Perturbation theory: degenerate

Read McIntyre 10.5-10.6

PH451/551

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From before:

Need to solve: $H|n> = En\,|n>$ *where* $H = H_0 + \lambda H'$

Energy: $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots$

State: $|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \ldots$

The solution to the unperturbed problem is known:

$$H_0|n^{(0)}> = E_n^{(0)}\,|n^{(0)}>$$

$\langle k^{(0)}|n^{(0)}\rangle = \delta_{kn}$ - unperturbed states are orthogonal and form a complete basis: $\sum_n |n^{(0)}\rangle\langle n^{(0)}| = 1$

Consider non-degenerate case: all $E_n^{(0)}$ are different !

Find first- and second-order energy corrections and first-order state correction

From last before:

First-order energy correction

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

First-order correction to the state

$$|n^{(1)}\rangle = \sum_{k \neq n} \frac{H'_{kn}}{\left(E_n^{(0)} - E_k^{(0)}\right)} |k^{(0)}\rangle$$

$H'_{kn} = \langle k^{(0)} | H' | n^{(0)} \rangle$ (matrix element of the perturbation)

From last time:

Second-order energy correction

$$E_n^{(2)} = \langle n^{(0)} | H' | n^{(1)} \rangle = \sum_{k \neq n} \frac{|H'_{kn}|^2}{\left(E_n^{(0)} - E_k^{(0)}\right)}$$

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$$|n^{(1)}\rangle = \sum_{k \neq n} \frac{H'_{kn}}{\left(E_n^{(0)} - E_k^{(0)}\right)} |k^{(0)}\rangle$$

Things to note

1. When there is an off-diagonal term in the perturbation Hamiltonian (i.e. H'_{kn}), there are first order corrections to the state/wave function.
2. Nearby (in energy) states "mix in" to a larger degree than far-away ones
3. Degeneracy presents problems in this formulation – denominator blows up (need new strategy)
4. "Small" means that off-diagonal matrix element is small relative to energy separations

Example: Stark effect in H

1. Dipole energy:

$$H' = -\mathbf{d} \bullet \mathbf{E}$$
$$= -(-e\mathbf{r}) \bullet \mathrm{E}\hat{\mathbf{z}}$$
$$= e\mathrm{E}z$$
$$= e\mathrm{E}r\cos\theta$$

2. States:

$$\left|n\ell m^{(0)}\right\rangle \doteq R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$$

3. Perturbation: $\left\langle n\ell m^{(0)}\right| H' \left|n\ell m^{(0)}\right\rangle$?

Matrix elements, n = 2

1. Dipole energy: $H' = e\mathrm{E}r\cos\theta$
2. States: $\left|2\ell m^{(0)}\right\rangle \doteq R_{2\ell}(r)Y_{\ell m}(\theta,\phi) = R_{2\ell}(r)P_{\ell}(\cos\theta)e^{im\phi}$
3. Perturbation: $\left\langle 2\ell m^{(0)}\right|H'\left|2\ell m^{(0)}\right\rangle$?
4. In groups – zero or non-zero?

$$\left\langle 200^{(0)}\right|H'\left|2\ell m^{(0)}\right\rangle$$

$$\left\langle 211^{(0)}\right|H'\left|2\ell m^{(0)}\right\rangle$$

$$\left\langle 210^{(0)}\right|H'\left|2\ell m^{(0)}\right\rangle$$

$$\left\langle 21-1^{(0)}\right|H'\left|2\ell m^{(0)}\right\rangle$$

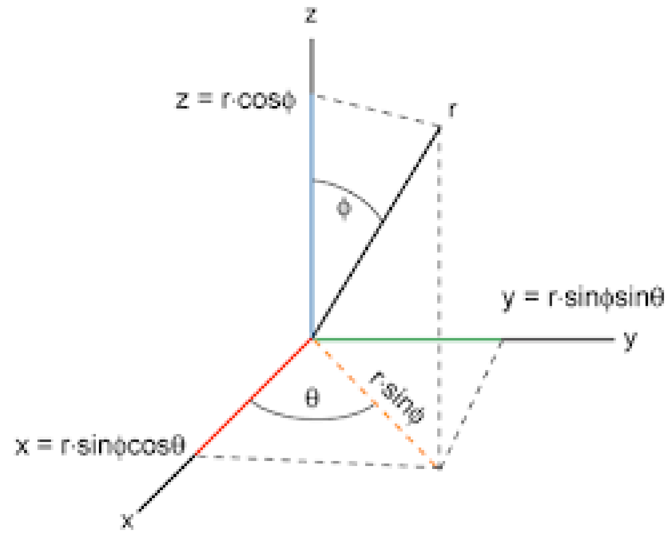
Parity – even or odd?

1. We can exploit parity!

$$\cos\theta = \sqrt{\frac{4\pi}{3}} Y_{1,0}$$

$$\vec{r} \to -\vec{r}$$

$$\begin{cases} x \to -x \\ y \to -y \\ z \to -z \end{cases} \qquad \begin{cases} r \to r \\ \theta \to \pi - \theta \\ \phi \to \phi + \pi \end{cases}$$



1. New integral:

$$\int_{\Omega} d\Omega \, Y^{*}_{\ell m_\ell}(\theta,\phi) Y_{1,0}(\theta,\phi) Y_{\ell' m_\ell'}(\theta,\phi) = ?$$