An ideal paramagnet satisfies the equation of state (Curie's law),

$$M = \frac{D}{T} H,$$

where M is the magnetization, H the magnetic field, T the absolute temperature, and D a constant. An internal energy U is independent of M, following that $dU = C_M dT$, where C_M is a constant heat capacity. Assume that the paramagnet is used to create a Carnot engine and that the engine operates between temperatures T_h and T_c such that $T_h > T_c$.

(a) The first law of Thermodynamics for the magnetic system is written as

$$dU = dQ - dW = TdS + HdM.$$

For an adiabatic process, show that

$$\frac{1}{2}(M_h^2 - M_c^2) = C_M D \ln \frac{T_h}{T_c} = C_M D \ln \frac{H_h M_c}{H_c M_h},$$

where M_h (M_c) is the magnetization at T_h (T_c), when the magnetic field is H_h (H_c).

The Carnot cycle takes the following steps:

- $1 \to 2$ isothermal demagnetization at $T = T_h$, $M_2 < M_1$
- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$
- $3 \to 4$ isothermal magnetization at $T = T_c$, $M_4 > M_3$
- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

(b) Determine the heat transfer ΔQ and the work performed by the system ΔW for each of the four steps.

(c) Sketch the Carnot cycle in the (M, T)-plane and in the (M, H)-plane.

(d) Prove that the engine has efficiency

$$\eta = 1 - \frac{T_c}{T_h}.$$

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where M_h (M_c) is the magnetization at T_h (T_c), when the magnetic field is H_h (H_c).

Solution:
For an adiabatic process, $dQ = 0$, therefore $dU = -dW$, where $dU = C_M dT$ and $-dW = HdM$. Using the equation of state, we obtain

$$C_M dT = HdM = \frac{T}{D}MdM \tag{11}$$

$$\Rightarrow \quad C_M D\frac{dT}{T} = MdM \tag{12}$$

$$\Rightarrow \quad \int_{T_c}^{T_h} C_M D\frac{dT}{T} = \int_{M_c}^{M_h} MdM \tag{13}$$

$$\Rightarrow \quad C_M D\ln\frac{T_h}{T_c} = \frac{1}{2}(M_h^2 - M_c^2) \tag{14}$$

The equation of state leads to

$$\frac{T_h}{T_c} = \frac{DH_h/M_h}{DH_c/M_c} = \frac{H_h M_c}{H_c M_h}. \tag{15}$$

Therefore,

$$\frac{1}{2}(M_h^2 - M_c^2) = C_M D\ln\frac{T_h}{T_c} = C_M D\ln\frac{H_h M_c}{H_c M_h} \tag{16}$$

The Carnot cycle takes the following steps:

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- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

(b) Determine the heat transfer ΔQ and the work performed by the system ΔW for each of the four steps.

Solution:

- $1 \to 2$ isothermal demagnetization at $T = T_h$, $M_2 < M_1$

For an isothermal process, $dU = C_M dT = 0$, therefore $dQ = dW = -HdM$. Using the equation of state, we obtain

$$dQ = dW = -\frac{T_h}{D} M dM \tag{18}$$

$$\Rightarrow \quad \Delta Q_{12} = \Delta W_{12} = -\frac{T_h}{D}\int_{M_1}^{M_2} M dM = \frac{T_h}{2D}(M_1^2 - M_2^2) > 0 \tag{19}$$

- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

For an adiabatic process, $dQ = 0$, therefore

$$\Delta Q_{23} = 0. \tag{20}$$

Since $dW = -dU = -C_M dT$,

$$\Delta W_{23} = -C_M \int_{T_h}^{T_c} dT = C_M(T_h - T_c). \tag{21}$$

- $3 \to 4$ isothermal magnetization at $T = T_c$, $M_4 > M_3$

$$dQ = dW = -\frac{T_c}{D} M dM \tag{22}$$

$$\Rightarrow \quad \Delta Q_{34} = \Delta W_{34} = -\frac{T_c}{D}\int_{M_3}^{M_4} M dM = -\frac{T_c}{2D}(M_4^2 - M_3^2) < 0 \tag{23}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Since the step is an adiabatic process,

$$\Delta Q_{41} = 0. \tag{24}$$

Since $dW = -dU = -C_M dT$,

$$\Delta W_{41} = -C_M \int_{T_c}^{T_h} dT = -C_M(T_h - T_c). \tag{25}$$

(c) Sketch the Carnot cycle in the (M, T)-plane and in the (M, H)-plane.

Solution:
In the (M, T)-plane,

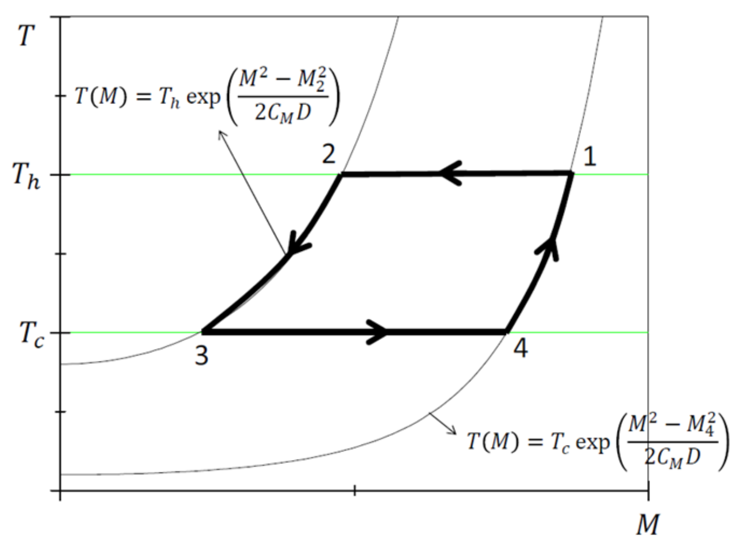
- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

$$\frac{1}{2}(M_2^2 - M^2) = C_M D \ln \frac{T_h}{T} \tag{25}$$
$$\Rightarrow \quad \frac{T_h}{T} = \exp\left(\frac{M_2^2 - M^2}{2C_M D}\right) \tag{26}$$
$$\Rightarrow \quad T(M) = T_h \exp\left(\frac{M^2 - M_2^2}{2C_M D}\right) \tag{27}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Similarly,

$$T(M) = T_c \exp\left(\frac{M^2 - M_4^2}{2C_M D}\right) \tag{28}$$



In the (M, H)-plane,

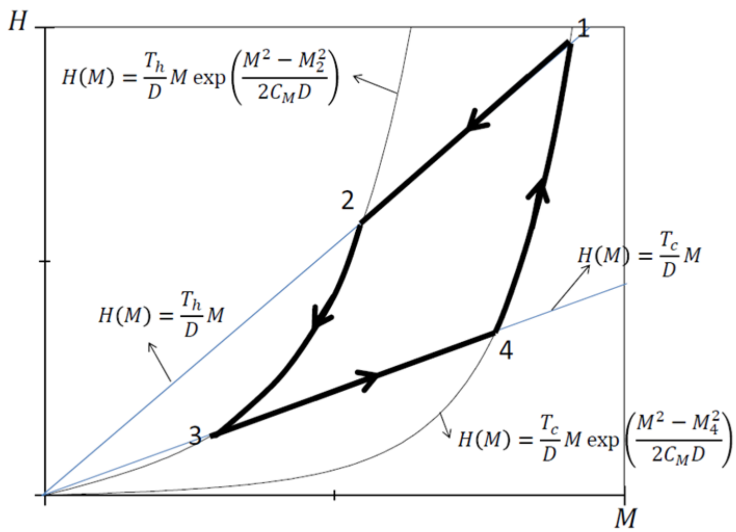
- $2 \to 3$ adiabatic demagnetization, $T_h \to T_c$, $M_3 < M_2$

$$\frac{1}{2}(M_2^2 - M^2) = C_M D \ln\left(\frac{T_h}{D}\frac{M}{H}\right) \tag{29}$$
$$\Rightarrow \quad \frac{T_h}{D}\frac{M}{H} = \exp\left(\frac{M_2^2 - M^2}{2C_M D}\right) \tag{30}$$
$$\Rightarrow \quad H(M) = \frac{T_h}{D} M \exp\left(\frac{M^2 - M_2^2}{2C_M D}\right) \tag{31}$$

- $4 \to 1$ adiabatic magnetization, $T_c \to T_h$, $M_1 > M_4$.

Similarly,

$$H(M) = \frac{T_c}{D} M \exp\left(\frac{M^2 - M_4^2}{2C_M D}\right) \tag{32}$$



(d) Prove that the engine has efficiency

$$\eta = 1 - \frac{T_c}{T_h}.$$

Solution:
The total work performed by the system is

$$\Delta W = \Delta W_{12} + \Delta W_{23} + \Delta W_{34} + \Delta W_{41}. \tag{33}$$

Since $\Delta W_{12} = \Delta Q_{12}$, $\Delta W_{34} = \Delta Q_{34}$, and $\Delta W_{23} = -\Delta W_{41}$,

$$\Delta W = \Delta Q_{12} + \Delta Q_{34}. \tag{34}$$

The engine efficiency is

$$\eta = \frac{\Delta W}{\Delta Q_{12}} \tag{35}$$

$$= 1 + \frac{\Delta Q_{34}}{\Delta Q_{12}} \tag{36}$$

$$= 1 + \frac{\frac{T_c}{2D}(M_3^2 - M_4^2)}{\frac{T_h}{2D}(M_1^2 - M_2^2)} \tag{37}$$

$$= 1 - \frac{T_c}{T_h}\frac{M_4^2 - M_3^2}{M_1^2 - M_2^2} \tag{38}$$

From Eq.(15), we obtain

$$M_1^2 - M_2^2 = M_4^2 - M_3^2 = 2C_M D \ln \frac{T_h}{T_c} \tag{39}$$

Thus,

$$\eta = 1 - \frac{T_c}{T_h} \tag{40}$$