CALCULUS OF GENERALIZED HYPERBOLIC TETRAHEDRON

REN GUO

Abstract. We calculate the Jacobian matrix of the dihedral angles of a generalized hyperbolic tetrahedron as functions of edge lengths and find the complete set of symmetries of this matrix.

1. Introduction

1.1. Tetrahedron. Motivated by studying the polyhedral geometry of triangulated 3-manifolds, Luo [Lu08] calculated the Jacobian matrix of the dihedral angles of a hyperbolic (Euclidean or spherical) tetrahedron as functions of edge lengths. This Jacobian matrix enjoys many symmetries. Some of the symmetries were discovered by Schl"afli, Wigner [Wi59], Taylor-Woodward [TW05]. Luo discovered the complete set of symmetries of the Jacobian matrix.

Denote by $v_1, v_2, v_3, v_4$ the vertexes of a hyperbolic (Euclidean or spherical) tetrahedron. Let $a_{ij}$ and $x_{ij}$ be the dihedral angles and the edge length at the edge $v_iv_j$. The angle $a_{ij}$ for $i,j \in \{1, 2, 3, 4\}$ is a function of the lengths $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$.

Theorem (Luo).

$$P_{rs}^{ij} = \frac{1}{\sin a_{ij} \sin a_{rs}} \frac{\partial a_{ij}}{\partial x_{rs}}$$

satisfies

1. (Schl"afli) $P_{rs}^{ij} = P_{ij}^{rs}$.
2. (Wigner, Taylor-Woodward) $P_{kl}^{ij} = P_{ij}^{ik} = P_{ij}^{il}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.
3. $P_{ik}^{ij} = -P_{ki}^{ji}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$.
4. $P_{ij}^{kl} = P_{kl}^{ij} w_{ij}$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$, where

$$w_{ij} = \frac{\cos a_{ij} \cos a_{ik} \cos a_{ki} + \cos a_{ij} \cos a_{jl} \cos a_{il} + \cos a_{ik} \cos a_{jl} + \cos a_{il} \cos a_{jk}}{\sin^2 a_{ij}}.$$

5. $P_{rs}^{ij} = P_{r's'}^{ij}$ for $\{i, j\} \neq \{r, s\}$ and $\{i, j, i', j'\} = \{r, s, r', s'\} = \{1, 2, 3, 4\}$.

Yakut, Savas and Kader [YSK09] calculated the Jacobian matrix for a hyperbolic or spherical tetrahedron and represented each entry of the matrix in terms of $x_{ij}$.

The symmetries of the Jacobian matrix are not obvious in their result.

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1.2. **Generalized hyperbolic tetrahedron.** In this paper we calculate the Jacobian matrix of the dihedral angles of a *generalized hyperbolic tetrahedron* as functions of edge lengths. We find a uniform way to deal with the 15 types of generalized hyperbolic tetrahedra. The complete set of symmetries of the Jacobian matrix are discovered. It is a generalization of Luo’s result. Our main theorem is the following.

Let $a_{ij}$ and $x_{ij}$ be the dihedral angle and the edge length at the edge $e_{ij}$ of a generalized hyperbolic tetrahedron. The angle $a_{ij}$ for $i,j \in \{1,2,3,4\}$ is a function of the lengths $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$. Denote by $G$ the Gram matrix of the generalized hyperbolic tetrahedron. Let $G_{ij}$ be the matrix obtained by deleting the $i$–th row and $j$–th column of the Gram matrix $G$.

**Theorem 1.** The Jacobian matrix is

$$\frac{\partial(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})}{\partial(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})} = \sqrt{\frac{\det G_{11} \det G_{22} \det G_{33} \det G_{44}}{-(\det G)^3}} D M D,$$

where

$$D = \text{diag}(\sin a_{12}, \sin a_{13}, \sin a_{14}, \sin a_{23}, \sin a_{24}, \sin a_{34})$$

is a diagonal matrix and

$$M = \begin{pmatrix}
    w_{12} & - \cos a_{23} & - \cos a_{24} & - \cos a_{13} & - \cos a_{14} & 1 \\
    - \cos a_{23} & w_{13} & - \cos a_{34} & - \cos a_{12} & 1 & - \cos a_{14} \\
    - \cos a_{24} & - \cos a_{34} & w_{14} & 1 & - \cos a_{12} & - \cos a_{13} \\
    - \cos a_{13} & - \cos a_{12} & 1 & w_{23} & - \cos a_{34} & - \cos a_{24} \\
    - \cos a_{14} & 1 & - \cos a_{12} & - \cos a_{34} & w_{24} & - \cos a_{23} \\
    1 & - \cos a_{14} & - \cos a_{13} & - \cos a_{24} & - \cos a_{23} & w_{34}
\end{pmatrix},$$

where

$$w_{ij} = \frac{\cos a_{ij} \cos a_{jk} \cos a_{ki} + \cos a_{ij} \cos a_{jl} \cos a_{li} + \cos a_{ik} \cos a_{jl} + \cos a_{il} \cos a_{jk}}{\sin^2 a_{ij}}.$$

Note that the matrix $D M D$ is the same for the 15 types of generalized hyperbolic tetrahedra. What is different is the factor in front of the matrix $D M D$. This factor depends only on the Gram matrix $G$. Luo’s result about the symmetries of $P_{rs}^{ij}$ can be interpreted as the symmetries of the matrix $M$ as follows:

1. $M$ is a symmetric matrix.
2. Any antidiagonal entry of $M$ is 1.
3. The $(ij, ik)$–th entry of $M$ is $- \cos a_{kj}$.
4. The $(ij, ij)$–th entry of $M$ is $w_{ij}$.
5. Except the diagonal entries, $M$ is symmetric about the antidiagonal axis.

Theorem 1 can be considered as a generalization from 2 dimensions to 3 dimensions of the derivative of cosine law of a generalized hyperbolic triangle which is studied systematically in [GL09] Lemma 3.5.

Heard [He05] calculated the Jacobian matrix of a generalized hyperbolic tetrahedron and represented each entry of the matrix in terms of $c_{ij} = (-1)^{i+j} \det G_{ij}$. The symmetries of the Jacobian matrix are not obvious in his result.

1.3. **Plan of the paper.** In section 2, we recall the definition of a generalized hyperbolic tetrahedron and some properties. In section 3, the derivative of the law of cosine for a link at a vertex of a generalized hyperbolic tetrahedron is summarized. In section 4, the derivative of the law of cosine for a face of a generalized hyperbolic
tetrahedron is summarized. In section 5, we calculate the determinant of the Gram matrix $G$. Theorem 1 is proved in section 6.

2. Definition and properties

The 4-dimensional Minkowski space is the real vector space $\mathbb{R}^4$ equipped the inner product:

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4,$$

where $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4)$.

The 3-dimensional hyperbolic space is identified with the positive sheet of the horoball of two sheets:

$$\mathbb{H}^3 = \{ X \in \mathbb{R}^4 : \langle X, X \rangle = -1, x_4 > 0 \}.$$

The positive half of the light cone is

$$C^+ = \{ X \in \mathbb{R}^4 : \langle X, X \rangle = 0, x_4 > 0 \}$$

and the positive half of the unit sphere is

$$S^+ = \{ X \in \mathbb{R}^4 : \langle X, X \rangle = 1, x_4 > 0 \}.$$

For any point $v_i \in \mathbb{H}^3 \cup C^+ \cup S^+$, the type of the point is the number

$$\varepsilon_i = -\langle v_i, v_i \rangle = \begin{cases} 
1, & \text{if } v_i \in \mathbb{H}^3, \\
0, & \text{if } v_i \in C^+, \\
-1, & \text{if } v_i \in S^+.
\end{cases}$$

For any point $v_i \in \mathbb{H}^3 \cup C^+ \cup S^+$, we associate $v_i$ a geometric object in $\mathbb{H}^3$ denoted by $P_{\varepsilon_i}(v_i)$ as follows.

1. If $v_i \in \mathbb{H}^3$, then $P_{\varepsilon_i}(v_i) = P_1(v_i) = v_i$, i.e., the point itself.

2. If $v_i \in C^+$, then $P_{\varepsilon_i}(v_i) = P_0(v_i) = \{ X \in \mathbb{H}^3 : \langle X, u \rangle \geq -\frac{1}{2} \}$, i.e., a horoball in $\mathbb{H}^3$.

3. If $v_i \in S^+$, then $P_{\varepsilon_i}(v_i) = P_{-1}(v_i) = \{ X \in \mathbb{H}^3 : \langle X, u \rangle \geq 0 \}$, i.e., a half space of $\mathbb{H}^3$.

Given two points $v_i, v_j \in \mathbb{H}^3 \cup C^+ \cup S^+$, if $P_{\varepsilon_i}(v_i) \cap P_{\varepsilon_j}(v_j) = \emptyset$, there is a geodesic segment in $\mathbb{H}^3$ whose length realizes the distance between $P_{\varepsilon_i}(v_i)$ and $P_{\varepsilon_j}(v_j)$. It is denoted by $e_{ij}$.

Given three points $v_i, v_j, v_k \in \mathbb{H}^3 \cup C^+ \cup S^+$, if $P_{\varepsilon_i}(v_i)$, $P_{\varepsilon_j}(v_j)$ and $P_{\varepsilon_k}(v_k)$ are disjoint with each other, draw the three geodesic segments $e_{ij}, e_{jk}, e_{ki}$. There is a totally geodesic polygon in $\mathbb{H}^3$ bounded by $e_{ij}, e_{jk}, e_{ki}$ and the boundary of $P_{\varepsilon_i}(v_i)$, $P_{\varepsilon_j}(v_j)$ and $P_{\varepsilon_k}(v_k)$. This polygon is denoted by $\triangle ijk$. In fact $\triangle ijk$ is a generalized hyperbolic triangle which is studied in [GL09].

Given four points $v_1, v_2, v_3, v_4 \in \mathbb{H}^3 \cup C^+ \cup S^+$ such that $P_{\varepsilon_1}(v_1), P_{\varepsilon_2}(v_2), P_{\varepsilon_3}(v_3)$ and $P_{\varepsilon_4}(v_4)$ are disjoint with each other, there are four polygons $\triangle 234, \triangle 134, \triangle 124, \triangle 123$.

If $P_{\varepsilon_i}(v_i)$ is a horoball, the intersections $\partial P_{\varepsilon_i}(v_i) \cap \triangle ijk$, $\partial P_{\varepsilon_i}(v_i) \cap \triangle ikl$, and $\partial P_{\varepsilon_i}(v_i) \cap \triangle ilj$ are three Euclidean line segments which bound a Euclidean triangle, i.e., the link at $v_i$, denoted by $LK(v_i)$.

If $P_{\varepsilon_i}(v_i)$ is a half space, the intersections $\partial P_{\varepsilon_i}(v_i) \cap \triangle ijk$, $\partial P_{\varepsilon_i}(v_i) \cap \triangle ikl$, and $\partial P_{\varepsilon_i}(v_i) \cap \triangle ilj$ are three hyperbolic geodesic segments which bound a hyperbolic triangle, i.e., the link at $v_i$, denoted by $LK(v_i)$. 


If the polygons $\triangle 234, \triangle 134, \triangle 124, \triangle 123$ and the links $LK(v_i)$ with $v_i \in C^+ \cup S^+$ for $i \in \{1, 2, 3, 4\}$ bound an object in $\mathbb{H}^3$ with positive volume, this object is a \textit{generalized hyperbolic tetrahedron} which is denoted by $T_{1234}$. Its edges are the geodesic segments $e_{ij}$ for $i, j \in \{1, 2, 3, 4\}$. And its faces are $\triangle 234, \triangle 134, \triangle 124, \triangle 123$.

If $v_i \in \mathbb{H}^3$, the link $LK(v_i)$ is a spherical triangle which is the intersection of $T_{1234}$ and a sufficiently small sphere centered at $v_i$.

A generalized hyperbolic tetrahedron is uniquely determined by the four points $v_1, v_2, v_3, v_4$. According to different types of the points $v_i$, there are 15 types of generalized hyperbolic tetrahedra.

3. Link

A link $LK(v_i)$ is a spherical, Euclidean or hyperbolic triangle if $v_i \in \mathbb{H}^3, C^+$ or $S^+$ respectively. If $v_i \in \mathbb{H}^3 \cup S^+$, denote by $b_{kl}^i, b_{ij}^i, b_{jk}^i$ the length of edges of $Lk(v_i)$: $\partial P_{e_i}(v_i) \cap \triangle ikl, \partial P_{e_i}(v_i) \cap \triangle ilj$ and $\partial P_{e_i}(v_i) \cap \triangle ijk$. If $v_i \in C^+$, let $b_{kl}^i, b_{ij}^i, b_{jk}^i$ be TWICE of the length of edges of $Lk(v_i)$. Denoted by $a_{ij}$ the dihedral angle between the face $\triangle ijk$ and $\triangle ilj$ for $\{i, j, k, l\} = \{1, 2, 3, 4\}$. The dihedral angles $a_{ij}, a_{ik}, a_{il}$ become the opposite inner angles of $Lk(v_i)$.

We introduce a function of $b_{jk}^i$ and its derivative as follows

$$\rho_{jk}^i = \int_0^{b_{jk}^i} \cos(\sqrt{\varepsilon}s)ds,$$

$$\rho_{jk}^i = \cos(\sqrt{\varepsilon}b_{jk}^i),$$

where $\varepsilon$ is the type of $v_i$.

The \textit{amplitude} of the link $LK(v_i)$ is defined as $[Fe89]

$$A^i = \rho_{jk}^i \rho_{jl}^i \sin a_{ij},$$

which only depends on the link $Lk(v_i)$.

The derivative of the law of cosine of a spherical, Euclidean or hyperbolic triangle is derived in [CL03, Lu06] and has the uniform formula.

\textbf{Lemma 2.}

$$\frac{\partial a_{ij}}{\partial b_{kl}^i} = \frac{\rho_{kl}^i}{A^i},$$

$$\frac{\partial a_{ij}}{\partial b_{ij}^i} = \frac{\rho_{ij}^i}{A^i}(- \cos a_{ij}).$$

4. Face

Let $x_{ij}$ be the length of the edge $e_{ij}$. We introduce a function of $x_{ij}$ and its derivative as follows:

$$\tau_{ij} = \frac{1}{2} x_{ij} + \frac{1}{2} \varepsilon_i \varepsilon_j e^{-x_{ij}},$$

$$\tau'_{ij} = \frac{1}{2} x_{ij} + \frac{1}{2} \varepsilon_i \varepsilon_j e^{-x_{ij}}.$$  

Each face $\triangle jkl$ of $T_{1234}$ is a generalized hyperbolic triangle. It has the edge lengths $x_{kl}, x_{ij}, x_{jk}$ and the opposite generalized angles $b_{kl}^i, b_{ij}^i, b_{jk}^i$.

The \textit{amplitude} of the face $\triangle jkl$ is defined as

$$A_{jkl} = \tau_{jk} \tau_{jl} \rho_{kl}^i.$$
which only depends on the face.

The derivative of the law of cosine of a generalized hyperbolic triangle is derived in [GL09] and has the uniform formula.

**Lemma 3** ([GL09] Lemma 3.5).

\[
\frac{\partial b_{jkl}}{\partial x_{kl}} = \frac{\tau_{kl}}{A_{jkl}},
\]

\[
\frac{\partial b_{jkl}}{\partial x_{lj}} = \frac{\tau_{kl}(-\rho'_{jk})}{A_{jkl}}.
\]

5. **GRAM MATRIX**

Given four points \(v_1, v_2, v_3, v_4 \in \mathbb{H}^3 \cup C^+ \cup S^+\), the Gram matrix of \(T_{1234}\) determined by \(v_1, v_2, v_3, v_4\) is defined as

\[
G = (\langle v_i, v_j \rangle)_{4\times4}.
\]

**Lemma 4.** If \(i \neq j\), then \(\langle v_i, v_j \rangle = -\tau'_{ij}\).

**Proof.** When \(\varepsilon_i \varepsilon_j \neq 0\), it is well-known. See, for example, [Ra06] pp 62-72. When \(\varepsilon_i \varepsilon_j = 0\), see [He05] pp 7-9. When \(\varepsilon_i = \varepsilon_j = 0\), the formula was obtained in [Pe87]. Note that we use a different convention in the definition of a horoball from the convention in [He05]. Due to our convention, we have the following.

If \(X \in C^+, Y \in \mathbb{H}^3 \cup S^+\), then \(\langle X, Y \rangle = -\frac{1}{2}e^d\) where \(d\) is the distance between the horoball associated to \(X\) and the vertex or the half space associated to \(Y\).

If \(X, Y \in C^+\), then \(\langle X, Y \rangle = -\frac{1}{2}e^d\) where \(d\) is the distance between the two horoballs associated to \(X\) and \(Y\). \(\square\)

Therefore the Gram matrix can be written as

\[
G = \begin{pmatrix}
-\varepsilon_1 & -\tau'_{12} & -\tau'_{13} & -\tau'_{14} \\
-\tau'_{12} & -\varepsilon_2 & -\tau'_{23} & -\tau'_{24} \\
-\tau'_{13} & -\tau'_{23} & -\varepsilon_3 & -\tau'_{34} \\
-\tau'_{14} & -\tau'_{24} & -\tau'_{34} & -\varepsilon_4
\end{pmatrix}.
\]

Before calculating \(\det G\), we recall an analogy in 2 dimensions. Recall that \(G_{ij}\) is the matrix obtained by deleting the \(i\)-th row and \(j\)-th column of the Gram matrix \(G\).

**Lemma 5** ([GL09] Lemma 3.3). \(G_{ii}\) is the Gram matrix of the face \(\triangle jkl\) and

\[
\sqrt{-\det G_{ii}} = A_{jkl}
\]

where \(A_{jkl}\) is the amplitude of the face \(\triangle jkl\) (6).

**Lemma 6.**

\[
\sqrt{-\det G} = \tau_{ij} \tau_{ik} \tau_{il} A^i,
\]

where \(A^i\) is the amplitude of the link \(LK(v_i)\) (3).
Proof. Case 1. If one of vertex is not in $C^+$, say $\varepsilon_1 = \pm 1$. Then

$$\det G = \frac{1}{\varepsilon_1} \det \begin{pmatrix} \varepsilon_1 & \tau'_{12} & \tau'_{13} & \tau'_{14} \\ \varepsilon_1 \tau_{12} & \varepsilon_1 \varepsilon_2 & \varepsilon_1 \tau_{13} & \varepsilon_1 \tau_{14} \\ \varepsilon_1 \tau'_{13} & \varepsilon_1 \tau_{13} & \varepsilon_1 \varepsilon_3 & \varepsilon_1 \tau'_{14} \\ \varepsilon_1 \tau'_{14} & \varepsilon_1 \tau_{14} & \varepsilon_1 \varepsilon_4 & \varepsilon_1 \tau'_{13} \end{pmatrix}$$

$$= \frac{1}{\varepsilon_1} \det \begin{pmatrix} \varepsilon_1 & \tau'_{12} & \tau'_{13} & \tau'_{14} \\ 0 & \varepsilon_1 \varepsilon_2 - (\tau'_{12})^2 & \varepsilon_1 \tau_{12} \tau_{13} - \tau'_{12} \tau'_{13} & \varepsilon_1 \tau_{12} \tau_{14} - \tau'_{12} \tau'_{14} \\ 0 & \varepsilon_1 \tau_{13} - \tau'_{12} \tau_{13} & \varepsilon_1 \varepsilon_3 - (\tau'_{13})^2 & \varepsilon_1 \tau_{13} \tau_{14} - \tau'_{13} \tau'_{14} \\ 0 & \varepsilon_1 \tau_{14} - \tau'_{12} \tau_{14} & \varepsilon_1 \varepsilon_4 - (\tau'_{14})^2 \\ & & & \end{pmatrix}$$

$$= \det \begin{pmatrix} \varepsilon_1 \varepsilon_2 - (\tau'_{12})^2 & \varepsilon_1 \tau_{12} \tau_{13} - \tau'_{12} \tau'_{13} & \varepsilon_1 \tau_{12} \tau_{14} - \tau'_{12} \tau'_{14} \\ \varepsilon_1 \tau_{13} - \tau'_{12} \tau_{13} & \varepsilon_1 \varepsilon_3 - (\tau'_{13})^2 & \varepsilon_1 \tau_{13} \tau_{14} - \tau'_{13} \tau'_{14} \\ \varepsilon_1 \tau_{14} - \tau'_{12} \tau_{14} & \varepsilon_1 \varepsilon_4 - (\tau'_{14})^2 & \end{pmatrix}$$

$$= (\tau_{12} \tau_{13} \tau_{14})^2 \det \begin{pmatrix} -1 & -\rho_{23}^1 & -\rho_{24}^1 \\ -\rho_{23}^1 & -1 & -\rho_{34}^1 \\ -\rho_{24}^1 & -\rho_{34}^1 & -1 \end{pmatrix}$$

$$= (\tau_{12} \tau_{13} \tau_{14})^2 (\rho_{23}^1)^2.$$ 

In the step (a) we use the fact $\varepsilon_i \varepsilon_j - (\tau'_{ij})^2 = -(\tau_{ij})^2$ which is easily verified using the definition (4) and (5). We also use the law of cosine for a generalized hyperbolic triangle([GL09] Lemma 3.1):

$$\rho_{kl}^j = \frac{\varepsilon_j \tau'_{kl} + \tau'_{jk} \tau'_{jl}}{\tau_{jk} \tau_{jl}}.$$ 

In the step (b) we use the law of cosine for a spherical or hyperbolic triangle. 

For details of calculations, see [Fe89] pp 167-171.

Case 2. If all vertexes are on $C^+$, i.e., $\varepsilon_i = 0$ for $i = 1, 2, 3, 4$, then, by definition (5),

$$\det G = \frac{1}{16} \det \begin{pmatrix} e^{x_{12}} & e^{x_{13}} & e^{x_{14}} \\ e^{x_{12}} & 0 & e^{x_{23}} \\ e^{x_{13}} & e^{x_{23}} & 0 \\ e^{x_{14}} & e^{x_{24}} & e^{x_{34}} \end{pmatrix}$$

$$= \frac{1}{16} \left( e^{2x_{12} + 2x_{23} + 2x_{34}} + e^{2x_{13} + 2x_{24} + 2x_{34}} + e^{2x_{14} + 2x_{23}} - 2e^{x_{12} + x_{23} + x_{13} + x_{24}} - 2e^{x_{13} + x_{24} + x_{14} + x_{23}} - 2e^{x_{14} + x_{23} + x_{12} + x_{34}} \right)$$

$$= \frac{1}{16} \left( \frac{1}{16} (b_{14}^1)^4 + \frac{1}{16} (b_{24}^1)^4 + \frac{1}{16} (b_{23}^1)^4 - 2 \frac{1}{16} (b_{34}^1 b_{23}^1)^2 - 2 \frac{1}{16} (b_{23}^1 b_{24}^1)^2 - 2 \frac{1}{16} (b_{14}^1 b_{24}^1)^2 \right)$$

$$= \left( \frac{e^{x_{12} e^{x_{13}}} e^{x_{14}}}{2} \right)^2 (-A^1)^2.$$
In the step (c) we use the law of cosine for an ideal hyperbolic triangle ([GL09] Appendix A):
\[
\frac{(b_{jk})^2}{4} = e^{x_{jk} - x_{ij} - x_{ik}}.
\]

In the step (d) we use the law of cosine for a Euclidean triangle. In fact, it is Heron’s formula of the area of a Euclidean triangle. □

6. JACOBIAN MATRIX

First, a generalized hyperbolic tetrahedron is determined by its edge lengths uniquely up to isometries. Therefore the dihedral angle $a_{ij}$ for $i, j \in \{1, 2, 3, 4\}$ is a function of the lengths $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$.

Proof of Theorem 1. To prove of the theorem, we need to calculate $\frac{\partial a_{ij}}{\partial x_{kl}}, \frac{\partial a_{ij}}{\partial x_{ik}}$ and $\frac{\partial a_{ij}}{\partial x_{ij}}$.

\[
\begin{align*}
\frac{\partial a_{ij}}{\partial x_{kl}} &= \frac{\partial a_{ij}}{\partial b_{kl}^i} \cdot \frac{\partial b_{kl}^i}{\partial x_{kl}} \\
&= \frac{\rho_{kl}^i}{A^i} \cdot \frac{\tau_{kl}}{A_{ikl}} \\
&= \frac{\tau_{kl}}{\tau_{il} \tau_{ik} A^i} \cdot \frac{1}{\sin a_{ij} \sin a_{kl}} \cdot \sin a_{ij} \sin a_{kl} \\
&= \frac{\tau_{kl}}{\tau_{il} \tau_{ik} A^i} \cdot \frac{\rho_{il}^i \rho_{ij}^j \rho_{jk}^k}{A^j A^k} \cdot \sin a_{ij} \sin a_{kl} \\
&= \sqrt{- \det G_{ii} \det G_{jj} \det G_{kk} \det G_{ll} \det G_{ij} \det G_{kl} \det G_{lj} \det G_{ik} - \det G_{ij}^3} \cdot \sin a_{ij} \sin a_{kl}.
\end{align*}
\]

In the step (e), Lemma 2 and Lemma 3 are used.
In the step (f), the definition (6) is used.
In the step (g), the definition (3) is used.
In the step (h), the definition (6) and Lemma 5 are used.
In the step (i), Lemma 6 is used.
\[
\frac{\partial a_{ij}}{\partial x_{ik}} = \frac{\partial a_{ij}}{\partial b_{jk}^i} \cdot \frac{\partial b_{jk}^i}{\partial x_{ik}}
\]
\[
= \rho_{kl}^j \left( - \cos a_{ik} \right) \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{1}{\sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right)}
\]
\[
(\ j) \quad \rho_{kl}^j \left( - \cos a_{ik} \right) \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{1}{\sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right)}
\]
\[
\begin{aligned}
&= \sqrt{- \det G_{ii} \cdot \det G_{jj} \cdot \det G_{kk} \cdot \det G_{ll}} \cdot \frac{1}{\left( - \det G \right)^3} \cdot \sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right) \\
&\quad \cdot \sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right) \\
\end{aligned}
\]
\[
\frac{\partial a_{ij}}{\partial x_{ij}} = \frac{\partial a_{ij}}{\partial b_{jk}^i} \cdot \frac{\partial b_{jk}^i}{\partial x_{ij}} + \frac{\partial a_{ij}}{\partial b_{jl}^i} \cdot \frac{\partial b_{jl}^i}{\partial x_{ij}}
\]

By the symmetry of \( k \) and \( l \), we only need to calculate the first term.

\[
\begin{aligned}
\frac{\partial a_{ij}}{\partial b_{jk}^i} \cdot \frac{\partial b_{jk}^i}{\partial x_{ij}} &= \rho_{kl}^j \left( - \cos a_{ik} \right) \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{\tau_{jk}}{\sqrt{- \det G_{ll}}} \cdot \frac{1}{\sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right)} \\
&= \sqrt{- \det G_{ii} \cdot \det G_{jj} \cdot \det G_{kk} \cdot \det G_{ll}} \cdot \frac{1}{\left( - \det G \right)^3} \cdot \sin a_{ij} \sin a_{ik} \cdot \left( - \cos a_{jk} \right) \\
\end{aligned}
\]

In the step \((k)\), the law of cosine for a hyperbolic or spherical triangle is used. For a Euclidean triangle, we use the fact:

\[
1 = \cos a_{ij} + \cos a_{ij} \cos a_{jk} \sin a_{ij} \sin a_{jk}
\]

In the step \((l)\), compare what we need to compute with the result of the step \((j)\) in the last formula. They are the same if we switch \( i \) and \( j \). Hence the step \((l)\) holds.
Therefore
\[
\frac{\partial a_{ij}}{\partial x_{ij}} = \sqrt{\det G_{ii} \det G_{jj} \det G_{kk} \det G_{ll} - (\det G)^3} \cdot (\cos a_{ik} \cos a_{jl} + \cos a_{ik} \cos a_{ij} \cos a_{jk} + \cos a_{il} \cos a_{ij} \cos a_{jl})
\]
\[
= \sqrt{\det G_{ii} \det G_{jj} \det G_{kk} \det G_{ll} - (\det G)^3} \cdot \sin^2 a_{ij} \cdot w_{ij}.
\]

\[\Box\]

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455
E-mail address: guoxx170@umn.edu