Name and ID Number: ______________________________

1. ____________

2. ____________

3. ____________

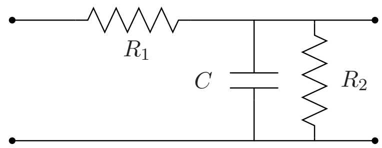
4. ____________

Total ____________

Instructions:

- Avoid using numbers in your answer until you have performed enough symbolic algebra to write an expression for the desired quantities. Check your equations for the correct dimensionality and make sure that they properly describe the expected physical behavior of the system.
- Ask for guidance if you are confounded by a question.

1. (25 points) Kirchoff's Laws, the potential divider and the Thevenin Theorem.



(a) Suppose that a constant (DC) potential $V_\circ$ is applied to the left side of this circuit. Use the concepts of conservation of energy and charge to derive the expression for the potential V_2 across R_2.

Answer (5 points): By conservation of energy and assuming Ohmic behavior, $V_\circ = V_1 + V_2 = I_1 R_1 + I_2 R_2$, and by conservation of charge $I_1 = I_2 = I$. So, $V_\circ = I(R_1 + R_2)$, or $I = V_\circ/(R_1 + R_2)$. Then,

$$V_2 = IR_2 = \frac{V_\circ}{R_1 + R_2} R_2 = V_\circ \frac{R_2}{R_1 + R_2} \ .$$

(b) Now, thinking in the frequency domain, consider the $R_2||C$ section and derive the expression for the total impedance of this section as a function of frequency ω.

Answer (5 points):

$$Z(\omega) = \frac{R_2/i\omega C}{R_2 + 1/i\omega C} = \frac{R_2}{1 + i\omega R_2 C}$$

(c) Using this result, write the expression for the potential $V_2(\omega)$ across R_2 when $V(\omega) = V_\circ e^{i\omega t}$ is applied to the left side of the circuit.

Answer (5 points):

$$A(\omega) = \frac{Z}{R_1 + Z} = \frac{R_2/(1 + i\omega R_2 C)}{R_1 + R_2/(1 + i\omega R_2 C)} = \frac{R_2}{R_1 + R_2 + i\omega R_1 R_2 C} = \frac{1}{1/F + i\omega R_1 C} \ ,$$

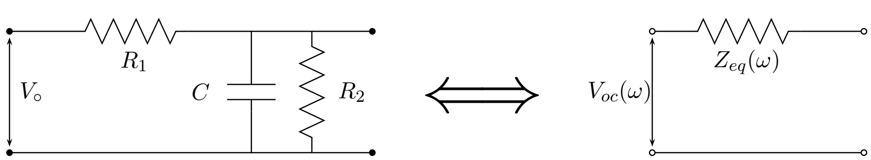
where

$$F = \frac{R_2}{R_1 + R_2} \ .$$

(d) If $R_1 = 100\ \Omega$, $R_2 = 1000\ \Omega$ and $C = 1\mu$F, at what frequency will $|V_2| = V_\circ/2$?

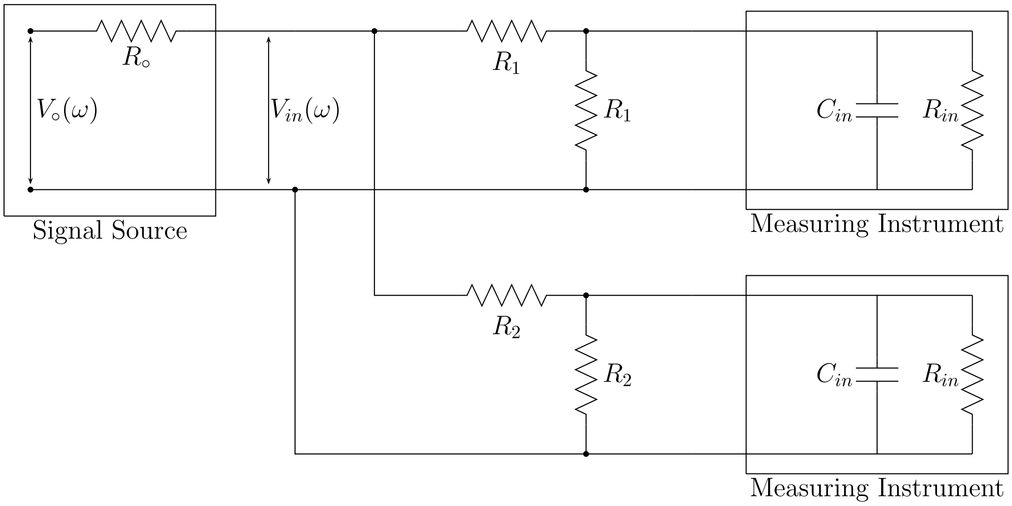
Answer (5 points): Since $F = 1000/(100+1000) = 10/11$,we need $\omega R_1 C = 9/10$. So, $\omega = 0.9/10^{-4} = 9$ kHz.

(e) Find the Thevenin equivalent open circuit potential V_{oc} and the equivalent resistance R_{eq}.



Answer (5 points): The open-circuit potential is simply $V_{oc}(\omega) = A(\omega)V_\circ$. The short-circuit current is $I_{sc} = V_\circ/R_1$, so $R_{eq}(\omega) = V_{oc}(\omega)/I_{sc} = A(\omega)R_1$.

2. (20 points) A sine-wave signal from a function generator with an output resistance $R_{out} = 50\ \Omega$ is split and passed through two potential dividers and then on to the two inputs of a real oscilloscope. The first divider consists of two 50 Ω resistors and the second consists of two 1 MΩ resistors. The oscilloscope has an input resistance of 1 MΩ and an input capacitance of 10 pF.



(a) When $\omega = 10^5$ Hz, will $Z_{in}(\omega)$ for the oscilloscope significantly effect the measured output of either divider?

Answer (10 points): The absolute value of the impedance of each input capacitor is $1/\omega C = 10^6\ \Omega$, the same value as R_{in}. The 50 Ω divider will not be noticeably effected. The 1 MΩ divider will be significantly effected by the fact that there are three 1 MΩ objects in parallel, with $|A(\omega)| = 0.33/(1 + 0.33) = 1/4$.

(b) What are the phase and amplitude relationships between the two waveforms observed on the oscilloscope when the frequency is $\nu = 10^5/2\pi$ Hz $\approxeq 16$ kHz?

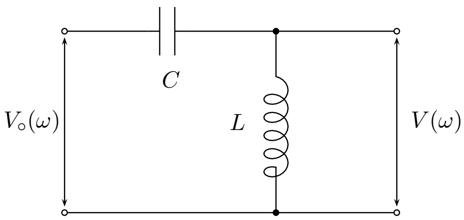
Answer (10 points): For $\omega < 100$ MHz or so, $V_{in}(\omega) = 2V_\circ(\omega)/3$, a value determined by the divider made of 50 Ω resistors. For the 50 Ω divider, $|A| = 1/2$, and the phase is defined as 0. For the 1 MΩ divider, define

$$R_{||}(\omega)^{-1} = R_2^{-1} + R_2^{-1} + i\omega C = R_2^{-1}(2 + i)$$

$$A(\omega) = \frac{R_{||}(\omega)}{R_2 + R_{||}(\omega)} = \frac{1}{3 + i} = \frac{1}{\sqrt{10}} e^{-i \tan^{-1}(1/3)}$$

3. (25 points) Analysis of an LC circuit.

(a) Analyze the circuit shown below and derive the transmission function $A(\omega) = V_{out}(\omega)/V_\circ(\omega)$ assuming that the *inductor is ideal* The component values are $L = 10^{-6}$ H and $C = 10^{-6}$ F.



Answer (6 points):

$$A(\omega) = \frac{i\omega L}{i\omega L + 1/i\omega C} = \frac{1}{1 - 1/\omega^2 LC} = \frac{1}{1 - \omega_\circ^2/\omega^2}$$

(b) Does this circuit have a resonant frequency, and if so what is it?

Answer (3 points):

$$\omega_\circ = \frac{1}{\sqrt{LC}} = 10^6 \text{ Hz.}$$

(c) Now add an internal resistance $r = 1.0\Omega$ to the inductor and derive the transmission function $A(\omega)$.

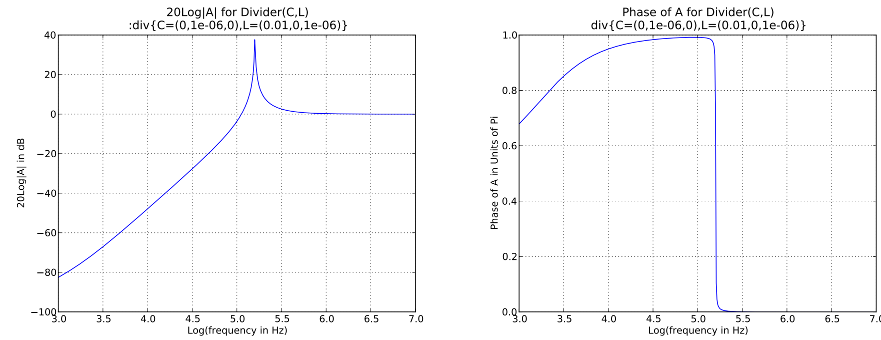
Answer (6 points):

$$A(\omega) = \frac{r + i\omega L}{r + i\omega L + 1/i\omega C} = \frac{r + i\omega L}{r + i(\omega L - 1/\omega C)} = \frac{r + i\omega L}{r + i\omega L(1 - \omega_\circ^2/\omega^2)}$$

$$= \frac{\sqrt{r^2 + \omega^2 L^2}\, e^{i \tan^{-1} \omega L/r}}{\sqrt{r^2 + (\omega L(1 - \omega_\circ^2/\omega^2))^2)}\, e^{i \tan^{-1} \omega L(1 - \omega_\circ^2/\omega^2)/r}}$$

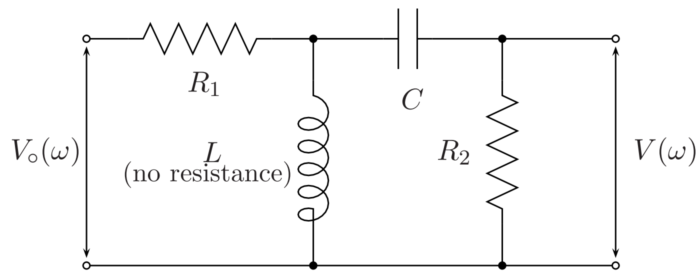
(d) Sketch $\log|A(\omega)|$ as a function of $\log\omega$. Using your expression for $A(\omega)$ with a real inductor, accurately depict the behavior at low and high frequencies and at any other important or interesting frequencies. Determine the slope of $\log|A(\omega)|$ as $\omega \to 0$ and $\omega \to \infty$. Explain or rationalize any odd or unexpected behavior.

Answer (10 points):



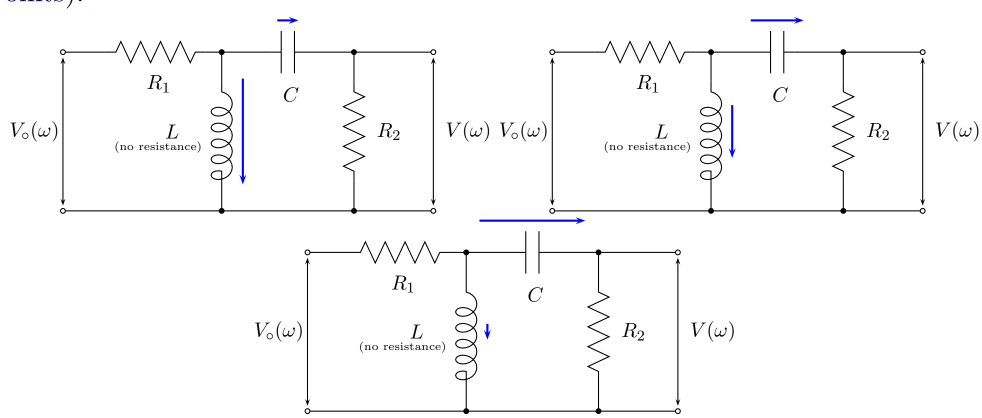
As $\omega \to 0$, $|A(\omega)| \to \omega^2/\omega_\circ^2$, so the slope is +40 dB per decade of frequency. The odd behavior is the peak at $\omega_\circ$, with $|A(\omega)| = 10^4$. This happens because, for this nearly-undamped resonantor, energy is continually added to the oscillator. A finite resistance $r = 0.01\ \Omega$ was required to keep the amplitude finite at $\omega_\circ$.

4. (30 points) Analyze the following circuit in the frequency domain. Assuming that $|R_2 + Z_C| >> Z_L$, the circuit can be considered to be two consecutive, independent filters with transmission functions $A_1(\omega)$ for the RL circuit and $A_2(\omega)$ for the CR circuit.



(a) Use physical reasoning and the concept of impedance to explain thoroughly the behavior of the each filter at low, intermediate and high frequencies. Clearly indicate where current (in the frequency domain) is or is not flowing at low, intermediate and high frequencies. Pictures would useful.

Answer (9 points):



The pictures describe low, medium and high frequency behavior. The length of the arrow represents the amplitude of the current.

(b) Derive the expressions for the transmission or response functions $A_1(\omega)$ and $A_2(\omega)$, each as a product of an amplitude and a phase factor.

Answer (4 points):

$$A_1(\omega) = \frac{i\omega L}{R_1 + i\omega L} = \frac{1}{1 - i\frac{R_1}{\omega L}} = \frac{1}{\sqrt{1 + \left(\frac{R_1}{\omega L}\right)^2}} e^{i \tan^{-1} \frac{R_1}{\omega L}}$$

$$A_2(\omega) = \frac{1/i\omega C}{R_2 + 1/i\omega C} = \frac{1}{1 + i\omega R_2 C} = \frac{1}{\sqrt{1 + (\omega R_2 C)^2}} e^{-i \tan^{-1} \omega R_2 C}$$

(c) There exist frequencies ν_1 or ν_2 for which $20 \log |A_1(\nu_1)| = -3$ dB and $20 \log |A_2(\nu_2)| = -3$ dB. Determine both of these for the case $R_1 = 2\pi \times 100\ \Omega$, $L = 1.0 \times 10^{-5}$ H, $R_2 = 1/2\pi \times 10^5\ \Omega$, and $C = 1.0 \times 10^{-12}$ F.

Answer (4 points): $\omega_1 = R_1/L = 2\pi \times 10^7$ Hz and $\omega_2 = 1/R_2 C = 2\pi \times 10^7$ Hz.

(d) Write the symbolic (no numbers) expression for $A_{total}(\omega)$ in terms of an amplitude $|A_{total}(\omega)|$ and a phase factor $e^{i\alpha}$.

Answer (3 points):

$$A(\omega) = A_1(\omega)A_2(\omega) = \frac{1}{\sqrt{1+(\omega_1/\omega)^2}}\frac{1}{\sqrt{1+(\omega/\omega_2)^2}}e^{i\left(\tan^{-1}\omega_1/\omega - \tan^{-1}\omega_2/\omega\right)}$$

(e) Accurately sketch $|A_1(\nu)|$, $|A_2(\nu)|$ and $|A_{total}(\nu)|$in dB versus $\log\nu$. Indicate the -3dB points and make the slopes of the functions reasonably accurate as $\nu \to 0$ and $\nu \to \infty$. Sketch the phase of $A_{total}(\nu)$ versus $\log\nu$.

Answer (10 points): The -6 dB point (-3 dB for each filter) occurs at 10 MHz. On the low frequency side, the slopes are +20 dB/decade for the individual filters and +40 dB/decade for the combination. Note that the powers curves are additive, a consequence of the decoupled filter assumption stated in the problem. The phase is wrong at $\nu \to 0$ because the inductor is ideal.