Solutions of Homework #4

(due Wednesday, February 4, 2026)

1. (10 pts) McIntyre 10.21

   a) The first-order correction is 0 for any state since the wavefunctions have a definitive parity – either even or odd – producing the even probability density whereas the perturbation (constant * X) is odd:

$$E_n^{(1)} = \left\langle n^{(0)}\left|H'\right|n^{(0)}\right\rangle = \left\langle n^{(0)}\left|-q\mathcal{E}x\right|n^{(0)}\right\rangle = -q\mathcal{E}\int_{-\infty}^{\infty}\varphi_n^*(x)\, x\, \varphi_n(x)\,dx$$

$$= -q\mathcal{E}\int_{-\infty}^{\infty}x\left|\varphi_n(x)\right|^2 dx = 0$$

The second-order correction is

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left|\left\langle n^{(0)}\left|H'\right|k^{(0)}\right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} = q^2\mathcal{E}^2 \sum_{k \neq n} \frac{\left|\left\langle n^{(0)}\left|x\right|k^{(0)}\right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}}$$

Use ladder operators to calculate the matrix elements

$$x = \sqrt{\tfrac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$\left\langle n^{(0)}\left|x\right|k^{(0)}\right\rangle = \langle n|x|k\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\left\langle n\right|\left(a^\dagger + a\right)\left|k\right\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\left[\langle n|a^\dagger|k\rangle + \langle n|a|k\rangle\right]$$

$$= \sqrt{\tfrac{\hbar}{2m\omega}}\left[\langle n|\sqrt{k+1}|k+1\rangle + \langle n|\sqrt{k}|k-1\rangle\right] = \sqrt{\tfrac{\hbar}{2m\omega}}\left[\sqrt{k+1}\,\delta_{n,k+1} + \sqrt{k}\,\delta_{n,k-1}\right]$$

Put this back into the energy correction to get

$$E_n^{(2)} = q^2\mathcal{E}^2 \frac{\hbar}{2m\omega} \sum_{k \neq n} \frac{\left[\sqrt{k+1}\,\delta_{n,k+1} + \sqrt{k}\,\delta_{n,k-1}\right]^2}{\left(n+\tfrac{1}{2}\right)\hbar\omega - \left(k+\tfrac{1}{2}\right)\hbar\omega} = \frac{q^2\mathcal{E}^2}{2m\omega^2} \sum_{k \neq n} \frac{\left[\sqrt{k+1}\,\delta_{n,k+1} + \sqrt{k}\,\delta_{n,k-1}\right]^2}{n-k}$$

$$= \frac{q^2\mathcal{E}^2}{2m\omega^2}\left[\frac{n}{n-(n-1)} + \frac{n+1}{n-(n+1)}\right] = \frac{q^2\mathcal{E}^2}{2m\omega^2}\left[n-(n+1)\right]$$

$$E_n^{(2)} = -\frac{q^2\mathcal{E}^2}{2m\omega^2}$$

The new energy levels to second order are thus

$$E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} = \left(n+\tfrac{1}{2}\right)\hbar\omega + 0 + \left(-\frac{q^2\mathcal{E}^2}{2m\omega^2}\right)$$

$$E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

b) Now solve the problem exactly by completing the square in the Hamiltonian:

$$H = H_0 + H' = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 - q\mathcal{E}x$$

$$= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2\left(x^2 - \frac{2q\mathcal{E}}{m\omega^2}x + \left(\frac{q\mathcal{E}}{m\omega^2}\right)^2\right) - \tfrac{1}{2}m\omega^2\left(\frac{q\mathcal{E}}{m\omega^2}\right)^2$$

$$= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2\left(x - \frac{q\mathcal{E}}{m\omega^2}\right)^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2}$$

$$= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 y^2 - \frac{q^2\mathcal{E}^2}{2m\omega^2} \quad ; \text{ where } y \equiv x - \frac{q\mathcal{E}}{m\omega^2}$$

This Hamiltonian represents a system with the position shifted and the energy levels shifted, both by constants. The energy spectrum is thus still a harmonic oscillator spectrum but shifted by the energy shift in the Hamiltonian:

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega - \frac{q^2 \mathcal{E}^2}{2m\omega^2} \quad,$$

which is exactly the same as the perturbation result above. Thus we know that all other orders in perturbation theory will yield zero (or at least all sum to zero).

2. (10 pts) McIntyre 10.10

a) The first-order correction to the energy is zero because the perturbation x^3 is odd and the energy eigenstates are either even or odd so that their squares are even. This is true for all states.

$$E_n^{(1)} = \left\langle n^{(0)} \middle| \hat{H}' \middle| n^{(0)} \right\rangle = \left\langle n^{(0)} \middle| \gamma x^3 \middle| n^{(0)} \right\rangle = \int_{-\infty}^{\infty} \varphi_n^{(0)*}(x) \gamma x^3 \varphi_n^{(0)}(x)\,dx = \gamma \int_{-\infty}^{\infty} x^3 \left| \varphi_n^{(0)}(x) \right|^2 dx = 0$$

b) The second-order correction to the energy is

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left| \left\langle n^{(0)} \middle| \hat{H}' \middle| k^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$$

Use the ladder operators to find the required matrix elements:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$x^3 = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger + a\right)^3$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + a a a\right)$$

This combination of ladder operators means that matrix elements of the x^3 operator are zero unless the two states differ in n by ± 1 or ± 3. Hence the energy shifts are

$$E_0^{(2)} = \sum_{k \neq 0} \frac{\left|\left\langle 0^{(0)}\left|\hat{H}'\right|k^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 0^{(0)}\left|\gamma x^3\right|1^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 0^{(0)}\left|\gamma x^3\right|3^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_3^{(0)}}$$

$$E_1^{(2)} = \sum_{k \neq 1} \frac{\left|\left\langle 1^{(0)}\left|\hat{H}'\right|k^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 1^{(0)}\left|\gamma x^3\right|0^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_0^{(0)}} + \frac{\left|\left\langle 1^{(0)}\left|\gamma x^3\right|2^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_2^{(0)}} + \frac{\left|\left\langle 1^{(0)}\left|\gamma x^3\right|4^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_4^{(0)}}$$

$$E_2^{(2)} = \sum_{k \neq 2} \frac{\left|\left\langle 2^{(0)}\left|\hat{H}'\right|k^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 2^{(0)}\left|\gamma x^3\right|1^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 2^{(0)}\left|\gamma x^3\right|3^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_3^{(0)}} + \frac{\left|\left\langle 2^{(0)}\left|\gamma x^3\right|5^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_5^{(0)}}$$

The required matrix elements are

$$\left\langle 0^{(0)}\left|\hat{H}'\right|1^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 0^{(0)}\left|\left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + \boxed{a a^\dagger a + a a a^\dagger} + aaa\right)\right|1^{(0)}\right\rangle$$
$$= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{2}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 3$$

$$\left\langle 0^{(0)}\left|\hat{H}'\right|3^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 0^{(0)}\left|(aaa)\right|3^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{2}\sqrt{3}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \sqrt{6}$$

$$\left\langle 1^{(0)}\left|\hat{H}'\right|0^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\left|\left(a^\dagger a a^\dagger + a a^\dagger a^\dagger\right)\right|0^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{1}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 3$$

$$\left\langle 1^{(0)}\left|\hat{H}'\right|2^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\left|\left(a^\dagger a a + a a^\dagger a + a a a^\dagger\right)\right|2^{(0)}\right\rangle$$
$$= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{2} + \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{3}\sqrt{3}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 6\sqrt{2}$$

$$\left\langle 1^{(0)}\left|\hat{H}'\right|4^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\left|(aaa)\right|4^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{3}\sqrt{4}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 2\sqrt{6}$$

$$\left\langle 2^{(0)}\left|\hat{H}'\right|1^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\left|\left(a^\dagger a^\dagger a + a^\dagger a a^\dagger + a a^\dagger a^\dagger\right)\right|1^{(0)}\right\rangle$$
$$= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{3}\sqrt{3}\sqrt{2}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 6\sqrt{2}$$

$$\left\langle 2^{(0)}\left|\hat{H}'\right|3^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\left|\left(a^\dagger a a + a a^\dagger a + a a a^\dagger\right)\right|3^{(0)}\right\rangle$$
$$= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{2}\sqrt{3} + \sqrt{3}\sqrt{3}\sqrt{3} + \sqrt{3}\sqrt{4}\sqrt{4}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 9\sqrt{3}$$

$$\left\langle 2^{(0)}\left|\hat{H}'\right|5^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\left|(aaa)\right|5^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{3}\sqrt{4}\sqrt{5}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 2\sqrt{15}$$

The energy shifts are

$$E_0^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{-\hbar\omega} + \frac{6}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-11}{\hbar\omega}\right)$$

$$E_1^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{+\hbar\omega} + \frac{72}{-\hbar\omega} + \frac{24}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-71}{\hbar\omega}\right)$$

$$E_2^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{72}{+\hbar\omega} + \frac{243}{-\hbar\omega} + \frac{60}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-191}{\hbar\omega}\right)$$

c) The first-order corrections to the eigenstates are

$$\left|n^{(1)}\right\rangle = \sum_{k\neq n} c_{nk}\left|n^{(0)}\right\rangle$$

where the expansion coefficients are the same matrix elements from above (note that they are all real)

$$c_{nk} = \frac{\left\langle k^{(0)}\left|H'\right|n^{(0)}\right\rangle}{E_n^{(0)} - E_k^{(0)}}$$

Thus

$$\left|0^{(1)}\right\rangle = \frac{\left\langle 1^{(0)}\left|H'\right|0^{(0)}\right\rangle}{E_0^{(0)} - E_1^{(0)}}\left|1^{(0)}\right\rangle + \frac{\left\langle 3^{(0)}\left|H'\right|0^{(0)}\right\rangle}{E_0^{(0)} - E_3^{(0)}}\left|3^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(-3\left|1^{(0)}\right\rangle - \sqrt{\tfrac{2}{3}}\left|3^{(0)}\right\rangle\right)$$

$$\left|1^{(1)}\right\rangle = \frac{\left\langle 0^{(0)}\left|H'\right|1^{(0)}\right\rangle}{E_1^{(0)} - E_0^{(0)}}\left|0^{(0)}\right\rangle + \frac{\left\langle 2^{(0)}\left|H'\right|1^{(0)}\right\rangle}{E_1^{(0)} - E_2^{(0)}}\left|2^{(0)}\right\rangle + \frac{\left\langle 4^{(0)}\left|H'\right|1^{(0)}\right\rangle}{E_1^{(0)} - E_4^{(0)}}\left|4^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(+3\left|0^{(0)}\right\rangle - 6\sqrt{2}\left|2^{(0)}\right\rangle - 2\sqrt{\tfrac{2}{3}}\left|4^{(0)}\right\rangle\right)$$

$$\left|2^{(1)}\right\rangle = \frac{\left\langle 1^{(0)}\left|H'\right|2^{(0)}\right\rangle}{E_2^{(0)} - E_1^{(0)}}\left|1^{(0)}\right\rangle + \frac{\left\langle 3^{(0)}\left|H'\right|2^{(0)}\right\rangle}{E_2^{(0)} - E_3^{(0)}}\left|3^{(0)}\right\rangle + \frac{\left\langle 5^{(0)}\left|H'\right|2^{(0)}\right\rangle}{E_2^{(0)} - E_5^{(0)}}\left|5^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(+6\sqrt{2}\left|1^{(0)}\right\rangle - 9\sqrt{3}\left|3^{(0)}\right\rangle - 2\sqrt{\tfrac{5}{3}}\left|5^{(0)}\right\rangle\right)$$

3. (10 pts) McIntyre 10.17

The first-order energy correction is:

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

With $H' = \beta x$ and $\varphi_n^{(0)} = \sqrt{2/L}\,\sin(n\pi x/L)$, we find for the ground state

$$E_1^{(1)} = \int_0^L \frac{2}{L}\sin\left(\frac{\pi x}{L}\right)\beta x \sin\left(\frac{\pi x}{L}\right)dx = \frac{2\beta}{L}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2\beta}{L}\left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi x}{L}\right)}{4(\pi/L)} - \frac{\cos\left(\frac{2\pi x}{L}\right)}{8(\pi/L)^2}\right]_0^L$$

$$= \frac{2\beta}{L}\left[\frac{L^2}{4} - \frac{L\sin(2\pi)}{4(\pi/L)} - \frac{\cos(2\pi)}{8(\pi/L)^2} - \left(-\frac{\cos(2\pi)}{8(\pi/L)^2}\right)\right]$$

$$= \frac{2\beta}{L}\left[\frac{L^2}{4} - \frac{L^2}{8\pi^2} - \left(-\frac{L^2}{8\pi^2}\right)\right] = \frac{\beta L}{2}$$

4. (10 pts) McIntyre 10.18

The first-order energy correction is:

$$E_n^{(1)} = \left\langle n^{(0)} \middle| H' \middle| n^{(0)} \right\rangle$$

With $H' = L V_0 \delta(x - L/2)$ and $\varphi_n^{(0)} = \sqrt{2/L}\,\sin(n\pi x/L)$, we find

$$E_n^{(1)} = \int_0^L \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right) L V_0 \delta\left(x - \frac{L}{2}\right) dx = 2V_0 \sin^2\left(\frac{n\pi}{2}\right)$$

For odd values of n, the correction is $2V_0$, while for even values of n, it is zero:

$$E_n^{(1)} = \begin{cases} 2V_0 & ;\, n \text{ odd} \\ 0 & ;\, n \text{ even} \end{cases}$$

b) The wave function for a state with an even value of n is zero at the location of the delta function, so it does not "sample" the perturbation, and the energy is therefore unaffected. Not so for states with odd values of n, where the energy levels are indeed shifted.

c) The new wavefunction, correct to first order is:

$$|1\rangle = |1^{(0)}\rangle + \sum_{k \neq 1} \frac{\left\langle k^{(0)} \middle| H' \middle| 1^{(0)} \right\rangle}{E_1^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

The matrix element in the numerator in the sum is:

$$\left\langle k^{(0)} \middle| H' \middle| 1^{(0)} \right\rangle = \int_0^L \frac{2}{L}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) L V_0 \delta\left(x - \frac{L}{2}\right) dx = 2V_0 \sin\left(\frac{k\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$= 2V_0 \sin\left(\frac{k\pi}{2}\right)$$

We see that for k even, there is no contribution – the states with even labels do not mix. For k odd, all terms have the same numerator (modulo a sign), but the denominator becomes progressively larger as k increases, because the energy difference between the

ground state and the state in question increases. Thus the largest contribution comes from the $\boxed{k=3}$ state, and it is

$$c_{1k,\max} = c_{13} = \frac{2V_0 \sin\left(\dfrac{3\pi}{2}\right)}{E_1^{(0)} - E_3^{(0)}} = \frac{-2V_0}{(1-9)\dfrac{\pi^2\hbar^2}{2mL^2}}$$

$$\left|1^{(1)}\right\rangle \cong \frac{V_0 mL^2}{2\pi^2\hbar^2}\left|3^{(0)}\right\rangle$$

$$\boxed{\left|1\right\rangle \cong \left|1^{(0)}\right\rangle + \frac{V_0 mL^2}{2\pi^2\hbar^2}\left|3^{(0)}\right\rangle}$$

d) For this square bump the first-order perturbation is

$$E_n^{(1)} = \left\langle n^{(0)}\left|\hat{H}'\right|n^{(0)}\right\rangle = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2} \varphi_n^*(x)\frac{V_0}{\varepsilon}\varphi_n(x)\,dx = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\varphi_1^*(x)\frac{V_0}{\varepsilon}\varphi_1(x)\right]dx$$

$$= \frac{V_0}{\varepsilon}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\right]dx = \frac{V_0}{\varepsilon}\frac{2}{L}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left(\frac{1}{2}\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]\right)dx$$

$$= \frac{V_0}{\varepsilon L}\left[x - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}$$

$$= \frac{V_0}{\varepsilon L}\left[\frac{L}{2}+\varepsilon\frac{L}{2} - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2}+\varepsilon\frac{L}{2}\right)\right) - \left(\frac{L}{2}-\varepsilon\frac{L}{2}\right) + \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2}-\varepsilon\frac{L}{2}\right)\right)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L - \left(\frac{L}{2\pi}\right)\sin(\pi+\varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\pi-\varepsilon\pi)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi)\right]$$

$$\boxed{E_1^{(1)} = V_0\left[1 + \frac{\sin(\varepsilon\pi)}{\varepsilon\pi}\right]}$$

e) In the limit of small ε, we get

$$E_1^{(1)} \cong V_0\left[1 + \frac{1}{\varepsilon\pi}\varepsilon\pi\right] = 2V_0$$

$$E_1^{(1)} \cong 2V_0$$

just as we got in part (a). This is to be expected because in the limit of $\varepsilon \to 0$, the square bump looks like a delta function, and we arranged its parameters at the beginning so that the area of the bump $\left((V_0/\varepsilon)\varepsilon L = LV_0\right)$ is the same as the area of the delta function.