MTH 655/659, Assignment 1

Show all the relevant work, including the new (pieces of) code you implemented. Solve two of the problems 1-3 (group work allowed) and at least one of 4-9 (individual work only). (You are welcome to solve more problems for extra credit).

Basic problems

1. To which spaces $C^k, L^p, H^m, W^{m,p}$ over $\Omega = (\frac{1}{4}, 2)$ does $g(x) = max(x, \sqrt{x})$ belong ? (compute those norms that are finite and weak derivatives when appropriate; find the best k, m, p). What if you consider $\Omega = (0, 2)$ or $\Omega = (1, 2)$?

2. Prove the estimates we used in class for $\| f - I_h f \|_{C^0(0,1)}$ and $|f - I_h f|_{C^1(0,1)} :=\| f' - (I_h f)' \|_{C^0(0,1)}$, where $I_h f$ is the linear interpolant of f over $(0, 1)$ on a uniform grid associated with parameter h.

3. Modify the code FEM1D.m to solve the problem $-u'' = f(x), u(0) = u(1) = 0$ where f is chosen so that a) $u(x) = x - x^3$ and b) $u(x) = \sin(\pi x)$.
   Compute the approximate solution and error for different values of h in H^1 and L^2 norms (use numerical integration). Show the error in function of h (log-log plot); verify the theoretical order of convergence.

   Compare with the discrete norm $\max_i |u(x_i) - u_h(x_i)|$ and discuss what you observe (phenomenon of superconvergence) in that norm.

Additional problems

4. Consider a uniform grid over interval $(0, 1)$ with parameter h and an associated linear FE space V_h.

   i) Find FE interpolant $I_h f$ for $f(x) = x^3$.

   ii) Compute directly the norm of the interpolation error $\| f - I_h f \|_{C^0(0,1)}$. Compare with the error estimate for linear interpolation that you know from class. (Hint: where is the maximum of $f(x) - mx - n, a < x < b$ attained ?)

   iii) Compute $\| f - I_h f \|_{L^2(0,1)}$, $\| f - I_h f \|_{H^1(0,1)}$. This can be done analytically or numerically. Pick $h = (1/2)^m$ for a few m to determine the order of the error.

5. Work out details of the example $f(x, y) = loglog\frac{2}{r}$ (eq.1.8/p31 in text).

6. To which spaces $C^k, L^p, H^m, W^{m,p}$ does $g(r) = |r|^\beta$ belong on unit disk: $\{|r| < 1\} \subset \mathbb{R}^d$ in $d = 1, d = 2$? Consider in particular $\beta = 1, 2, -1, 1/2, -1/2$.

7. Work out details of how to set-up stiffness matrix and rhs when V_h is the space of piecewise quadratic functions. Implement it in FEM1d.m (Hint: work with reference element (-1,1) and use transformation to actual element). Placeholders are available in in FEM1d.m. Test convergence for functions from problem 3.

8. Implement the use of nonuniform grid and solve the problem as in 3) with $u(x) = \frac{1}{100}x(1-x)e^{10x^2}$. Experiment with uniform and with nonuniform grid: find how many nodes you have to use for each in order for the energy error to be less than 10^{-2} and 10^{-3}. Comment on advantages using nonuniform grid.

9. Consider the problem $-au''+bu'+cu = f$ with homogeneous Dirichlet boundary conditions, where a, b, c are nonnegative constants, and additionally $a \neq 0$. Derive variational and FE formulation for the problem using linear elements, and compute elements of the matrix of linear system A. Do you expect superconvergence in this case ? **Extra:** solve it numerically for $u = sin(x)$ and $a = c = 1$ and $b = .1$ and test for convergence. Do you see superconvergence ?
   Now change it to $a = \epsilon, b = 1, c = 0, f = 1$. Can you derive analytical solution ? Observe instability when $h = 0.1$ and when ϵ goes from 0.1 to 0.01 (interpretation is that classical FE requires fine mesh for high Peclet numbers and may be unstable).