Review of measurements

Recall:

$$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle \Rightarrow \{|\varphi_n\rangle\}$$

Hermitian operator (represents physical observable) → real eigenvalue

$\{|\varphi_n\rangle\}$ complete & orthonormal set of eigenvalues

$|\psi\rangle = \sum_n C_n|\varphi_n\rangle$

$C_n = \langle\varphi_n|\psi\rangle$

$\sum_n |C_n|^2 = 1$

$$\langle\varphi_n|\varphi_m\rangle = \delta_{mn}$$

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = 1$$

measurement ⇕ applying projection operator $P_n = |\varphi_n\rangle\langle\varphi_n|$

Initial state: $|\psi\rangle \Longrightarrow$ measure A $\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{Bmatrix}$

$a_1 \neq a_2 \neq \dots$ non-degenerate

$\mathcal{P} = |\langle\varphi_n|\psi\rangle|^2 = |C_n|^2$ — probability to get a_n — outcome of measurement

$P_n|\psi\rangle = |\varphi_n\rangle\underbrace{\langle\varphi_n|\psi\rangle}_{=C_n}$

State after measurement: $|\varphi_n\rangle$

What if $a_1 = a_2 = \dots$ ← degenerate eigenvalues?

g_n-fold degenerate

$$P_n = \sum_{i=1}^{g_n} |\varphi_n^i\rangle\langle\varphi_n^i|$$

$$\mathcal{P} = \sum_{i=1}^{g_n} |\langle\varphi_n^i|\psi\rangle|^2$$

Expect. value

$$\langle\hat{A}\rangle = \frac{\langle\psi|\hat{A}|\psi\rangle}{\langle\psi|\psi\rangle}$$

Example 1 Stern – Gerlach experiment

Initial state: $|\psi\rangle$

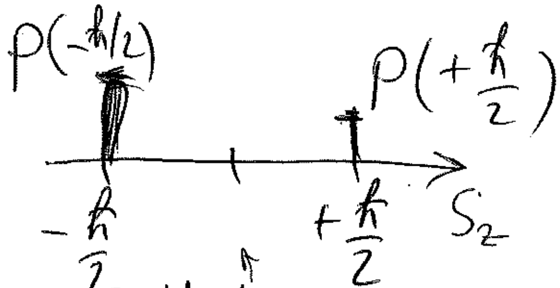
$s = 1/2$; measure S_z $\Rightarrow$ $\underset{\hat{A}}{\hat{S}_z} |\pm\rangle = \underset{a_n}{\pm\frac{\hbar}{2}} \underset{|\varphi_n\rangle}{|\pm\rangle}$

these properties are independent of $|\psi\rangle$ ← initial state

Probability to measure $\pm\frac{\hbar}{2}$:

$$\underbrace{|\langle \pm|\psi\rangle|^2}_{= P(\pm\frac{\hbar}{2})} \rightarrow \text{depends on } |\psi\rangle!$$

$$P(+\tfrac{\hbar}{2}) + P(-\tfrac{\hbar}{2}) = 1$$



Histogram of probabilities

$$\underset{\text{expectation value}}{\langle S_z \rangle} = \langle \psi | S_z | \psi \rangle = + \frac{\hbar}{2} P(+\tfrac{\hbar}{2}) - \frac{\hbar}{2} P(-\tfrac{\hbar}{2})$$

→ depends on $|\psi\rangle$!

↓ can be any real number between $-\frac{\hbar}{2}$ & $+\frac{\hbar}{2}$

Time evolution: $|\psi(t)\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n t} |\varphi_n\rangle$

Ex. $|\psi\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle$, $H|\varphi_n\rangle = E_n|\varphi_n\rangle$, $n = 1, 2$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}E_1 t} |\varphi_1\rangle + \frac{1}{\sqrt{2}} e^{-\frac{i}{\hbar}E_2 t} |\varphi_2\rangle$

Example 2

Consider a system whose state is given in terms of a complete and orthonormal set $\{|\varphi_n\rangle\}$ $\Rightarrow$

$$|\psi\rangle = \frac{1}{3}|\varphi_1\rangle + \frac{1}{6}|\varphi_2\rangle + \frac{2}{3}|\varphi_3\rangle,$$

where $|\varphi_n\rangle$ are eigenstates of the system's Hamiltonian, $H|\varphi_n\rangle = n\varepsilon_0|\varphi_n\rangle$

(a) measure energy $\rightarrow$ what values? with what probabilities?

(b) find average energy

Solution:

(a) possible values for energy: $E_n = n\varepsilon_0$, so $\varepsilon_0, 2\varepsilon_0, 3\varepsilon_0$

Probabilities: $$\mathcal{P}_n(n\varepsilon) = \frac{|\langle\varphi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

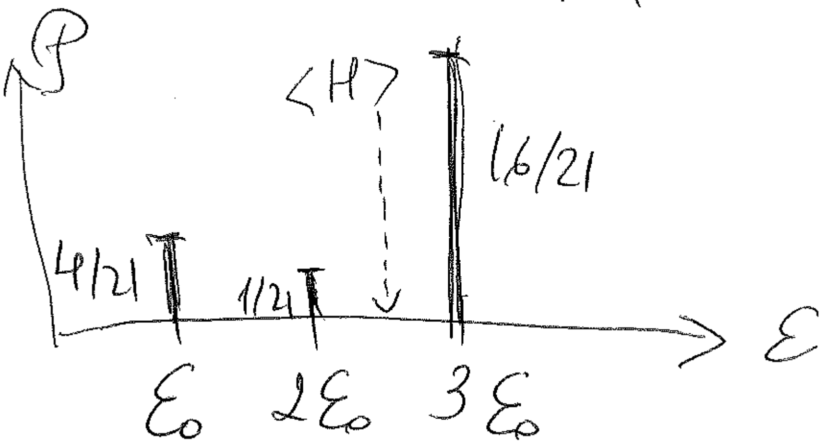
$$\langle\psi|\psi\rangle = \frac{1}{9} + \frac{1}{36} + \frac{4}{9} = \frac{5}{9} + \frac{1}{36} = \frac{21}{36} = \frac{7}{12}$$

$$\mathcal{P}_{n=1} = \frac{1}{9}\cdot\frac{12}{7} = \frac{4}{21}; \quad \mathcal{P}_{n=2} = \frac{1}{36}\cdot\frac{12}{7} = \frac{1}{21},$$

$$\mathcal{P}_{n=3} = \frac{4}{9} \cdot \frac{12}{7} = \frac{16}{21}$$

Check: $\mathcal{P}_{n=1} + \mathcal{P}_{n=2} + \mathcal{P}_{n=3} = \frac{4}{21} + \frac{1}{21} + \frac{16}{21} = 1$

(b) $$\langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \mathcal{P}_{n=1} \cdot \mathcal{E}_0 + \mathcal{P}_{n=2} \cdot 2\mathcal{E}_0 + \mathcal{P}_{n=3} \cdot 3\mathcal{E}_0 = \frac{4}{21} \cdot \mathcal{E}_0 + \frac{1}{21} \cdot 2\mathcal{E}_0 + \frac{16}{21} \cdot 3\mathcal{E}_0 = \frac{54}{21}\mathcal{E}_0 = 2\tfrac{4}{7}\mathcal{E}_0 \approx 2{,}5\,\mathcal{E}_0$$



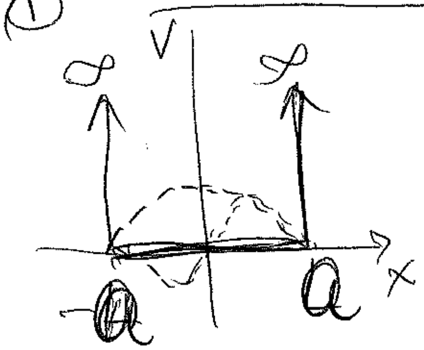
Histogram of probabilities

(c) What is the state of the system after some time t?

$$|\psi(t)\rangle = \left(\frac{1}{3} e^{-\frac{i}{\hbar}\mathcal{E}_0 t} |\varphi_1\rangle + \frac{1}{6} e^{-\frac{2i}{\hbar}\mathcal{E}_0 t} |\varphi_2\rangle + \frac{2}{3} e^{-\frac{3i}{\hbar}\mathcal{E}_0 t} |\varphi_3\rangle \right) \cdot \sqrt{\frac{12}{7}}$$

Particle-in-a-box $\hat{H} \doteq \frac{\hat{P}^2}{2m} + \hat{V}$; $\hat{P} \doteq -i\hbar \frac{d}{dx}$

1



$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\varphi(x) = E\,\varphi(x)$$

$$V(x) = \begin{cases} 0, & -a < x < a \\ \infty, & x > a,\ x < -a \end{cases}$$

$\boxed{\text{infinite box}}$

$-a < x < a$

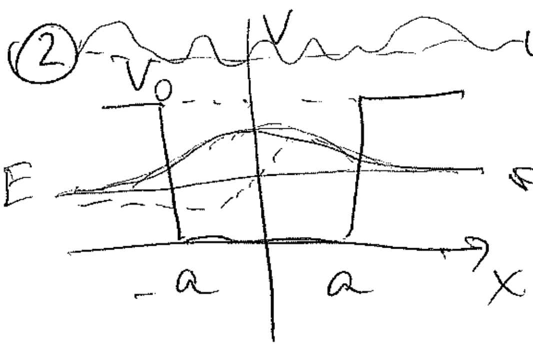
$$\varphi_n(x) = \sqrt{\frac{2}{2a}} \cos\frac{\pi n x}{2a} \quad (n = 1, 3, 5, \ldots) \quad \text{even solution}$$

$$\varphi_n(x) = \sqrt{\frac{2}{2a}} \sin\frac{\pi n x}{2a} \quad (n = 2, 4, 6, \ldots) \quad \text{odd solution}$$

$\varphi_n(\pm a) = 0 \leftarrow$ boundary condition ($\varphi_n(x) = 0$ for $|x| > a$)

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2m(2a)^2} \leftarrow \text{quantized energy}$$

2



unbound (scattering states)

bound states (cos/sin at $|x| < a$, decaying exponentials @ $|x| > a$)

$\boxed{\text{finite box}}$

scattering states: e^{ikx}

k: $\sqrt{2mE}$ @ $|x| < a$

$\sqrt{2m(E - V_0)}$ @ $|x| > a$

H-atom

$$\hat{H}|nlm\rangle = \underbrace{E_n}_{-\frac{E_I}{n^2}}|nlm\rangle$$

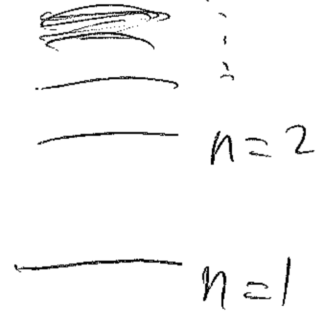
a quantum number per degree of freedom

$$\hat{\vec{L}}^2|nlm\rangle = \hbar^2 l(l+1)|nlm\rangle$$

$$\hat{L}_z|nlm\rangle = \hbar m|nlm\rangle$$

E_I ← ionization energy

$$\psi_{nlm}(r,\theta,\varphi) = \langle\vec{r}|nlm\rangle \qquad E_I = 13{,}6\ \text{eV}$$



$$= R_{nl}(r)\, Y_{lm}(\theta,\varphi)$$

R_{nl}: assoc. Laguerre polynomials

Y_{lm}: spherical harmonics

Note: $[\hat{H}, \hat{\vec{L}}^2] = [\hat{H}, \hat{L}_z] = [\hat{\vec{L}}^2, \hat{L}_z] = 0$

(so they share common eigenbasis $\{|nlm\rangle\}$)