_2\rangle}_{\text{uncoupled basis}}$$

set of commuting observables

? $\Updownarrow$

$$\{\vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z\} \Rightarrow \underbrace{|j_1 j_2 J M\rangle \equiv |JM\rangle}_{\text{coupled basis}}$$

Recall that we established the connection between coupled & uncoupled bases for a specific case of $j_1 = j_2 = 1/2 \Rightarrow$

$J = 1, 0 \Rightarrow$

$$|11\rangle = |++\rangle \equiv |\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \equiv \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + |\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$$

$$|1\,-1\rangle = |--\rangle \equiv |\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \equiv \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\tfrac{1}{2}\,-\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$$

Is there a generalised approach?

$\Downarrow$

yes! $\Rightarrow$

What is the dimensionality of the uncoupled & coupled bases? (must match!) ③

Changing basis for convenience should not affect physics

$\Downarrow$

$$|j_1 j_2 m_1 m_2\rangle \Rightarrow \underset{\text{dimensionality of basis}}{N_{j_1 j_2}} = (2j_1+1)\cdot(2j_2+1)$$

$\Downarrow$

for $j_{1,2}$ there are $2j_{1,2}+1$ states ($|j_1 m_1\rangle$ or $|j_2 m_2\rangle$)

On the other hand, for coupled basis $\Rightarrow$

$$|JM\rangle \Rightarrow N_J = 2J+1$$

$$\underbrace{N_{j_1 j_2}}_{(2j_1+1)(2j_2+1)} \overset{\text{show!}}{=} N_J = \sum_{J=|j_1-j_2|}^{j_1+j_2} (2J+1)$$ same dimensionality!

Any vector $|JM\rangle$ can be presented as a superposition of $\underbrace{|j_1 j_2 m_1 m_2\rangle}_{\text{form complete \& orthonormal basis}}$ vectors

How does completeness look like here?

$$I = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2|$$

$\uparrow$
identity

So: $|JM\rangle = I|JM\rangle =$

$$= \sum_{m_1}\sum_{m_2} |j_1 j_2 m_1 m_2\rangle \underbrace{\langle j_1 j_2 m_1 m_2 | JM\rangle}$$

relate coupled & uncoupled bases $\Leftarrow$ Clebsch-Gordan coefficients $\nearrow$ $C^{j_1 j_2 J}_{m_1 m_2 M}$

$\uparrow$ Tabulated, also can be derived

Properties of C.-G. coefficients:

1) $\langle j_1 j_2 m_1 m_2 | JM\rangle \neq 0$ only if
$M = m_1 + m_2$ & $|j_1 - j_2| \leq J \leq j_1 + j_2$

2) coeff. are <u>real</u> (convention)

3) $\langle j_1 j_2 \underbrace{j_1}_{m_1} \underbrace{J-j_1}_{m_2} | \underbrace{J\,J}_{M=J} \rangle > 0$ (convention)

max value $\rightarrow m_1$ $\nwarrow$ max value

Example Recall: $j_1 = j_2 = 1/2$ case

(from hyperfine discussion $S = I = 1/2$)

$$\overset{\swarrow F}{J}\ \overset{\swarrow M_F}{M}$$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|\overset{m_1 m_2}{+-}\rangle - |-+\rangle\right) = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle - |\tfrac{1}{2}\tfrac{1}{2}\,{-\tfrac{1}{2}}\tfrac{1}{2}\rangle\right)$$

C.-G. coeff. based on this:

$$\langle \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\,{-\tfrac{1}{2}}|00\rangle = \frac{1}{\sqrt{2}}$$ This must be >0 by convention

$$\langle \tfrac{1}{2}\tfrac{1}{2}\,{-\tfrac{1}{2}}\,\tfrac{1}{2}|00\rangle = -\frac{1}{\sqrt{2}}$$

(also; see the Table)

Why is it $(|+-\rangle - |-+\rangle)$ and not $(|-+\rangle - |+-\rangle)$ (given that the global phase doesn't matter)

$\Downarrow$

convention (C.-G. property #3 ← p. 4) $\Rightarrow$ $\langle \tfrac{1}{2}\tfrac{1}{2}\underset{\substack{\uparrow\\ m_1 = j_1}}{\tfrac{1}{2}}\,\underset{\substack{\uparrow\\ m_2 = J - j_1}}{-\tfrac{1}{2}}|\overset{J}{0}\,\overset{M=J}{0}\rangle > 0$

$m_2 = \underset{0}{J} - \underset{1/2}{j_1}$

Spectroscopic notation

Specifies atom state

$\vec{J} = \vec{L} + \vec{S}$

$^{2S+1}L_J$

$L = 0 \quad 1 \quad 2 \quad 3$

total orb.

$S \quad P \quad D \quad F \quad \ldots$

$^2S_{1/2} \rightarrow$ $L=0$, $S=1/2$, $J=1/2$

for single electron:

$l = 0 \Rightarrow s$
$1 \quad p$
$2 \quad d$
$\ldots$

$j = \underset{\underset{1/2}{\uparrow}}{l+s}, \ldots |l-s|$

So $l+\frac{1}{2}$, $l-\frac{1}{2}$ if $l \geq 1$ or $\frac{1}{2}$ for $l=0$

Example

A (H)-atom is in a $^2P_{1/2}$ state with total ang. momentum up along z-axis. With what probability will the e^- be found with spin down?

Solution! $^2P_{1/2} \Rightarrow$
$l = 1 \quad (=j_1)$
$s = 1/2 \quad (=j_2)$
$J = 1/2, \; M = 1/2$
given!

Need:
Prob. that e^- is in the $m_s = -\frac{1}{2}$ state!

Table 11.3

$$|\overset{J}{\tfrac{1}{2}}\;\overset{M}{\tfrac{1}{2}}\rangle \doteq \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}\;\tfrac{1}{2}\;1\;\overset{m_2}{-\tfrac{1}{2}}\rangle - \tfrac{1}{\sqrt{3}}\,|\tfrac{1}{2}\;\tfrac{1}{2}\;0\;\tfrac{1}{2}\rangle \Rightarrow \boxed{P = \tfrac{2}{3}}$$