Physics 451 **Winter 2026**

Solutions of Homework #4

(due Wednesday, February 4, 2026)

1. (10 pts) McIntyre 10.7

   For a spin-1 system:

$$H_0 = -\boldsymbol{\mu}\bullet\mathbf{B}_0 = \omega_0 S_z \doteq \begin{pmatrix} \hbar\omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 \end{pmatrix}$$

where we have defined the Larmor frequency $\omega_0 = g_N e B_0/2m_p$. The zeroth-order energies are $E_1^{(0)} = \hbar\omega_0$, $E_2^{(0)} = 0$, and $E_3^{(0)} = -\hbar\omega_0$. The perturbation Hamiltonian H' is determined by the field $\mathbf{B}' = B_1\hat{\mathbf{z}}$ and is characterized by a different Larmor frequency $\omega_1 = g_N e B_1/2m_p$:

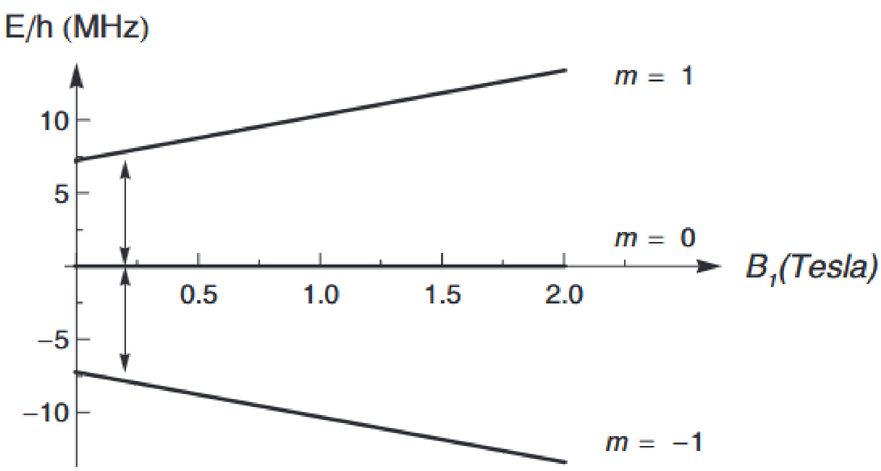
$$H' = -\boldsymbol{\mu}\bullet\mathbf{B}' = \omega_2 S_z \doteq \begin{pmatrix} \hbar\omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_1 \end{pmatrix}$$

Perturbation theory tells us that the first-order correction to the energy is the expectation value of the perturbation in the unperturbed state:

$$E_n^{(1)} = H'_{nn} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

These are the diagonal elements of the matrix representing H' in the basis of zeroth-order energy eigenstates. The matrix above thus yields the first-order energy shifts due to the perturbation:

$$E_1^{(1)} = \hbar\omega_1$$
$$E_2^{(1)} = 0$$
$$E_3^{(1)} = -\hbar\omega_1$$



2. (10 pts) McIntyre 10.10

a) The first-order correction to the energy is zero because the perturbation x^3 is odd and the energy eigenstates are either even or odd so that their squares are even. This is true for all states.

$$E_n^{(1)} = \langle n^{(0)} | \hat{H}' | n^{(0)} \rangle = \langle n^{(0)} | \gamma x^3 | n^{(0)} \rangle = \int_{-\infty}^{\infty} \varphi_n^{(0)*}(x) \gamma x^3 \varphi_n^{(0)}(x) dx = \gamma \int_{-\infty}^{\infty} x^3 \left| \varphi_n^{(0)}(x) \right|^2 dx = 0$$

b) The second-order correction to the energy is

$$E_n^{(2)} = \sum_{k \neq n} \frac{\left| \langle n^{(0)} | \hat{H}' | k^{(0)} \rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$$

Use the ladder operators to find the required matrix elements:

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a^\dagger + a \right)$$

$$x^3 = \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \left(a^\dagger + a \right)^3$$

$$= \left(\frac{\hbar}{2m\omega} \right)^{\frac{3}{2}} \left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + a a^\dagger a + a a a^\dagger + a a a \right)$$

This combination of ladder operators means that matrix elements of the x^3 operator are zero unless the two states differ in n by ± 1 or ± 3. Hence the energy shifts are

$$E_0^{(2)} = \sum_{k\neq 0} \frac{\left|\left\langle 0^{(0)}\middle|\hat{H}'\middle|k^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 0^{(0)}\middle|\gamma x^3\middle|1^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 0^{(0)}\middle|\gamma x^3\middle|3^{(0)}\right\rangle\right|^2}{E_0^{(0)} - E_3^{(0)}}$$

$$E_1^{(2)} = \sum_{k\neq 1} \frac{\left|\left\langle 1^{(0)}\middle|\hat{H}'\middle|k^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 1^{(0)}\middle|\gamma x^3\middle|0^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_0^{(0)}} + \frac{\left|\left\langle 1^{(0)}\middle|\gamma x^3\middle|2^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_2^{(0)}} + \frac{\left|\left\langle 1^{(0)}\middle|\gamma x^3\middle|4^{(0)}\right\rangle\right|^2}{E_1^{(0)} - E_4^{(0)}}$$

$$E_2^{(2)} = \sum_{k\neq 2} \frac{\left|\left\langle 2^{(0)}\middle|\hat{H}'\middle|k^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_k^{(0)}} = \frac{\left|\left\langle 2^{(0)}\middle|\gamma x^3\middle|1^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_1^{(0)}} + \frac{\left|\left\langle 2^{(0)}\middle|\gamma x^3\middle|3^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_3^{(0)}} + \frac{\left|\left\langle 2^{(0)}\middle|\gamma x^3\middle|5^{(0)}\right\rangle\right|^2}{E_2^{(0)} - E_5^{(0)}}$$

The required matrix elements are

$$\begin{aligned}
\left\langle 0^{(0)}\middle|\hat{H}'\middle|1^{(0)}\right\rangle &= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 0^{(0)}\middle|\left(a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a + a^\dagger a a^\dagger + a^\dagger a a + a a^\dagger a^\dagger + \boxed{a a^\dagger a + a a a^\dagger} + aaa\right)\middle|1^{(0)}\right\rangle \\
&= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{1}\sqrt{2}\sqrt{2}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 3
\end{aligned}$$

$$\left\langle 0^{(0)}\middle|\hat{H}'\middle|3^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 0^{(0)}\middle|(aaa)\middle|3^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{2}\sqrt{3}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \sqrt{6}$$

$$\left\langle 1^{(0)}\middle|\hat{H}'\middle|0^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\middle|\left(a^\dagger a a^\dagger + a a^\dagger a^\dagger\right)\middle|0^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{1}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 3$$

$$\begin{aligned}
\left\langle 1^{(0)}\middle|\hat{H}'\middle|2^{(0)}\right\rangle &= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\middle|\left(a^\dagger a a + a a^\dagger a + a a a^\dagger\right)\middle|2^{(0)}\right\rangle \\
&= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{1}\sqrt{1}\sqrt{2} + \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{3}\sqrt{3}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 6\sqrt{2}
\end{aligned}$$

$$\left\langle 1^{(0)}\middle|\hat{H}'\middle|4^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 1^{(0)}\middle|(aaa)\middle|4^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{3}\sqrt{4}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 2\sqrt{6}$$

$$\begin{aligned}
\left\langle 2^{(0)}\middle|\hat{H}'\middle|1^{(0)}\right\rangle &= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\middle|\left(a^\dagger a^\dagger a + a^\dagger a a^\dagger + a a^\dagger a^\dagger\right)\middle|1^{(0)}\right\rangle \\
&= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{1}\sqrt{1} + \sqrt{2}\sqrt{2}\sqrt{2} + \sqrt{3}\sqrt{3}\sqrt{2}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 6\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
\left\langle 2^{(0)}\middle|\hat{H}'\middle|3^{(0)}\right\rangle &= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\middle|\left(a^\dagger a a + a a^\dagger a + a a a^\dagger\right)\middle|3^{(0)}\right\rangle \\
&= \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{2}\sqrt{2}\sqrt{3} + \sqrt{3}\sqrt{3}\sqrt{3} + \sqrt{3}\sqrt{4}\sqrt{4}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 9\sqrt{3}
\end{aligned}$$

$$\left\langle 2^{(0)}\middle|\hat{H}'\middle|5^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left\langle 2^{(0)}\middle|(aaa)\middle|5^{(0)}\right\rangle = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} \left(\sqrt{3}\sqrt{4}\sqrt{5}\right) = \left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}} 2\sqrt{15}$$

The energy shifts are

$$E_0^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{-\hbar\omega} + \frac{6}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-11}{\hbar\omega}\right)$$

$$E_1^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{9}{+\hbar\omega} + \frac{72}{-\hbar\omega} + \frac{24}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-71}{\hbar\omega}\right)$$

$$E_2^{(2)} = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{72}{+\hbar\omega} + \frac{243}{-\hbar\omega} + \frac{60}{-3\hbar\omega}\right) = \gamma^2 \left(\tfrac{\hbar}{2m\omega}\right)^3 \left(\frac{-191}{\hbar\omega}\right)$$

c) The first-order corrections to the eigenstates are

$$\left|n^{(1)}\right\rangle = \sum_{k\neq n} c_{nk}\left|n^{(0)}\right\rangle$$

where the expansion coefficients are the same matrix elements from above (note that they are all real)

$$c_{nk} = \frac{\left\langle k^{(0)}\middle|H'\middle|n^{(0)}\right\rangle}{E_n^{(0)} - E_k^{(0)}}$$

Thus

$$\left|0^{(1)}\right\rangle = \frac{\left\langle 1^{(0)}\middle|H'\middle|0^{(0)}\right\rangle}{E_0^{(0)} - E_1^{(0)}}\left|1^{(0)}\right\rangle + \frac{\left\langle 3^{(0)}\middle|H'\middle|0^{(0)}\right\rangle}{E_0^{(0)} - E_3^{(0)}}\left|3^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(-3\left|1^{(0)}\right\rangle - \sqrt{\tfrac{2}{3}}\left|3^{(0)}\right\rangle\right)$$

$$\left|1^{(1)}\right\rangle = \frac{\left\langle 0^{(0)}\middle|H'\middle|1^{(0)}\right\rangle}{E_1^{(0)} - E_0^{(0)}}\left|0^{(0)}\right\rangle + \frac{\left\langle 2^{(0)}\middle|H'\middle|1^{(0)}\right\rangle}{E_1^{(0)} - E_2^{(0)}}\left|2^{(0)}\right\rangle + \frac{\left\langle 4^{(0)}\middle|H'\middle|1^{(0)}\right\rangle}{E_1^{(0)} - E_4^{(0)}}\left|4^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(+3\left|0^{(0)}\right\rangle - 6\sqrt{2}\left|2^{(0)}\right\rangle - 2\sqrt{\tfrac{2}{3}}\left|4^{(0)}\right\rangle\right)$$

$$\left|2^{(1)}\right\rangle = \frac{\left\langle 1^{(0)}\middle|H'\middle|2^{(0)}\right\rangle}{E_2^{(0)} - E_1^{(0)}}\left|1^{(0)}\right\rangle + \frac{\left\langle 3^{(0)}\middle|H'\middle|2^{(0)}\right\rangle}{E_2^{(0)} - E_3^{(0)}}\left|3^{(0)}\right\rangle + \frac{\left\langle 5^{(0)}\middle|H'\middle|2^{(0)}\right\rangle}{E_2^{(0)} - E_5^{(0)}}\left|5^{(0)}\right\rangle$$

$$= \gamma\left(\tfrac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\left(\tfrac{1}{\hbar\omega}\right)\left(+6\sqrt{2}\left|1^{(0)}\right\rangle - 9\sqrt{3}\left|3^{(0)}\right\rangle - 2\sqrt{\tfrac{5}{3}}\left|5^{(0)}\right\rangle\right)$$

3. (10 pts) McIntyre 10.17

The first-order energy correction is:

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

With $H' = \beta x$ and $\varphi_n^{(0)} = \sqrt{2/L}\sin(n\pi x/L)$, we find for the ground state

$$E_1^{(1)} = \int_0^L \frac{2}{L}\sin\left(\tfrac{\pi x}{L}\right)\beta x \sin\left(\tfrac{\pi x}{L}\right)dx = \frac{2\beta}{L}\int_0^L x\sin^2\left(\tfrac{\pi x}{L}\right)dx$$

$$= \frac{2\beta}{L}\left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi x}{L}\right)}{4(\pi/L)} - \frac{\cos\left(\frac{2\pi x}{L}\right)}{8(\pi/L)^2}\right]_0^L$$

$$= \frac{2\beta}{L}\left[\frac{L^2}{4} - \frac{L\sin(2\pi)}{4(\pi/L)} - \frac{\cos(2\pi)}{8(\pi/L)^2} - \left(-\frac{\cos(2\pi)}{8(\pi/L)^2}\right)\right]$$

$$= \frac{2\beta}{L}\left[\frac{L^2}{4} - \frac{L^2}{8\pi^2} - \left(-\frac{L^2}{8\pi^2}\right)\right] = \frac{\beta L}{2}$$

4. (10 pts) McIntyre 10.18

The first-order energy correction is:

$$E_n^{(1)} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

With $H' = LV_0\delta(x - L/2)$ and $\varphi_n^{(0)} = \sqrt{2/L}\sin(n\pi x/L)$, we find

$$E_n^{(1)} = \int_0^L \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right) LV_0\delta\left(x - \frac{L}{2}\right)dx = 2V_0\sin^2\left(\frac{n\pi}{2}\right)$$

For odd values of n, the correction is $2V_0$, while for even values of n, it is zero:

$$E_n^{(1)} = \begin{cases} 2V_0 & ; n \ odd \\ 0 & ; n \ even \end{cases}$$

b) The wave function for a state with an even value of n is zero at the location of the delta function, so it does not "sample" the perturbation, and the energy is therefore unaffected. Not so for states with odd values of n, where the energy levels are indeed shifted.

c) The new wavefunction, correct to first order is:

$$|1\rangle = |1^{(0)}\rangle + \sum_{k \neq 1} \frac{\langle k^{(0)} | H' | 1^{(0)} \rangle}{E_1^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

The matrix element in the numerator in the sum is:

$$\langle k^{(0)} | H' | 1^{(0)} \rangle = \int_0^L \frac{2}{L}\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) LV_0\delta\left(x - \frac{L}{2}\right)dx = 2V_0\sin\left(\frac{k\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$
$$= 2V_0\sin\left(\frac{k\pi}{2}\right)$$

We see that for k even, there is no contribution – the states with even labels do not mix. For k odd, all terms have the same numerator (modulo a sign), but the denominator becomes progressively larger as k increases, because the energy difference between the ground state and the state in question increases. Thus the largest contribution comes from the $\boxed{k=3}$ state, and it is

$$c_{1k,\max} = c_{13} = \frac{2V_0 \sin\left(\frac{3\pi}{2}\right)}{E_1^{(0)} - E_3^{(0)}} = \frac{-2V_0}{(1-9)\frac{\pi^2\hbar^2}{2mL^2}}$$

$$\left|1^{(1)}\right\rangle \cong \frac{V_0 mL^2}{2\pi^2\hbar^2}\left|3^{(0)}\right\rangle$$

$$\boxed{\left|1\right\rangle \cong \left|1^{(0)}\right\rangle + \frac{V_0 mL^2}{2\pi^2\hbar^2}\left|3^{(0)}\right\rangle}$$

d) For this square bump the first-order perturbation is

$$E_n^{(1)} = \left\langle n^{(0)}\middle|\hat{H}'\middle|n^{(0)}\right\rangle = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2} \varphi_n^*(x)\frac{V_0}{\varepsilon}\varphi_n(x)\,dx = \int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\varphi_1^*(x)\frac{V_0}{\varepsilon}\varphi_1(x)\right]dx$$

$$= \frac{V_0}{\varepsilon}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left[\frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)\right]dx = \frac{V_0}{\varepsilon}\frac{2}{L}\int_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}\left(\frac{1}{2}\left[1-\cos\left(\frac{2\pi x}{L}\right)\right]\right)dx$$

$$= \frac{V_0}{\varepsilon L}\left[x - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi x}{L}\right)\right]_{L/2-\varepsilon L/2}^{L/2+\varepsilon L/2}$$

$$= \frac{V_0}{\varepsilon L}\left[\frac{L}{2} + \varepsilon\frac{L}{2} - \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2}+\varepsilon\frac{L}{2}\right)\right) - \left(\frac{L}{2} - \varepsilon\frac{L}{2}\right) + \left(\frac{L}{2\pi}\right)\sin\left(\frac{2\pi}{L}\left(\frac{L}{2}-\varepsilon\frac{L}{2}\right)\right)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L - \left(\frac{L}{2\pi}\right)\sin(\pi+\varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\pi-\varepsilon\pi)\right]$$

$$= \frac{V_0}{\varepsilon L}\left[\varepsilon L + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi) + \left(\frac{L}{2\pi}\right)\sin(\varepsilon\pi)\right]$$

$$\boxed{E_1^{(1)} = V_0\left[1 + \frac{\sin(\varepsilon\pi)}{\varepsilon\pi}\right]}$$

e) In the limit of small ε, we get

$$E_1^{(1)} \cong V_0\left[1 + \frac{1}{\varepsilon\pi}\varepsilon\pi\right] = 2V_0$$

$$E_1^{(1)} \cong 2V_0$$

just as we got in part (a). This is to be expected because in the limit of ε -> 0, the square bump looks like a delta function, and we arranged its parameters at the beginning so that the area of the bump $\left((V_0/\varepsilon)\varepsilon L = LV_0\right)$ is the same as the area of the delta function.