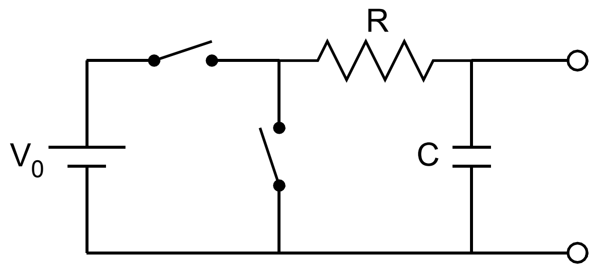
RC Circuits

Concepts

The addition of a simple capacitor to a circuit of resistors allows two related phenomena to occur. The observation that the time-dependence of a complex waveform is altered by the circuit is referred to as a time-domain analysis. On the other hand, observation that a single-frequency wave undergoes an amplitude and phase shift upon passage through the circuit is referred to as a frequency-domain analysis.

Time-Domain Analysis of the *RC* Circuit



The behavior of this "*RC*" circuit can be analyzed in the time-domain by solving an appropriate differential equation with the appropriate boundary conditions. Begin with Kirchoff's Potential Law, which is a consequence of conservation of energy:

$$V_0 = V_R + V_C = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}.$$

Consider the case in which initially the capacitor is charged to V_0 through the horizontal switch while the vertical switch is open. The boundary or initial condition is that at $t = 0$ $Q_0 = V_0 / C$. Now the horizontal switch is opened and the vertical switch is closed. Charge flows from one side of the capacitor to the other, and the differential equation to solve is simply

$$V_R + V_C = R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \Rightarrow \quad R\frac{dQ(t)}{dt} = -\frac{Q(t)}{C}.$$

The solution is

$$Q(t) = Q_0 e^{-\frac{t}{RC}} \quad \text{with} \quad Q_0 = V_0 C.$$

The quantity $\tau = RC$ is the *time constant* or *characteristic time* or *1/e time*, and it is the time for the charge to decay from Q_0 to Q_0 / e. The potentials across the resistor and capacitor are

$$V_R(t) = I(t)R = R\frac{dQ}{dt} = -V_0 e^{-\frac{t}{RC}}$$

and

$$V_C(t) = Q(t)/C = V_0 e^{-\frac{t}{RC}}.$$

Now consider the case in which initially *RC* is grounded through closure of the vertical switch. When this switch is opened and the horizontal switched is closed, current begins to flow and the capacitor begins to charge. The initial or boundary condition is that at $t = 0$, $V_R + V_C = V_0$, and the equation to solve is

$$V_0 = V_R + V_C = IR + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}.$$

The solution is the solution above (the solution to the homogeneous equation) plus whatever is necessary to satisfy the initial condition. Thus, the solution is

$$Q(t) = Q_0\left(1 - e^{-\frac{t}{RC}}\right) \quad \text{with} \quad Q_0 = V_0 C.$$

In time $\tau = RC$, Q rises from 0 to $Q_0(1-1/e)$. Notice that the current decreases with time as

$$I(t) = \frac{dQ}{dt} = \frac{V_0}{R}e^{-\frac{t}{RC}}.$$

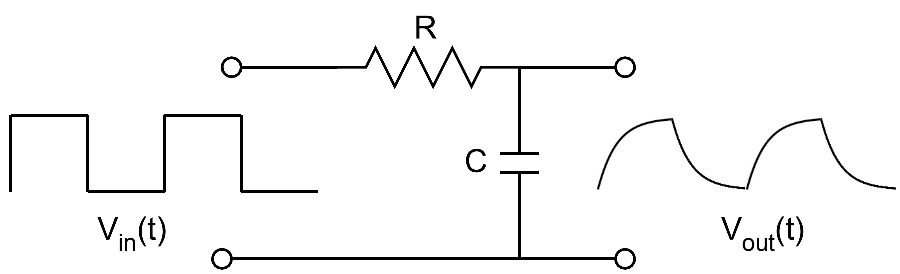
The potentials across the resistor and capacitor are

$$V_R(t) = I(t)R = R\frac{dQ}{dt} = V_0 e^{-\frac{t}{RC}},$$

and

$$V_C(t) = Q(t)/C = V_0\left(1 - e^{-\frac{t}{RC}}\right).$$

Instead of using manual switches, it is easier to use a square wave from a function generator to alternate the applied potential between 0 and V_0. This circuit and the resulting $V_{out}(t) = V_C(t)$ measured across the output terminals for a low frequency square wave are shown in the figure below.



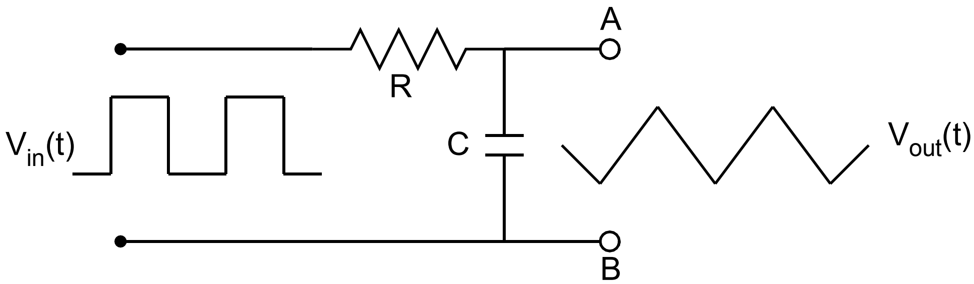
As the frequency of the applied square wave increases, the output waveform is diminished because there is insufficient time for the capacitor to charge to the applied voltage. Examining the expression for $V_C(t)$ at times *t* small compared to τ, we find

$$V_C(t) = V_0\left(1 - e^{-\frac{t}{RC}}\right) \simeq V_0\left(1 - \left(1 - \frac{t}{RC}\right)\right) = V_0\frac{t}{RC}$$

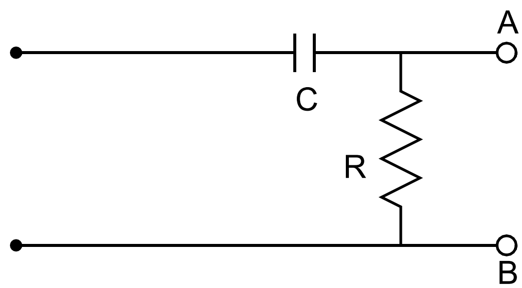
using the expansion

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \simeq 1 + x.$$

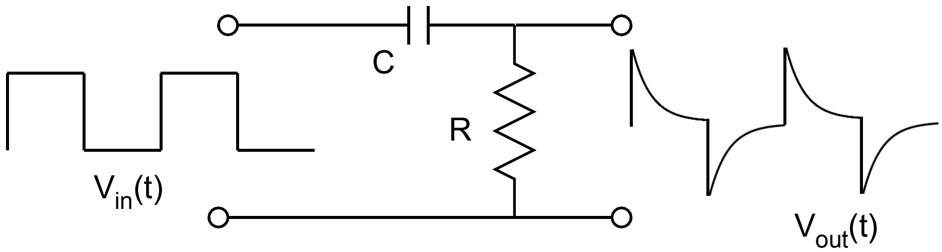
Thus $V_C(t)$ is linear in *t* with a slope $1/RC$. At high frequencies, the *RC* circuit acts as an integrator of a square wave, as shown below.



Time-Domain Analysis of the *CR* Circuit

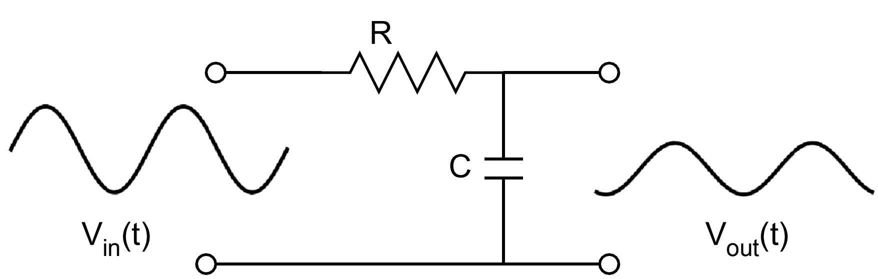


If we swap the positions of the resistor and capacitor, we have a *CR* circuit, where we measure the output voltage across the resistor. The Kirchoff loop analysis from above is still valid here, the only change being that $V_{out} = V_R$. Using the results from above, the output signal response to a square wave input is shown below. In this case, the input frequency is low and the circuit behaves as a differentiator.



Frequency Domain Analysis

The behavior of these circuits in the frequency-domain can be determined by solving the same differential equation using a single-frequency applied potential $V_{in}(t) = V_0 e^{i\omega t}$. As pictured below, the resulting $V_{out}(t)$ has a smaller amplitude and is shifted in phase.



The analysis begins with

$$V_{in}(t) = V_R(t) + V_C(t) = I(t)R + \frac{Q(t)}{C} = R\frac{dQ}{dt} + \frac{Q(t)}{C}.$$

In steady state, $Q(t)$ oscillates at the same frequency ω as the applied signal but with a different phase: $Q(t) = Q_0 e^{i(\omega t+\alpha)}$. Thus,

$$V_o e^{i\omega t} = \left(i\omega R + \frac{1}{C}\right) Q_0 e^{i(\omega t+\alpha)}$$

or

$$V_0 = \left(i\omega R + \frac{1}{C}\right) Q_0 e^{i\alpha}$$

V_0 is the amplitude of the input potential at frequency ω, and Q_0 / C is the amplitude of the output signal across the capacitor at frequency ω. At this point, we have two unknowns, Q_0 and α, but only one equation. However, there are actually two equations here. V_0 is a real number and must be equal to the real part of the right hand side of the equation, and the imaginary part of the right hand side must be zero. Rewriting this expression as

$$\begin{aligned}\frac{V_0 C}{Q_0} &= (1 + i\omega RC)e^{i\alpha} = \sqrt{1+\omega^2 R^2 C^2}\, e^{i\tan^{-1}\omega RC} e^{i\alpha} \\ &= \sqrt{1+\omega^2 R^2 C^2}\, e^{i(\alpha+\tan^{-1}\omega RC)}\end{aligned},$$

the fact that the imaginary part of the right hand side must be zero yields

$$\alpha = -tan^{-1}\omega RC.$$

Then,

$$\frac{V_0 C}{Q_0} = \sqrt{1+\omega^2 R^2 C^2} \rightarrow Q_0 = \frac{V_0 C}{\sqrt{1+\omega^2 R^2 C^2}}$$

The final result for the output signal across the capacitor is

$$V_C(\omega) = \frac{V_0(\omega)}{\sqrt{1+\omega^2 R^2 C^2}} e^{-i\tan^{-1}\omega RC}$$

The frequency-dependent *transmission function* or *response function* for this circuit is

$$A(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{V_C(\omega)}{V_0(\omega)} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} e^{-i\tan^{-1}\omega RC}.$$

We define the corner, or 3dB, frequency as

$$\omega_c = \frac{1}{RC},$$

yielding the response function

$$A(\omega) = \frac{1}{\sqrt{1+(\omega/\omega_c)}} e^{-i\tan^{-1}(\omega/\omega_c)}.$$

Notice that the phase difference $\alpha \rightarrow 0$ as $\omega \rightarrow 0$ and $\alpha \rightarrow -\pi/2$ for $\omega \gg \omega_c$ and that the amplitude $|A(\omega)|$ is 1 at low frequency but tends to zero at high frequency. The conclusion is that this circuit is a *low-pass filter*, meaning that it does not transmit high frequencies very well.

Concept of Impedance

The current-potential relationship across the capacitor is interesting. The current in the circuit is

$$I(t) = \frac{dQ}{dt} = i\omega Q_0 e^{i(\omega t+\alpha)}$$

so, in the frequency domain,

$$I(\omega) = \omega Q_0(\omega) e^{i(\alpha+\pi/2)}.$$

The interesting result is that

$$\frac{V_C(\omega)}{I(\omega)} = \frac{1}{i\omega C}.$$

Thus, the potential across a capacitor is

$$V(\omega) = I(\omega) Z(\omega)$$

where the *impedance* of the capacitor is

$$Z(\omega) = \frac{1}{i\omega C}$$

The expression $V(\omega) = I(\omega)Z(\omega)$ looks like Ohm's law, but it is not. It simply states that there is a linear relationship between the frequency-dependent complex potential and the frequency-dependent complex current. Capacitors are non-dissipative elements, so the time-average power $Re\langle V(t)I(t)\rangle = 0$. For a resistor, the impedance is $Z(\omega) = R$, a real quantity independent of frequency.

It is important to understand the physical significance of the impedance $Z(\omega)$ of a capacitor. At low frequencies, the impedance $Z \to \infty$ and the capacitor acts as an open circuit. At high frequencies, the impedance $Z \to 0$ and the capacitor acts as an short circuit. These two limiting behaviors are useful in determining the behavior of a circuit without performing a detailed analysis.

Concept of Impedance

With this definition of the complex impedance of a capacitor, it is now easy to analyze an *RC* circuit in the frequency domain. The expression for a *potential divider* can be used to determine the potential across the capacitor in the *RC* circuit:

$$V_C(\omega) = V_0(\omega)\frac{Z_C(\omega)}{Z_C(\omega)+R} = V_0(\omega)\frac{1/i\omega C}{1/i\omega C + R}$$

$$= V_0(\omega)\frac{1}{1+i\omega RC} = V_0(\omega)\frac{1}{\sqrt{1+\omega^2R^2C^2}}e^{-i\tan^{-1}\omega RC}$$

This expression is consistent with the previous conclusion that the *RC* circuit is a low-pass filter.

For the *CR* circuit, the potential divider expression yields the voltage across the resistor:

$$V_R(\omega) = V_0(\omega)\frac{R}{Z_C(\omega)+R} = V_0(\omega)\frac{R}{1/i\omega C + R}$$

$$= V_0(\omega)\frac{1}{1 - i\dfrac{1}{\omega RC}} = V_0(\omega)\frac{1}{\sqrt{1+\dfrac{1}{\omega^2 R^2 C^2}}} e^{i\tan^{-1}\frac{1}{\omega RC}}$$