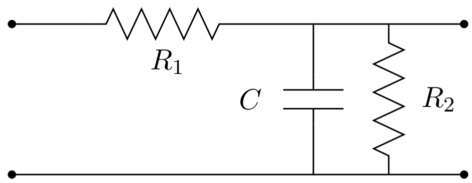
Name and ID Number: ____________________

Ask for guidance if you are confounded by a question.

1. (35 points) Kirchoff's Laws and the potential divider.



(a) Suppose that a constant (DC) potential $V_\circ$ is applied to the left side of this circuit. Use the concepts of conservation of energy and charge to derive the expression for the potential V_2 across R_2.

(7 points) By conservation of energy and Ohm's Law, $V_\circ = V_1 + V_2 = I_1R_1 + I_2R_2$, and by conservation of charge $I_1 = I_2 = I$. So, $V_\circ = I(R_1 + R_2)$, or $I = V_\circ/(R_1 + R_2)$. Then,

$$V_2 = IR_2 = \frac{V_\circ}{R_1 + R_2} R_2 = V_\circ \frac{R_2}{R_1 + R_2} \ .$$

(b) Now consider the $R_2||C$ section and derive the expression for the total impedance of this section as function of frequency ω.

(7 points) Since the potential across a resistor is linear in I by Ohm's Law and since the potential acoss a capacitor is linear in the current as well, for the parallel combination of a capacitor and resistor we can write $V = IZ_{C||R}$. By conservation of charge, $I = I_R + I_C = V/R + V/Z_C = V(1/R + 1/Z_c)$. Then,

$$Z_{C||R} = \frac{V}{I} = \frac{V}{V(1/R + 1/Z_c)} = \left[\frac{1}{R} + \frac{1}{Z_C}\right]^{-1} = \frac{RZ_C}{R + Z_C} =$$

$$= \frac{R/i\omega C}{R + 1/i\omega C} = \frac{R}{1 + i\omega RC} = \frac{R}{\sqrt{1 + \omega^2R^2C^2}} e^{-i \tan^{-1} \omega RC}$$

(c) Using this result, write the expression for the potential $V_2(\omega)$ across R_2 when $V(\omega) = V_\circ e^{i\omega t}$ is applied to the left side of the circuit.

(7 points)

$$V_2 = V_\circ \frac{Z_{R_2||C}}{R_1 + Z_{R_2||C}} \rightarrow A = \frac{Z_{R_2||C}}{R_1 + Z_{R_2||C}} \ .$$

One way to express A is

$$A = \frac{R_2}{(1 + i\omega R_2C)(R_1 + R_2/(1 + i\omega R_2C))} = \frac{R_2}{R_1 + R_2 + i\omega R_1R_2C}$$

$$= \frac{R_2}{\sqrt{(R_1 + R_2)^2 + (\omega R_1R_2C)^2}} e^{i\beta} \ , \text{ where } \beta = -\tan^{-1} \frac{\omega R_1R_2C}{R_1 + R_2}$$

Another more awkward way is to use

$$B = \sqrt{1 + \omega^2 R_2^2 C^2} \text{ and } \alpha = -\tan^{-1} \omega R_2 C \ ,$$

$$A = \frac{R_2 e^{i\alpha}}{R_1 B + R_2 e^{i\alpha}} \ .$$

(d) If $R_1 = 100\ \Omega$, $R_2 = 1000\ \Omega$ and $C = 1\mu$F, at what frequency will $|V_2| = V_\circ/2$?

(7 points) Using the first form above,

$$|A| = \frac{1}{2} \rightarrow \frac{R_2}{\sqrt{(R_1 + R_2)^2 + (\omega R_1 R_2 C)^2}} = \frac{1}{2}$$

$$\frac{R_2^2}{(R_1 + R_2)^2 + (\omega R_1 R_2 C)^2} = \frac{1}{4} \rightarrow (R_1 + R_2)^2 + (\omega R_1 R_2 C)^2 = 4R_2^2$$

$$\omega^2 = \frac{4R_2^2 - (R_1 + R_2)^2}{R_1^2 R_2^2 C^2} = \frac{4^6 - 1.21 \times 10^6}{10^6 \times 10^4 \times 10^{-12}} = 2.8 \times 10^8$$

$$\omega \simeq 1.7 \times 10^4 \rightarrow \nu \simeq 2.8 \times 10^3 \text{ Hz}$$

(e) At this frequency, how much power is dissipated in each element of the circuit when $V_\circ = 10$ V?

(7 points) Consider only the instantaneous power dissipated at the peak of the waveform, since we have not adquately discussed time-averaged power. P_1 and P_2 are the powers dissipated by R_1 and R_2. The capacitor is a non-dissipative object from a physical perspective and from a mathematical perspective because the current and potential differ in phase by $\pi/2$.

$$P_1 = \frac{V_\circ}{2} I_1 = \frac{(V_\circ/2)^2}{R_1} = \frac{25 \text{ V}^2}{100\Omega} = 0.25 \text{ W}$$

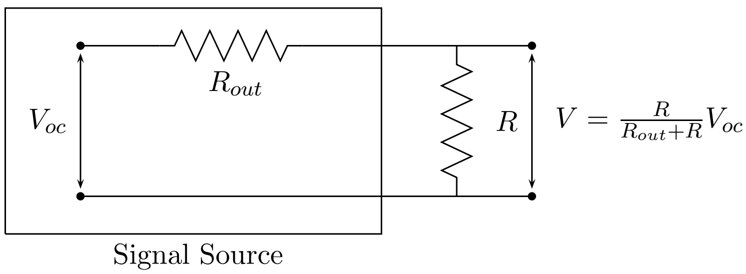
$$P_2 = \frac{V_\circ}{2} I_2 = \frac{(V_\circ/2)^2}{R_2} = \frac{25 \text{ V}^2}{1000\Omega} = 0.025 \text{ W}$$

2. (20 points) Input and output impedance.

   (a) Explain how to measure the open-circuit potential and the output resistance of some signal source at a particular frequency. Draw a clear picture, and explictly write any equations you would use.

   (10 points) Determine V_{oc}, the open-circuit potential, by measuring the output potential without a load resistor, that is, with $R = \infty$. Then, choosing $R \simeq R_{out}$ (an inspired guess), measure the output potential V and calculate R_{out} using

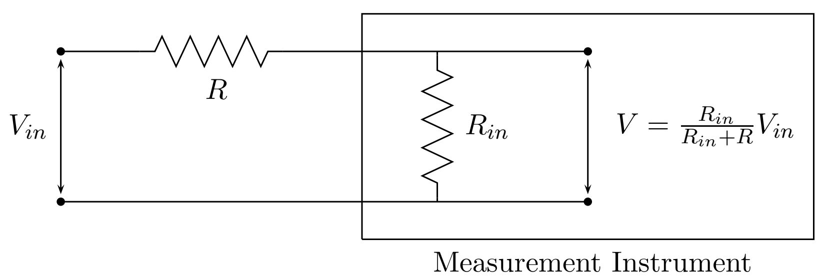
$$R_{out} = R\left(\frac{V_{oc}}{V} - 1\right) \ .$$



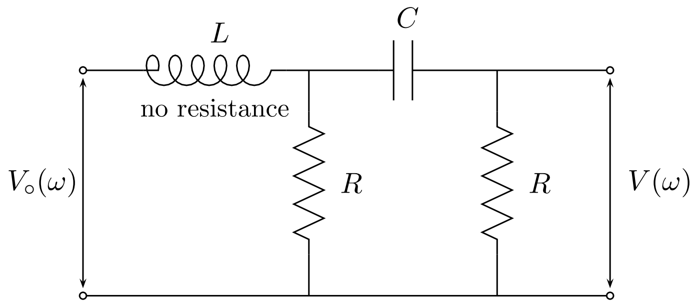
   (b) Explain how to measure the input resistance of an instrument at a particular frequency. Draw a clear picture, and explictly write any equations you would use.

   (10 points) Determine V_{oc}, the input potential, by measuring the potential without a resistor, that is, with $R = 0$. Then, choosing $R \simeq R_{in}$ (another inspired guess), measure the potential V and calculate R_{in} using

$$R_{in} = \frac{R}{\frac{V_{in}}{V} - 1} \ .$$

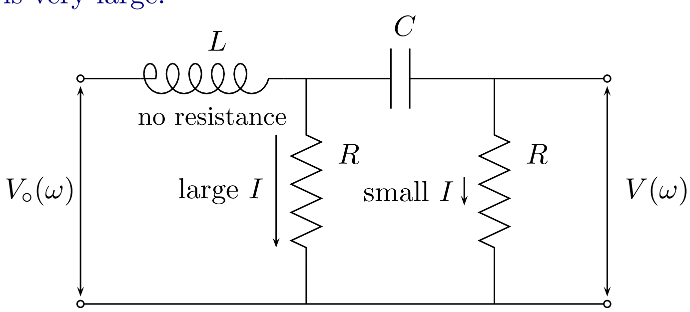


3. (45 points) Analyze the following circuit in the frequency domain, assuming that the inductor is ideal.

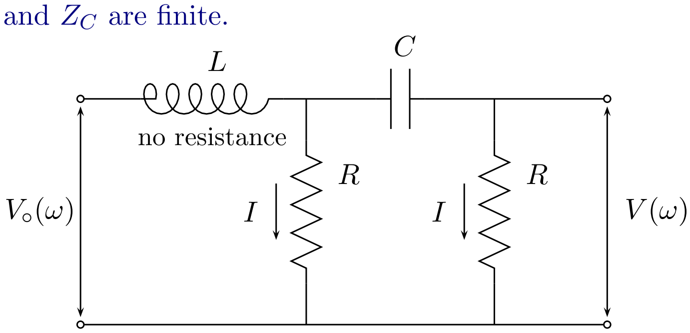


(a) Consider the circuit to be two consecutive filters with transmission functions $A_1(\omega)$ and $A_2(\omega)$. Use physical reasoning and the concept of impedance to explain thoroughly the behavior of the each filter at low, intermediate and high frequencies. Clearly indicate where current is or is not flowing at low, intermediate and high frequencies.

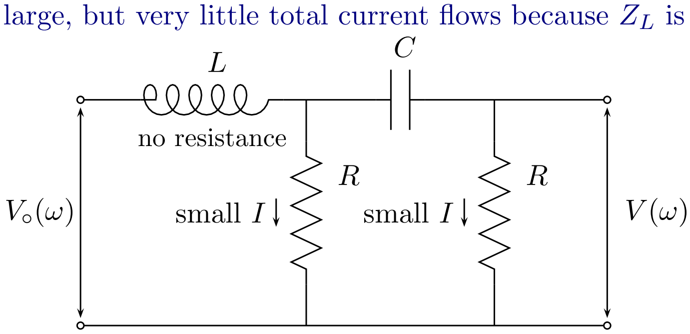
(4 points) At low ω, the current only flows through R_1 because Z_L is very small and Z_C is very large.



(4 points) At medium ω, the current flows through R_1 and R_2 about equally because Z_L and Z_C are finite.



(4 points) At high ω, the current flows equally through R_1 and R_2 because Z_C is very large, but very little total current flows because Z_L is very small.



(b) Derive the expressions for the transmission or response functions $A_1(\omega)$ and $A_2(\omega)$. Write the expression for $A_{total}(\omega)$ in terms of an amplitude $|A_{total}(\omega)|$ and a phase factor $e^{i\alpha}$.
(9 points)

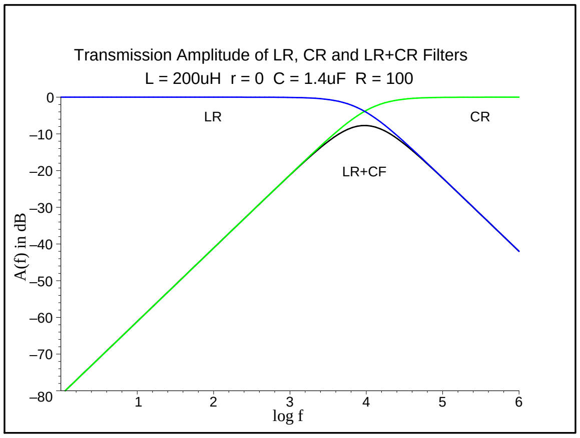
$$A_1 = \frac{R}{R + i\omega L} = \frac{1}{1 + i\omega L/R} = \frac{1}{\sqrt{1 + \omega^2 L^2/R^2}} e^{-i \tan^{-1} \omega L/R}$$

$$A_2 = \frac{R}{R + 1/i\omega C} = \frac{1}{1 - i/\omega RC} = \frac{1}{\sqrt{1 + 1/\omega^2 R^2 C^2}} e^{i \tan^{-1} 1/\omega RC}$$

$$A = A_1 A_2 = \frac{1}{\sqrt{1 + \omega^2 L^2/R^2}} \frac{1}{\sqrt{1 + 1/\omega^2 R^2 C^2}} e^{i(\tan^{-1} 1/\omega RC - \tan^{-1} \omega L/R)}$$

Notice that when $\omega = 1/\sqrt{LC} = \omega_\circ$, the phase of A is $\tan^{-1} \frac{1}{R}\sqrt{\frac{L}{C}} - \tan^{-1} \frac{1}{R}\sqrt{\frac{L}{C}} = 0$.

(c) Sketch $|A_1(\omega)|$, $|A_2(\omega)|$ and $|A_{total}(\omega)|$in dB versus $\log \omega$. Sketch the phase of $A_{total}(\omega)$.
(9 points) Since $A = A_1 \times A_2$, $\log A = \log A_1 + \log A_2$.
This graph was generated by *lr+cr_filters.mws*.



(3 points)
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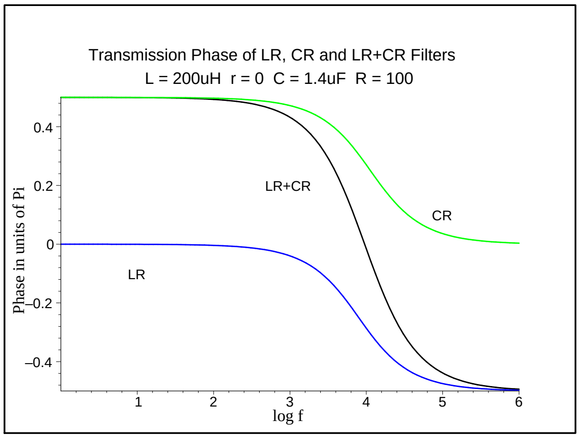

(d) Prove that the frequency at which the maximum in the transmission occurs is $\omega_\circ = 1/\sqrt{LC}$.

(6 points) At the peak in $\log|A|$ in the previous graph, $\log|A_1| = \log|A_2|$. So,

$$|A_1(\omega_\circ)| = |A_2(\omega_\circ)| \rightarrow \left|\frac{1}{1+i\omega L/R}\right| = \left|\frac{1}{1-i/\omega RC}\right|$$

$$|1+i\omega L/R| = |1-i/\omega RC| \rightarrow \frac{\omega L}{R} = \frac{1}{\omega RC} \rightarrow \omega^2 = \frac{1}{LC}$$

(e) What is the slope of $\log|A_{total}(\omega)|$ versus $\log\omega$ as $\omega \rightarrow \infty$ and as $\omega \rightarrow 0$

(6 points)

$$|A| = \frac{1}{\sqrt{1+\omega^2 L^2/R^2}} \frac{1}{\sqrt{1+1/\omega^2 R^2 C^2}}$$

$$\log A = -\frac{1}{2}[\log(1+\omega^2 L^2/R^2) + \log(1+1/\omega^2 R^2 C^2)]$$

As $\omega \rightarrow 0$,

$$\log A \rightarrow -\frac{1}{2}\log(1/\omega^2 R^2 C^2) = \log\omega + \log RC\ ,$$

so the slope is +1.

As $\omega \rightarrow \infty$,

$$\log A \rightarrow -\frac{1}{2}\log(\omega^2 L^2/R^2)) = -\log\omega - \log L/R\ ,$$

so the slope is -1.