PH 451/551
QM

Lecture # 15

Hyperfine interaction (ctd.)

Consider 4-fold degenerate state

$$|\psi_{1s}\rangle \doteq \underbrace{\psi_{1s}(r,\theta,\varphi)}_{|100\rangle} \underbrace{|\pm\rangle_e}_{\text{spin state of } e^-} \underbrace{|\pm\rangle_p}_{\text{spin state of proton}}$$

For both e^- & p:

$$S_z|\pm\rangle_e = \pm\frac{\hbar}{2}|\pm\rangle_e \ ; \ \vec{S}^2|\pm\rangle_e = \frac{3\hbar^2}{4}|\pm\rangle_e$$

$$I_z|\pm\rangle_p = \pm\frac{\hbar}{2}|\pm\rangle_p \ ; \ \vec{I}^2|\pm\rangle_p = \frac{3\hbar^2}{4}|\pm\rangle_p$$

Our perturbation $H'_{hf} = \frac{g_e g_p \mu_B \mu_N}{\hbar^2}\delta(\vec{r})\vec{S}\cdot\vec{I}$

Note: Our degeneracy is in the spin states of e^- & proton!

To find energy corrections we will need a 4×4 matrix of H'_{hf} built on states

$$|\overset{m_s}{+}\overset{m_I}{+}\rangle \equiv |+\rangle_e|+\rangle_p \qquad |-+\rangle \equiv |-\rangle_e|+\rangle_p$$

$$|+-\rangle \equiv |+\rangle_e|-\rangle_p \qquad |--\rangle \equiv |-\rangle_e|-\rangle_p$$

uncoupled basis

We will need to calculate

$$\langle m_s' m_I' ; 100 | H'_{hf} | 100 ; m_s m_I \rangle$$

for $m_s, m_I, m_s', m_I' = \pm\frac{1}{2}$

Note: $\langle m_s' m_I' ; 100 | \frac{g_e g_p \mu_B \mu_N}{\hbar^2} \delta(\vec{r}) \vec{S}\cdot\vec{I} | m_s m_I ; 100 \rangle =$

$$= \frac{g_e g_p \mu_B \mu_N}{\hbar^2} \underbrace{\langle 100 | \delta(\vec{r}) | 100 \rangle}_{\displaystyle \int |\psi_{100}(\vec{r},\theta,\varphi)|^2 \delta(\vec{r})\, dV = |\psi_{100}(0)|^2 = \frac{1}{\pi a_0^3}} \langle m_s' m_I' | \vec{S}\cdot\vec{I} | m_s m_I \rangle$$

$$\int |\psi_{100}(\vec{r},\theta,\varphi)|^2 \delta(\vec{r})\, dV = \underbrace{|\psi_{100}(0)|^2}_{} = \frac{1}{\pi a_0^3}$$

$$= \underbrace{\frac{g_e g_p \mu_B \mu_N}{\hbar^2} \frac{1}{\pi a_0^3}}_{= \frac{A}{\hbar^2} \leftarrow \text{const}} \underbrace{\langle m_s' m_I' | \vec{S}\cdot\vec{I} | m_s m_I \rangle}_{}$$

need to find these matrix elements to build a 4x4 matrix

$$\vec{S}\cdot\vec{I} = S_x I_x + S_y I_y + S_z I_z =$$

$$= \frac{1}{2}\left(S_+ I_- + S_- I_+\right) + S_z I_z$$

show!

Now need to act with this on our $|m_s\, m_I\rangle$ states!

Ex. $S_z |m_s m_I\rangle = \hbar m_s |m_s m_I\rangle$

acting on this only!

$I_z |m_s m_I\rangle = \hbar m_I |m_s m_I\rangle$

$S_+ |m_s m_I\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|m_s+1\, m_I\rangle$

So $\Rightarrow$ e.g. $S_+|\overset{m_s}{-}\ \overset{m_I}{+}\rangle = \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}\,\cdot |{+}\,{+}\rangle =$

$= \hbar|{+}{+}\rangle$

④

$$S_z \doteq \frac{\hbar}{2}\begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix} \end{array}$$

$$I_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & 0 & \\ & & 1 & \\ & 0 & & -1 \end{pmatrix}$$

What about $\vec{S}\cdot\vec{I}$?

$$\vec{S}\cdot\vec{I}\,|++\rangle = \frac{1}{2}\left(S_+ I_- + S_- I_+\right)|++\rangle +$$

$$+ \underbrace{S_z I_z|++\rangle}_{\frac{\hbar}{2}\cdot\frac{\hbar}{2}|++\rangle} = \frac{1}{2}(0+0)\,|++\rangle + \frac{\hbar^2}{4}|++\rangle =$$

$$= \frac{\hbar^2}{4}|++\rangle$$

$$\vec{S}\cdot\vec{I}\,|--\rangle = \frac{1}{2}(0+0) + \left(-\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)|--\rangle =$$

$$= \frac{\hbar^2}{4}|--\rangle$$

$$\vec{S}\cdot\vec{I}\,|+-\rangle = 0 + \frac{1}{2}\hbar\cdot\hbar\,|-+\rangle +$$

$$+\frac{\hbar}{2}\left(-\frac{\hbar}{2}\right)|+-\rangle = \frac{\hbar^2}{2}|-+\rangle - \frac{\hbar^2}{4}|+-\rangle$$

$$\vec{S}\cdot\vec{I}\,|-+\rangle = 0 + \frac{1}{2}\hbar\cdot\hbar\,|+-\rangle + \frac{\hbar}{2}\frac{-\hbar}{2}\,|-+\rangle =$$

$$= \frac{\hbar^2}{2}|+-\rangle - \frac{\hbar^2}{4}|-+\rangle$$

Then,

$$\vec{S}\cdot\vec{I} \doteq \frac{\hbar^2}{4}\begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Not diagonal!

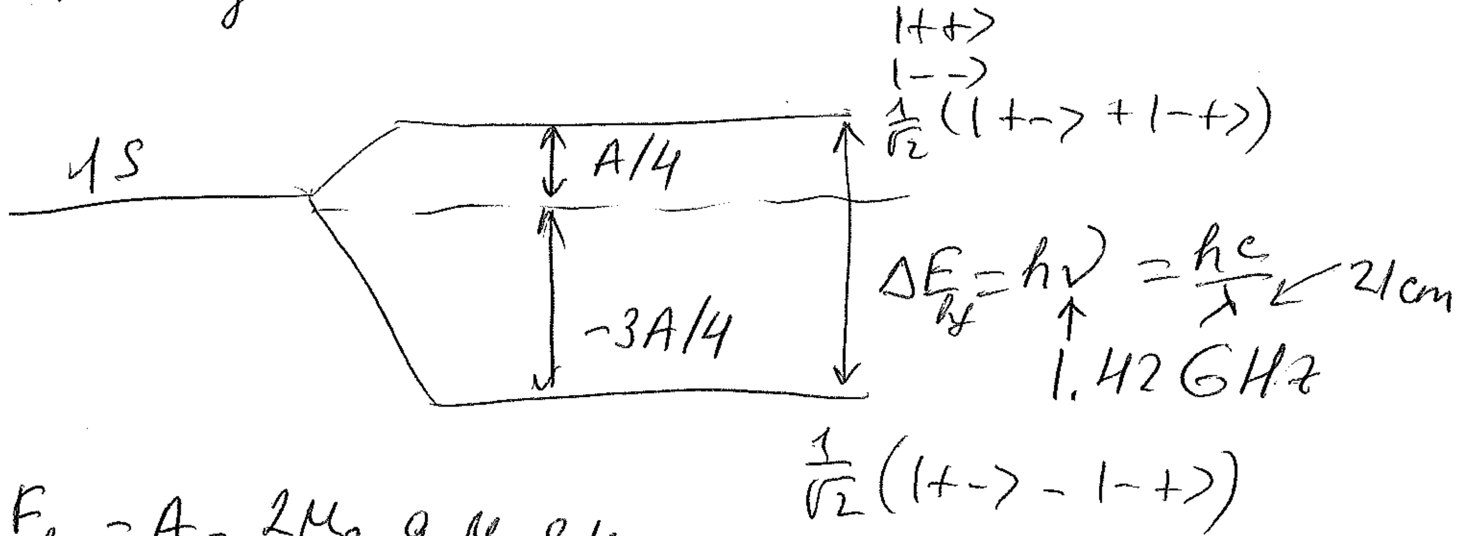
$|\pm\pm\rangle$ are eigenstates of $\vec{S}\cdot\vec{I}$, but $|\pm\mp\rangle$ are not!

Diagonalize $H'_{hf} = \frac{A}{\hbar^2}\vec{S}\cdot\vec{I} \Rightarrow$ 4 eigenvalues (energy corrections)

$E = \frac{A}{4} \leftarrow$ three-fold degenerate $\rightarrow |++\rangle$, $|--\rangle$, $\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

$E = -\frac{3A}{4} \leftarrow$ non-degenerate $\rightarrow \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$

Altogether:



$$\Delta E_{hf} = A = \frac{2\mu_0}{3} \frac{g_e \mu_B g_p \mu_N}{\pi a_0^3} = \alpha^4 m_e c^2 \frac{2}{3} g_e g_p \left(\frac{m_e}{m_p}\right) \sim$$

$$\sim \alpha^2 \frac{m_e}{m_p} |E_1^{(0)}|$$

$\uparrow$ $-\frac{E_I}{1^2}$

Note: could we have predicted that H'_{hf} is not diagonal in the $\{\underbrace{|m_s m_I\rangle}\}$ basis? $\Leftarrow$ eigenbasis of S_z & I_z (p. 4)

$\Downarrow$

$$[\vec{S}\cdot\vec{I}, S_z] = [S_x I_x + S_y I_y + S_z I_z, S_z] =$$

$$= \underbrace{[S_x, S_z]}_{= -i\hbar S_y} I_x + \underbrace{[S_y, S_z]}_{= i\hbar S_x} I_y = \underbrace{-i\hbar S_y I_x + i\hbar S_x I_y}_{\neq 0 \,!}$$

$\Downarrow$

so $\vec{S}\cdot\vec{I}$ & S_z do not share common eigenbasis!

matrix representing $\vec{S}\cdot\vec{I}$ $\Leftarrow$ would not be diagonal in this uncoupled basis $\leftarrow$ as seen from p. 5