PH 451/551 **Solutions of HW #3** **Winter 2026**

(due January 28, 2026)

Problem 1. McIntyre

9.16 The two measured energies are the $n = 1, 2$ states, so the measurement tells us that:

$$\mathcal{P}_{E_n} = \left|\langle n|\psi(t)\rangle\right|^2 = |c_n|^2$$
$$\mathcal{P}_{E_1} = \left|\langle 1|\psi(t)\rangle\right|^2 = |c_1|^2 = 0.36 = \tfrac{9}{25} = \left(\tfrac{3}{5}\right)^2 \Rightarrow c_1 = \tfrac{3}{5}e^{i\theta_1}$$
$$\mathcal{P}_{E_2} = \left|\langle 2|\psi(t)\rangle\right|^2 = |c_2|^2 = 0.64 = \tfrac{16}{25} = \left(\tfrac{4}{5}\right)^2 \Rightarrow c_2 = \tfrac{4}{5}e^{i\theta_2}$$

Thus we can construct the original state and the time evolved state:

$$|\psi(0)\rangle = \tfrac{1}{5}\left[3e^{i\theta_1}|1\rangle + 4e^{i\theta_2}|2\rangle\right]$$
$$|\psi(t)\rangle = \tfrac{1}{5}\left[3e^{i\theta_1}e^{-iE_1t/\hbar}|1\rangle + 4e^{i\theta_2}e^{-iE_2t/\hbar}|2\rangle\right] = \tfrac{1}{5}\left[3e^{i\theta_1}e^{-i\frac{3\omega t}{2}}|1\rangle + 4e^{i\theta_2}e^{-i\frac{5\omega t}{2}}|2\rangle\right]$$
$$= \tfrac{1}{5}e^{i\theta_1}e^{-i\frac{3\omega t}{2}}\left[3|1\rangle + 4e^{i(\theta_2-\theta_1)}e^{-i\omega t}|2\rangle\right]$$

Now use this to find the expectation value of the position:

$$\langle x\rangle = \langle\psi(t)|x|\psi(t)\rangle = \sqrt{\tfrac{\hbar}{2m\omega}}\langle\psi(t)|a^\dagger + a|\psi(t)\rangle$$
$$= \sqrt{\tfrac{\hbar}{2m\omega}}e^{+i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{-i\theta_1}\langle 1| + 4e^{-i\theta_2}e^{+i\omega t}\langle 2|\right)\left(a^\dagger + a\right)e^{-i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{i\theta_1}|1\rangle + 4e^{i\theta_2}e^{-i\omega t}|2\rangle\right)$$
$$= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{1}{25}\left[12e^{-i\theta_2}e^{+i\omega t}e^{i\theta_1}\langle 2|a^\dagger|1\rangle + 12e^{-i\theta_1}e^{i\theta_2}e^{-i\omega t}\langle 1|a|2\rangle\right]$$
$$= \sqrt{\tfrac{\hbar}{2m\omega}}\tfrac{12}{25}\sqrt{2}\left[e^{+i\omega t+i\theta_1-i\theta_2} + e^{-i\omega t-i\theta_1+i\theta_2}\right] = \sqrt{\tfrac{\hbar}{m\omega}}\tfrac{24}{25}\cos\left(\omega t + \theta_1 - \theta_2\right)$$

If $\langle x\rangle$ is a minimum at $t = 0$, then $\cos(\theta_1 - \theta_2) = -1$ and $(\theta_1 - \theta_2) = \pi$. The overall phase is unknown but doesn't matter (cannot be measured). The time dependent wave function is

$$\psi(x,t) = e^{-i\frac{3\omega t}{2}}\tfrac{1}{5}\left(3e^{i\theta_1}\varphi_1(x) + 4e^{i\theta_2}e^{-i\omega t}\varphi_2(x)\right) = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) + 4e^{i(\theta_2-\theta_1)}e^{-i\omega t}\varphi_2(x)\right)$$
$$= e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) + 4e^{-i\pi}e^{-i\omega t}\varphi_2(x)\right) = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3\varphi_1(x) - 4e^{-i\omega t}\varphi_2(x)\right)$$

In Dirac notation, the state is

$$|\psi(t)\rangle = e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left[3|1\rangle - 4e^{-i\omega t}|2\rangle\right]$$

b) The expectation value of the momentum is:

$$\langle p\rangle = \langle\psi(t)|p|\psi(t)\rangle = i\sqrt{\tfrac{m\omega\hbar}{2}}\langle\psi(t)|a^\dagger - a|\psi(t)\rangle$$
$$= i\sqrt{\tfrac{m\omega\hbar}{2}}e^{+i\frac{3\omega t}{2}}e^{-i\theta_1}\tfrac{1}{5}\left(3\langle 1| - 4e^{+i\omega t}\langle 2|\right)\left(a^\dagger - a\right)e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3|1\rangle - 4e^{-i\omega t}|2\rangle\right)$$
$$= \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{i}{25}\left(-12e^{+i\omega t}\langle 2|a^\dagger|1\rangle + 12e^{-i\omega t}\langle 1|a|2\rangle\right) = \sqrt{\tfrac{m\omega\hbar}{2}}\tfrac{12i}{25}\sqrt{2}\left(-e^{+i\omega t} + e^{-i\omega t}\right)$$
$$= \sqrt{m\omega\hbar}\,\tfrac{24}{25}\sin\omega t$$

c) The expectation value of the energy is

$$\langle E\rangle = \sum_n E_n \mathcal{P}_{E_n} = \sum_n \left(n+\tfrac{1}{2}\right)\hbar\omega\mathcal{P}_{E_n}$$
$$\mathcal{P}_{E_n} = \left|\langle n|\psi(t)\rangle\right|^2 = \left|\langle n|e^{-i\frac{3\omega t}{2}}e^{i\theta_1}\tfrac{1}{5}\left(3|1\rangle - 4e^{-i\omega t}|2\rangle\right)\right|^2 = \tfrac{1}{25}\left|3\langle n|1\rangle - 4\langle n|2\rangle\right|^2$$
$$= \tfrac{1}{25}\left|9\delta_{n1} + 16\delta_{n2}\right|$$
$$\langle E\rangle = \sum_n \left(n+\tfrac{1}{2}\right)\hbar\omega\tfrac{1}{25}\left|9\delta_{n1} + 16\delta_{n2}\right| = \tfrac{1}{25}\hbar\omega\left[9\tfrac{3}{2} + 16\tfrac{5}{2}\right] = \tfrac{1}{25}\hbar\omega\left[\tfrac{107}{2}\right]$$
$$= \tfrac{107}{50}\hbar\omega = 2.14\hbar\omega$$

Problem 2: A charged particle subject to a 1D harmonic oscillator potential is initially in a state $|\psi(0)\rangle$ that is an equal superposition of two (arbitrary) energy eigenstates (but the coefficients are not necessarily real, so there could be a relative phase shift).

(a) Write down the state $|\psi(t)\rangle$.

$|\psi(0)\rangle = c_n|n\rangle + c_m|m\rangle$ where the basis states are those of the quantum HO. The coefficients are equal in magnitude (equal superposition), but may differ by a phase. Thus

$$c_n = \frac{1}{\sqrt{2}}; c_m = \frac{1}{\sqrt{2}}e^{i\delta}; |c_n|^2 + |c_m|^2 = 1$$

Time dependence comes from the energy eigenvalue of each represented state:

$$|\psi(0)\rangle = c_n|n\rangle e^{-iE_n t/\hbar} + c_m|m\rangle e^{-iE_m t/\hbar}$$

and for the HO $E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$.

So $|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}e^{-i\omega t/2}\left\{|n\rangle e^{-in\omega t} + e^{i\delta}|m\rangle e^{-im\omega t}\right\}$

Notice the different frequencies associated with the evolution of the two states and the static relative phase shift. The overall phase (from the "1/2" in the energy eigenvalue) is not measureable and is irrelevant. You can further simplify if you like to

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}e^{-i\left(n+\frac{1}{2}\right)\omega t}\left\{|n\rangle + |m\rangle e^{-i\left[(m-n)\omega t - \delta\right]}\right\}$$

(b) Find the expectation value of the electric dipole moment operator qx , where q is the charge and x is the position operator.

The expectation value of the electric dipole moment operator is

$$\langle\psi(t)|qx|\psi(t)\rangle = \left|\tfrac{1}{\sqrt{2}}e^{-i\left(n+\frac{1}{2}\right)\omega t}\right|^2 \left\{\langle n| + \langle m|e^{+i[(m-n)\omega t-\delta]}\right\} qx \left\{|n\rangle + |m\rangle e^{-i[(m-n)\omega t-\delta]}\right\}$$

The phase factor out front evaluates to 1 and there is a factor of ½ from the constants. Write x in terms of the raising and lowering operators

$$\langle\psi(t)|qx|\psi(t)\rangle = \tfrac{1}{2}q\sqrt{\frac{\hbar}{2m\omega}}\left\{\langle n| + \langle m|e^{+i[(m-n)\omega t-\delta]}\right\}\left(a+a^\dagger\right)\left\{|n\rangle + |m\rangle e^{-i[(m-n)\omega t-\delta]}\right\}$$

This is a sum of 8 terms of the type $\langle n|a|m\rangle = \sqrt{m}\quad \delta_{n,m-1}; \langle n|a^\dagger|m\rangle = \sqrt{m+1}\,\delta_{n,m+1}$

The only ones to survive the projection are states that differ by 1.

$$\langle\psi(t)|qx|\psi(t)\rangle = \tfrac{1}{2}q\sqrt{\frac{\hbar}{2m\omega}}\left\{\langle n|a+a^\dagger|m\rangle e^{-i[(m-n)\omega t-\delta]} + \langle m|a+a^\dagger|n\rangle e^{+i[(m-n)\omega t-\delta]}\right\}$$

$$= \tfrac{1}{2}q\sqrt{\frac{\hbar}{2m\omega}}\left\{\langle n|a|m\rangle e^{-i[(m-n)\omega t-\delta]} + \langle n|a^\dagger|m\rangle e^{-i[(m-n)\omega t-\delta]} + \langle m|a|n\rangle e^{+i[(m-n)\omega t-\delta]} + \langle m|a^\dagger|n\rangle e^{+i[(m-n)\omega t-\delta]}\right\}$$

$$= \tfrac{1}{2}q\sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{m}e^{-i[\omega t-\delta]} + \sqrt{m+1}e^{+i[\omega t-\delta]} + \sqrt{m+1}e^{-i[\omega t-\delta]} + \sqrt{m}e^{+i[\omega t-\delta]}\right\}$$

$$= q\sqrt{\frac{\hbar}{2m\omega}}\left\{\delta_{n,m-1}\sqrt{m}\cos(\omega t-\delta) + \delta_{n,m+1}\sqrt{m+1}\cos(\omega t-\delta)\right\}$$

(c) Interpret your result.

This says that the dipole moment oscillates with frequency ω, provided the superposition is of states that differ by 1. The amplitude is $q\sqrt{\frac{\hbar}{2m\omega}}$.

(d) If you make a measurement of the energy of the particle at $t = 0$, what will the outcome be?

The energy will be either $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ or $E_m = \left(m+\frac{1}{2}\right)\hbar\omega; m = n \pm 1$. Equal probability of each state, regardless of time.

Problem 3.

$$\langle n|X^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle n|(a+a^+)^4|n\rangle \circled{=}$$

$$(a+a^+)^4 = (a^2 + \underbrace{a^+a}_{N} + \underbrace{aa^+}_{N+1} + a^{+2})^2 =$$

$$= (a^2 + 2N + 1 + a^{+2})^2 = a^4 + a^{+4} + \underline{(2N+1)^2} +$$

$$+ a^2(2N+1) + a^{+2}(2N+1) + (2N+1)a^2 +$$

$$+ (2N+1)a^{+2} + \underline{a^2a^{+2}} + \underline{a^{+2}a^2}$$

For $\langle n|\ldots|n\rangle$ objects $\Rightarrow$ only underlined terms ~~are~~ non-zero will produce results

$\Downarrow$

$$\circled{=} \langle n|(2N+1)^2 + a^2a^{+2} + a^{+2}a^2|n\rangle \cdot \left(\frac{\hbar}{2m\omega}\right)^2 =$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \Big[(2n+1)^2 + \langle n|a\underbrace{a\,a^+}_{N+1}a^+|n\rangle +$$

$$+ \langle n|a^+\overbrace{a^+a}^{N}a|n\rangle\Big] = \left(\frac{\hbar}{2m\omega}\right)^2 \cdot \Big[(2n+1)^2 +$$

(2)

$$+ \langle n | \underbrace{a a^+}_{N+1} | n \rangle + \langle n | a \underbrace{N a^+}_{a^+ + a^+ N} | n \rangle +$$

$$+ \langle n | a^+ \underbrace{N a}_{-a + aN} | n \rangle \Big] = \left(\frac{\hbar}{2m\omega}\right)^2 \Big[(2n+1)^2 + n + 1 +$$

$$+ n + 1 + n(n+1) + n^2 - n \Big] = \left(\frac{\hbar}{2m\omega}\right)^2 \Big[4n^2 + 4n + 1 +$$

$$+ n^2 + n^2 + 2n + 2 \Big] = \left(\frac{\hbar}{2m\omega}\right)^2 \Big[6n^2 + 6n + 3 \Big]$$