MTH 655/659-LAB 9

1. **(Demonstrates that even a descent direction and a decrease of residual is not always leading to success. You can solve this problem analytically or on the computer.)**

Consider finding $\min f(x)$ where $f(x) = x^2$ with the following iteration

$$x^{(k+1)} = x^{(k)} + \lambda_k p_k$$

i) use $x^{(0)} = 2$, directions $p_k = (-1)^{k+1}$ and step sizes $\lambda_k = 2 + 3(2^{-(k+1)})$

ii) use $x^{(0)} = 2$, directions $p_k = -1$ and step sizes $\lambda_k = 2^{-k+1}$

Show that each p_k is a descent direction. Find $x^{(k)}$ and show that $f(x^{(k)})$ decreases monotonically. However, what is the $\lim_{k\to\infty} x^{(k)}$. Is it the minimizer of $f(x)$ or not ?

2. **Newton's method for unconstrained minimization**

For a given smooth $f : \mathbb{R}^n \mapsto \mathbb{R}$, find the minimum using the algorithm:

$$x^{(n+1)} = x^{(n)} - (\nabla^2 f(x^{(n)}))^{-1} \nabla f(x^{(n)})$$

(Use your code for Newton's method which finds the root of $F(\mathbf{x}) = \nabla f(\mathbf{x})$).

Apply the algorithm to

$$f(\mathbf{x}) = (x_1 - 2)^4 + (x_1 - 2)^2 x_2^2 + (x_2 + 1)^2$$

which has a minimum at $x_* = (2, -1)$. Use initial guess $\mathbf{x}^{(0)} = (1, 1)$. Try to assess the rate of convergence of the algorithm during the entire iteration. Point out disadvantages and advantages of the Newton's method as perceived from this example.

Extra: at each point compare (plot) Newton's direction and the steepest descent direction (normalize both directions).

3. **Globalized Newton with Armijo rule:**

We sometimes want to ensure that in each Newton step for finding the root of $f(x) = 0$ where $f : \mathbb{R} \mapsto \mathbb{R}$, we actually *sufficiently decrease* the magnitude of residual that is, that

$$|f(x_{new})| < (1 - \alpha\lambda)|f(x_{old})|, \ \lambda > 0, \alpha \in (0, \frac{1}{\lambda}), \tag{1}$$

Here $x_{new} = x_{old} + \lambda p$ where in this case we choose p to be the Newton step $p = -\frac{f(x_{old})}{f'(x_{old})}$.

i) Use your original Newton's method from LAB2 to find the root of $f(x) = 0$ using initial guess $x^{(0)} = 10$. Let $f(x) = arctan(x)$. Problems ?????

ii) Modify your algorithm to include line search with backtracking: that is, instead of taking the full Newton's step $x_{new} = x_{old} + p$, take the smaller step $x_{new} = x_{old} + \lambda p$. Use $\alpha = 10^{-4}$, try $\lambda = 1, 1/2, 1/4...$ until you are succesful and (1) holds. Record your steps and the size(s) of residual. Plot progress of your iteration $f(x_k)$ versus k.

Repeat for some other functions (this is problem 8.5.3/p151 from [Kelley]).

4. Extra: ask me for problems that involve a trust region algorithm.