PH 551 **Winter 2026**

Solutions of Homework #5

Problem 1

10.15 The state $|\psi_-\rangle$ is

$$|\psi_-\rangle = \tfrac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right)$$

The expectation value of the dipole moment $\mathbf{d} = -e\mathbf{r}$ is

$$\begin{aligned}\langle\psi_-|\mathbf{d}|\psi_-\rangle &= \langle\psi_-|-e\mathbf{r}|\psi_-\rangle = -e\tfrac{1}{\sqrt{2}}\left(\langle 200| + \langle 210|\right)\mathbf{r}\,\tfrac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right)\\ &= -e\tfrac{1}{2}\left(\langle 200|\mathbf{r}|200\rangle + \langle 200|\mathbf{r}|210\rangle + \langle 210|\mathbf{r}|200\rangle + \langle 210|\mathbf{r}|210\rangle\right)\end{aligned}$$

The first and last terms are zero by symmetry and the other two terms are complex conjugates of each other. Hence we need to find

$$\begin{aligned}\langle 200|\mathbf{r}|210\rangle &= \int \psi_{200}^{(0)*}(r,\theta,\phi)\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\psi_{210}^{(0)}(r,\theta,\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi\\ &= \frac{2}{(2a_0)^{3/2}}\frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{3}a_0(2a_0)^{3/2}}\sqrt{\frac{3}{4\pi}}\int_0^\infty r\left(1-\frac{r}{2a_0}\right)e^{-r/a_0}\left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)r^2\,dr\\ &\quad \int_0^\pi \cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\phi\\ &= \frac{1}{16\pi a_0^4}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^4\left(1-\frac{r}{2a_0}\right)e^{-r/a_0}\left(\sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}\right)\cos\theta\sin\theta\,d\theta\,d\phi\,dr\end{aligned}$$

The azimuthal integrals $\int_0^{2\pi}\cos\phi\,d\phi$ and $\int_0^{2\pi}\sin\phi\,d\phi$ are both zero. The remaining polar integral gives 2/3, yielding

$$\begin{aligned}\langle 200|\mathbf{r}|210\rangle &= \frac{\hat{\mathbf{k}}}{16\pi a_0^4}\frac{2}{3}2\pi\int_0^\infty r^4\left(1-\frac{r}{2a_0}\right)e^{-r/a_0}\,dr\\ &= \frac{\hat{\mathbf{k}}}{12a_0^4}\left[\int_0^\infty r^4 e^{-r/a_0}\,dr - \frac{1}{2a_0}\int_0^\infty r^5 e^{-r/a_0}\,dr\right] = \frac{\hat{\mathbf{k}}}{12a_0^4}\left[4!\,a_0^5 - \frac{1}{2a_0}5!\,a_0^6\right]\\ &= -3a_0\hat{\mathbf{k}}\end{aligned}$$

The expectation value of the dipole moment is

$$\begin{aligned}\langle\psi_-|\mathbf{d}|\psi_-\rangle &= -e\tfrac{1}{2}\left(\langle 200|\mathbf{r}|210\rangle + \langle 210|\mathbf{r}|200\rangle\right)\\ &= -e\tfrac{1}{2}\left(-3a_0\hat{\mathbf{k}} + -3a_0\hat{\mathbf{k}}\right)\\ &= 3ea_0\hat{\mathbf{k}}\end{aligned}$$

which means it is aligned with the applied electric field, as expected [see Fig. 8.9(b)].

Problem 2

10.23 a) For the unperturbed case ($\varepsilon = 0$) we have

$$H_0 \doteq V_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

with eigenvalues $E_1 = V_0$, $E_2 = V_0$, $E_3 = 4V_0$ and eigenvectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $|1\rangle$ and $|2\rangle$ are degenerate and $|3\rangle$ is nondegenerate.

b) Now look at the perturbation of the nondegenerate $|3\rangle$ state. First we need to write the perturbation Hamiltonian $H' = H - H_0$

$$H' \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon & 0 \\ 2\varepsilon & 0 & 3\varepsilon \\ 0 & 3\varepsilon & 0 \end{pmatrix}$$

The first-order energy correction is

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$
$$E_3^{(1)} = \langle 3^{(0)} | H' | 3^{(0)} \rangle = 0$$
$$\boxed{E_3^{(1)} = 0}$$

The second-order energy correction is

$$E_3^{(2)} = \sum_{k \neq 3} \frac{\left|\langle 3^{(0)} | H' | k^{(0)} \rangle\right|^2}{E_3^{(0)} - E_k^{(0)}} = \frac{\left|\langle 3^{(0)} | H' | 1^{(0)} \rangle\right|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{\left|\langle 3^{(0)} | H' | 2^{(0)} \rangle\right|^2}{E_3^{(0)} - E_2^{(0)}}$$
$$= \frac{|0|^2}{4V_0 - V_0} + \frac{|3\varepsilon V_0|^2}{4V_0 - V_0} = 3\varepsilon^2 V_0$$

Hence the corrected energy is

$$E_3 = E_3^{(0)} + E_3^{(1)} + E_3^{(2)} = 4V_0 + 0 + 3\varepsilon^2 V_0$$
$$\boxed{E_3 = V_0\left[4 + 3\varepsilon^2\right]}$$

c) Now look at the perturbation of the degenerate $|1\rangle$ and $|2\rangle$ states. Here we need to diagonalize the perturbation Hamiltonian within that 2x2 space:

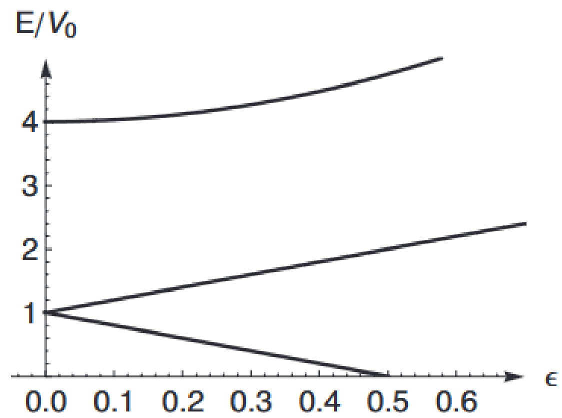
$$H' \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon & 0 \\ 2\varepsilon & 0 & 3\varepsilon \\ 0 & 3\varepsilon & 0 \end{pmatrix} \Rightarrow H'_{1,2} \doteq V_0 \begin{pmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{pmatrix}$$

Diagonalizing gives

$$\begin{vmatrix} -\lambda & 2\varepsilon V_0 \\ 2\varepsilon V_0 & -\lambda \end{vmatrix} = 0$$
$$(-\lambda)(-\lambda) - (2\varepsilon V_0)^2 = 0$$
$$(\lambda^2 - 4\varepsilon^2 V_0^2) = 0$$
$$\lambda = \pm\, 2\varepsilon V_0$$

$$\boxed{\begin{aligned} E_1 &= E_1^{(0)} + E_1^{(1)} = V_0 + 2\varepsilon V_0 = V_0(1 + 2\varepsilon) \\ E_2 &= E_2^{(0)} + E_2^{(1)} = V_0 - 2\varepsilon V_0 = V_0(1 - 2\varepsilon) \end{aligned}}$$

d) The degenerate levels split linearly, while the nondegenerate level has a quadratic dependence and is repelled by the one lower level it is coupled to, as expected.



Problem 3

10.24

$$H \doteq V_0 \begin{pmatrix} 3 & \varepsilon & 0 & 0 \\ \varepsilon & 3 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 5 & \varepsilon \\ 0 & 0 & \varepsilon & 7 \end{pmatrix}$$

a) For the unperturbed case we have

$$H_0 \doteq V_0 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

with eigenvalues $E_1 = 3V_0$, $E_2 = 3V_0$, $E_3 = 5V_0$, $E_4 = 7V_0$ and eigenvectors

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\; |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\; |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\; |4\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that $|1\rangle$ and $|2\rangle$ are degenerate and $|3\rangle$ and $|4\rangle$ are nondegenerate.

b) Let's first do nondegenerate perturbation theory for the $|3\rangle$ and $|4\rangle$ states. First we need to write the perturbation Hamiltonian:

$$H' \doteq V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix}$$

The first-order corrections are

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}' | n^{(0)} \rangle$$
$$E_3^{(1)} = \langle 3^{(0)} | \hat{H}' | 3^{(0)} \rangle = 0$$
$$E_4^{(1)} = \langle 4^{(0)} | \hat{H}' | 4^{(0)} \rangle = 0$$

So we need to go to second order for these states

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left| \langle n^{(0)} | \hat{H}' | k^{(0)} \rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E_3^{(2)} = \sum_{k \neq n} \frac{\left| \langle 3^{(0)} | \hat{H}' | k^{(0)} \rangle \right|^2}{E_3^{(0)} - E_k^{(0)}} = \frac{\left| \langle 3^{(0)} | \hat{H}' | 1^{(0)} \rangle \right|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{\left| \langle 3^{(0)} | \hat{H}' | 2^{(0)} \rangle \right|^2}{E_3^{(0)} - E_2^{(0)}} + \frac{\left| \langle 3^{(0)} | \hat{H}' | 4^{(0)} \rangle \right|^2}{E_3^{(0)} - E_4^{(0)}}$$
$$= \frac{|0|^2}{5V_0 - 3V_0} + \frac{|2\varepsilon V_0|^2}{5V_0 - 3V_0} + \frac{|\varepsilon V_0|^2}{5V_0 - 7V_0} = \varepsilon^2 V_0 \left(\frac{4}{2} + \frac{1}{-2} \right) = \frac{3}{2} \varepsilon^2 V_0$$

$$E_4^{(2)} = \sum_{k \neq n} \frac{\left| \langle 4^{(0)} | \hat{H}' | k^{(0)} \rangle \right|^2}{E_4^{(0)} - E_k^{(0)}} = \frac{\left| \langle 4^{(0)} | \hat{H}' | 1^{(0)} \rangle \right|^2}{E_4^{(0)} - E_1^{(0)}} + \frac{\left| \langle 4^{(0)} | \hat{H}' | 2^{(0)} \rangle \right|^2}{E_4^{(0)} - E_2^{(0)}} + \frac{\left| \langle 4^{(0)} | \hat{H}' | 3^{(0)} \rangle \right|^2}{E_4^{(0)} - E_3^{(0)}}$$
$$= \frac{|0|^2}{7V_0 - 3V_0} + \frac{|0|^2}{7V_0 - 3V_0} + \frac{|\varepsilon V_0|^2}{7V_0 - 5V_0} = \frac{1}{2} \varepsilon^2 V_0$$

Thus the energies to second order are

$$\boxed{\begin{aligned} E_3 &= V_0 \left[5 + \frac{3}{2} \varepsilon^2 \right] \\ E_4 &= V_0 \left[7 + \frac{1}{2} \varepsilon^2 \right] \end{aligned}}$$

Now look at the perturbation of the degenerate $|1\rangle$ and $|2\rangle$ states. Here we need to diagonalize the perturbation Hamiltonian within that 2x2 space:

$$H' \doteq V_0 \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 0 & \varepsilon \\ 0 & 0 & \varepsilon & 0 \end{pmatrix} \Rightarrow H'_{1,2} \doteq V_0 \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

Diagonalize this

$$\begin{vmatrix} -\lambda & \varepsilon V_0 \\ \varepsilon V_0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left(\varepsilon V_0\right)^2 = 0$$

$$\lambda = \pm \varepsilon V_0$$

The corrected energies are

$$\boxed{\begin{aligned} E_1 &= E_1^{(0)} + E_1^{(1)} = 3V_0 + \varepsilon V_0 \\ E_2 &= E_2^{(0)} + E_2^{(1)} = 3V_0 - \varepsilon V_0 \end{aligned}}$$

Problem 4

(a)

11.9 Recall the angular momentum commutation relations

$$\left[J_x,J_y\right]=i\hbar J_z \;;\; \left[J_y,J_z\right]=i\hbar J_x \;;\; \left[J_z,J_x\right]=i\hbar J_y$$
$$\left[\mathbf{J}^2,J_x\right]=\left[\mathbf{J}^2,J_y\right]=\left[\mathbf{J}^2,J_z\right]=0$$

Taking commutators with the hyperfine Hamiltonian gives (recall that the electron and proton spin operators commute)

$$\left[H'_{hf},\mathbf{S}^2\right]=\left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I},\mathbf{S}^2\right]=\frac{A}{\hbar^2}\mathbf{I}\bullet\left[\mathbf{S},\mathbf{S}^2\right]=\frac{A}{\hbar^2}\mathbf{I}\bullet\left[S_x\mathbf{i}+S_y\mathbf{j}+S_z\mathbf{k},\mathbf{S}^2\right]=$$
$$=\frac{A}{\hbar^2}\mathbf{I}\bullet\left\{\mathbf{i}\left[S_x,\mathbf{S}^2\right]+\mathbf{j}\left[S_y,\mathbf{S}^2\right]+\mathbf{k}\left[S_z,\mathbf{S}^2\right]\right\}=0$$
$$\left[H'_{hf},\mathbf{I}^2\right]=\left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I},\mathbf{I}^2\right]=\frac{A}{\hbar^2}\mathbf{S}\bullet\left[\mathbf{I},\mathbf{I}^2\right]=\frac{A}{\hbar^2}\mathbf{S}\bullet\left[I_x\mathbf{i}+I_y\mathbf{j}+I_z\mathbf{k},\mathbf{I}^2\right]=$$
$$=\frac{A}{\hbar^2}\mathbf{S}\bullet\left\{\mathbf{i}\left[I_x,\mathbf{I}^2\right]+\mathbf{j}\left[I_y,\mathbf{I}^2\right]+\mathbf{k}\left[I_z,\mathbf{I}^2\right]\right\}=0$$
$$\left[H'_{hf},S_z\right]=\left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I},S_z\right]=\frac{A}{\hbar^2}\left[S_xI_x+S_yI_y+S_zI_z,S_z\right]$$
$$=\frac{A}{\hbar^2}\left\{I_x\left[S_x,S_z\right]+I_y\left[S_y,S_z\right]+I_z\left[S_z,S_z\right]\right\}$$
$$=\frac{A}{\hbar^2}\left\{-i\hbar S_yI_x+i\hbar S_xI_y\right\}=\frac{-iA}{\hbar}\left\{S_yI_x-S_xI_y\right\}\neq 0$$
$$\left[H'_{hf},I_z\right]=\left[\frac{A}{\hbar^2}\mathbf{S}\bullet\mathbf{I},I_z\right]=\frac{A}{\hbar^2}\left[S_xI_x+S_yI_y+S_zI_z,I_z\right]$$
$$=\frac{A}{\hbar^2}\left\{S_x\left[I_x,I_z\right]+S_y\left[I_y,I_z\right]+S_z\left[I_z,I_z\right]\right\}$$
$$=\frac{A}{\hbar^2}\left\{-i\hbar I_yS_x+i\hbar I_xS_y\right\}=\frac{-iA}{\hbar}\left\{I_yS_x-I_xS_y\right\}\neq 0$$

(b)

11.7 The ladder operators are

$$S_+ = S_x + iS_y \quad I_+ = I_x + iI_y$$
$$S_- = S_x - iS_y \quad I_- = I_x - iI_y$$

Solve these for the Cartesian components:

$$S_x = \tfrac{1}{2}\left(S_+ + S_-\right) \quad I_x = \tfrac{1}{2}\left(I_+ + I_-\right)$$
$$S_y = \tfrac{-i}{2}\left(S_+ - S_-\right) \quad I_y = \tfrac{-i}{2}\left(I_+ + I_-\right)$$

and substitute to get

$$\begin{aligned}\mathbf{S}\bullet\mathbf{I} &= S_xI_x + S_yI_y + S_zI_z = \tfrac{1}{2}\left(S_+ + S_-\right)\tfrac{1}{2}\left(I_+ + I_-\right) + \tfrac{-i}{2}\left(S_+ - S_-\right)\tfrac{-i}{2}\left(I_+ - I_-\right) + S_zI_z \\ &= \tfrac{1}{4}\left(S_+I_+ + S_-I_+ + S_+I_- + S_-I_-\right) - \tfrac{1}{4}\left(S_+I_+ - S_-I_+ - S_+I_- + S_-I_-\right) + S_zI_z \\ &= \tfrac{1}{2}\left(S_-I_+ + S_+I_-\right) + S_zI_z\end{aligned}$$

Problem 5

11.4 a) A spin 3/2 system has s (or j) equal to 3/2 and so has four possible states with m_s (or m_j) equal to +3/2, +1/2, -1/2, and -3/2. The four eigenstates are

$$\left|sm_s\right\rangle = \left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{-1}{2}\right\rangle,\ \left|\tfrac{3}{2},\tfrac{-3}{2}\right\rangle$$

The eigenvalue equations are

$$\mathbf{S}^2\left|sm_s\right\rangle = s\left(s+1\right)\hbar^2\left|sm_s\right\rangle = \tfrac{15}{4}\hbar^2\left|\tfrac{3}{2},m_s\right\rangle$$

$$S_z\left|sm_s\right\rangle = m_s\hbar\left|sm_s\right\rangle = \begin{cases} \tfrac{3}{2}\hbar\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle \\ \tfrac{1}{2}\hbar\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle \\ -\tfrac{1}{2}\hbar\left|\tfrac{3}{2},\tfrac{-1}{2}\right\rangle \\ -\tfrac{3}{2}\hbar\left|\tfrac{3}{2},\tfrac{-3}{2}\right\rangle \end{cases}$$

b) The matrices representing $\mathbf{S}^2$ and S_z are diagonal in their own basis—the $|sm_s\rangle$ basis, with the diagonal eigenvalues from above:

$$\mathbf{S}^2 \doteq \begin{pmatrix} \frac{15}{4}\hbar^2 & 0 & 0 & 0 \\ 0 & \frac{15}{4}\hbar^2 & 0 & 0 \\ 0 & 0 & \frac{15}{4}\hbar^2 & 0 \\ 0 & 0 & 0 & \frac{15}{4}\hbar^2 \end{pmatrix}$$

$$S_z \doteq \begin{pmatrix} \frac{3}{2}\hbar & 0 & 0 & 0 \\ 0 & \frac{1}{2}\hbar & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\hbar & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\hbar \end{pmatrix}$$

c) Generate the matrices representing S_x and S_y by noting that $S_\pm = S_x \pm iS_y$, which gives

$$S_x = \tfrac{1}{2}(S_+ + S_-)$$
$$S_y = \tfrac{-i}{2}(S_+ - S_-)$$

First generate $S_\pm$ using

$$S_\pm|sm_s\rangle = \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}|s, m_s \pm 1\rangle$$

which gives the matrix elements

$$\begin{aligned}\langle s'm_s'|S_\pm|sm_s\rangle &= \langle s'm_s'|\hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}|s, m_s \pm 1\rangle \\ &= \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\langle s'm_s'|s, m_s \pm 1\rangle \\ &= \hbar\sqrt{\left(s(s+1) - m_s(m_s \pm 1)\right)}\delta_{ss'}\delta_{m_s', m_s \pm 1}\end{aligned}$$

Hence

$$S_+ \doteq \begin{pmatrix} 0 & \sqrt{3}\hbar & 0 & 0 \\ 0 & 0 & 2\hbar & 0 \\ 0 & 0 & 0 & \sqrt{3}\hbar \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_- \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3}\hbar & 0 & 0 & 0 \\ 0 & 2\hbar & 0 & 0 \\ 0 & 0 & \sqrt{3}\hbar & 0 \end{pmatrix}$$

This gives

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -i2 & 0 \\ 0 & i2 & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$$

d) To find the eigenvalues of S_x, diagonalize the matrix

$$\det(S_x - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & \sqrt{3}\hbar/2 & 0 & 0 \\ \sqrt{3}\hbar/2 & -\lambda & \hbar & 0 \\ 0 & \hbar & -\lambda & \sqrt{3}\hbar/2 \\ 0 & 0 & \sqrt{3}\hbar/2 & -\lambda \end{vmatrix}$$

$$= -\lambda\left(-\lambda\left(\lambda^2 - 3\hbar^2/4\right) - \hbar(-\lambda\hbar)\right) - \sqrt{3}\hbar/2\left(\sqrt{3}\hbar/2\left(\lambda^2 - 3\hbar^2/4\right) - \hbar(0)\right)$$

$$= \lambda^2\left(\lambda^2 - 3\hbar^2/4\right) - 3\hbar^2/4\left(\lambda^2 - 3\hbar^2/4\right) - \lambda^2\hbar^2$$

$$0 = \left(\lambda^2 - 3\hbar^2/4\right)^2 - \lambda^2\hbar^2$$

Solve to get

$$\left(\lambda^2 - 3\hbar^2/4\right) = \pm\lambda\hbar$$

$$\lambda^2 \mp \lambda\hbar - 3\hbar^2/4 = 0$$

$$\lambda = \pm\frac{\hbar}{2} \pm \sqrt{\left(\frac{\hbar}{2}\right)^2 + \frac{3\hbar^2}{4}} = \pm\frac{\hbar}{2} \pm \hbar$$

Which gives the four values

$$S_x = \frac{3\hbar}{2},\quad \frac{\hbar}{2},\quad -\frac{\hbar}{2},\quad -\frac{3\hbar}{2}$$

as expected.