PH 457/557
QM

Lecture # 14

Angular momentum ladder operators

Introduce $J_\pm = J_x \pm iJ_y$

$(J_+)^+ = J_-$ ← not Hermitian

$[J_+, J_-] = 2\hbar J_z$, $[\vec{J}^2, J_\pm] = 0$

(WS13) $[J_x, J_y] = i\hbar J_z$

What do $J_\pm$ do to the state $|j\, m_j\rangle$?

Consider $\vec{J}^2 \underbrace{J_\pm |j\, m_j\rangle}_{|?\rangle} = J_\pm \vec{J}^2 |j\, m_j\rangle =$

recall H.O. a, a^+ acting on $|n\rangle$ $\quad [\vec{J}^2, J_\pm] = 0$

$= \hbar^2 j(j+1) \underbrace{J_\pm |j\, m_j\rangle}_{|?\rangle} \Rightarrow$ $J_\pm$ do not change j

HW#5 ⇒ show $[J_z, J_\pm] = \pm\hbar J_\pm$
(here we will use the result) ⇒

$$J_z \underbrace{J_\pm |j m_j\rangle}_{?} = J_\pm \underbrace{J_z |j m_j\rangle}_{\hbar m_j |j m_j\rangle} \pm$$

$$\pm \hbar J_\pm |j m_j\rangle = \hbar (m \pm 1) \underbrace{J_\pm |j m_j\rangle}_{|?\rangle}$$

$$J_z |?\rangle = \hbar (m \pm 1) |?\rangle$$

$$\Rightarrow c\, |j\ m \pm 1\rangle$$

HW#5 $\Rightarrow$ $J_\mp J_\pm = \vec{J}^2 - J_z^2 \mp \hbar J_z$

$$\langle j m_j | J_\mp J_\pm | j m_j \rangle = \langle j m_j | J_\pm^\dagger J_\pm | j m_j \rangle$$

$$= \| J_\pm | j m_j \rangle \|^2 = \langle j m_j | \vec{J}^2 - J_z^2 \mp$$

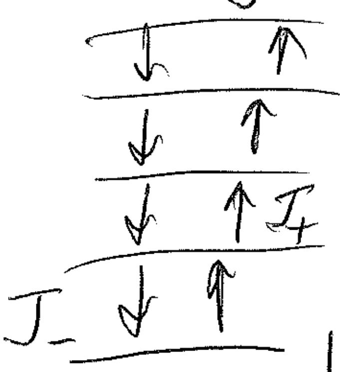
$$\mp \hbar J_z | j m_j \rangle = \hbar^2 j(j+1) - \hbar^2 m_j^2 \mp \hbar^2 m_j =$$

$$= \hbar^2 \left(j(j+1) - m(m \pm 1) \right) = c^2 \Rightarrow$$

$$c = \hbar \sqrt{j(j+1) - m(m \pm 1)}$$

So, $J_\pm |j\ m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)} \cdot |j\ m_j \pm 1\rangle$

E.g. $j = 2 \Rightarrow m_j = -2, -1, 0, 1, 2$



$|2\ 2\rangle$

$|2\ 1\rangle$

$|2\ 0\rangle$

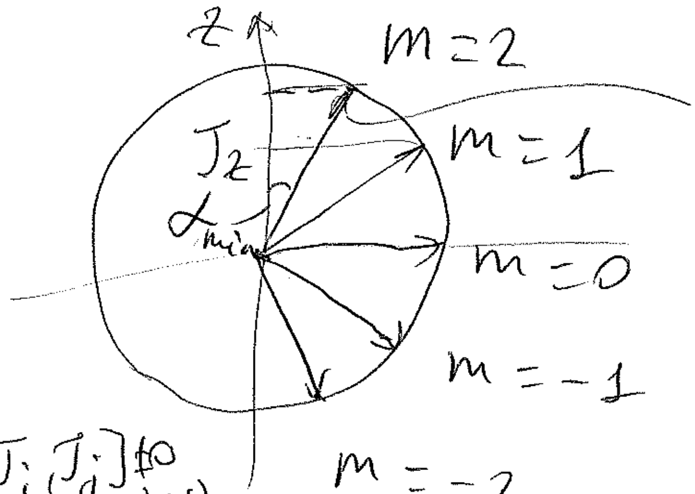
$|2\ -1\rangle$

$|2\ -2\rangle$

$J_+ |j\ j\rangle = 0$

$J_- |j\ -j\rangle = 0$

Another visualisation:



$|\vec{J}| = \hbar\sqrt{j(j+1)} = \sqrt{6}\,\hbar$

$\uparrow$ 2

$J_\pm \Rightarrow$ don't rescale $\vec{J}$, but step through available projections on z

$[J_i, J_j] \neq 0$ $(i \neq j)$

can't know <u>all</u> comp.

Note: $|J_z| < |\vec{J}| \Leftrightarrow m_j^2 < j(j+1)$

$m_{j\,max} = j$

There is a minimal angle $\alpha_{min} \Rightarrow \cos\alpha_{min} = \frac{J_z}{|\vec{J}|} = \frac{m_{j\,max}}{\sqrt{j(j+1)}} = \frac{j}{\sqrt{j(j+1)}}$

← consequence of uncertainty principle

if $\alpha_{min} = 0 \Rightarrow J_x = J_y = 0,\ J_z = \pm\hbar j$ ← know all comp.

Hyperfine interaction

$$\vec{\mu}_e = -g_e \frac{e}{2m_e} \vec{S} = -\underbrace{g_e}_{\text{gyromagn. ratio } (\approx 2)} \underbrace{\mu_B}_{\text{Bohr magneton}} \frac{\vec{S}}{\hbar}$$

$\vec{\mu}_e$: magnetic moment of e^-

$\mu_B = \frac{e\hbar}{2m_e}$

$$\vec{\mu}_p = g_p \frac{e}{2m_p} \vec{I} = g_p \underbrace{\mu_N}_{\text{nuclear magneton}} \frac{\vec{I}}{\hbar} \qquad \mu_N = \frac{e\hbar}{2m_p}$$

$\vec{\mu}_p$: magnetic moment of proton

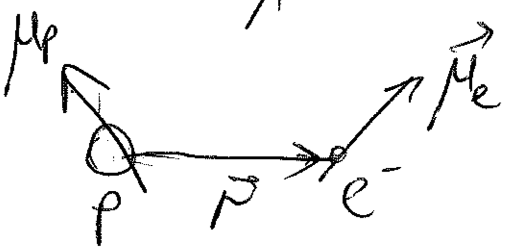
<u>Note:</u> $\mu_N << \mu_B$

0 for s-states

orb. ang. mom. of e^-

$$H'_{hf} = \vec{\mu}_p \cdot \frac{\mu_0}{4\pi} \frac{e\vec{L}}{mr^3} + \underbrace{\frac{\mu_0}{4\pi} \frac{1}{r^3} \left[\vec{\mu}_e \cdot \vec{\mu}_p - 3 \frac{(\vec{\mu}_e \cdot \vec{r})(\vec{\mu}_p \cdot \vec{r})}{r^2} \right]} - \underbrace{\frac{\mu_0}{4\pi} \frac{8\pi}{3} \vec{\mu}_e \cdot \vec{\mu}_p \, \delta(\vec{r})}$$

less obvious, but also 0 for s-states



dipole-dipole interaction (like in E&M)

Fermi's contact term (when e^- is very close to p)

$\Downarrow$

s-states ($\ell = 0$)

We will focus on H'_{hf} effect on the 1s state of H-atom (ground state)

$|100\rangle$

$$H'_{hf} = -\frac{\mu_0}{4\pi}\frac{8\pi}{3}\vec{\mu}_e \cdot \vec{\mu}_p\,\delta(\vec{r}) = \frac{\mu_0}{4\pi}\frac{g_e g_p \mu_B \mu_N}{\hbar^2}\cdot \delta(\vec{r})\vec{S}\cdot\vec{I}$$

$$-\frac{g_e g_p \mu_B \mu_N}{\hbar^2}\vec{S}\cdot\vec{I}$$

Next, use $\hat{H}'_{hf}$ as a perturbation to $\hat{H}_0$ $\Rightarrow$

$\underline{n=1}$ $\Rightarrow$?

$|100\rangle$

$H_0 + H'_{hf}$

Note; $H'_{hf} \sim \vec{S}\cdot\vec{I}$

$\downarrow$

acts on spin variables for e^- & p!

For a complete description of the state, need

$$|100\rangle \otimes |S\,m_s\rangle \otimes |I\,m_I\rangle$$

$S = 1/2$, $m_s = \pm 1/2$ for electron; $I = 1/2$, $m_I = \pm 1/2$ for proton

or, simply,

$|100;\, m_s m_I\rangle \leftarrow$ 4-fold degenerate