OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #99

September 26 and 27, 2005

Comprehensive examination for Fall 2005

PART I, Monday September 26, 9:00 am

General Instructions

This Comprehensive Examination for Fall 2005 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 26, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 27, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1.

Consider an ideal 2-dimensional Fermi gas consisting of N non-interacting particles of spin $S = \frac{1}{2}$ and mass m enclosed in a 2D "box" of size $L \times L$. The external magnetic field is zero. Find:

(a) the density of states function $D(\epsilon)$, where ϵ is the energy of a single particle;

(b) the Fermi energy ϵ_f of the gas (i.e., the chemical potential at zero temperature: $\epsilon_f = \mu(T = 0)$);

Then,

(c) show that at finite temperatures the chemical potential satisfies the equation:

$$\mu + k_B T \ln\left[1 + \exp\left(-\frac{\mu}{k_B T}\right)\right] = \epsilon_f,$$

Now, consider the same gas in external magnetic field H at $T = 0$. Ignore the interaction of the field with the orbit of the electrons, assume it acts only on the spin. Assume that the particles are electrons (so that the magnetic moment of an individual particle can be expressed in terms of the Bohr magneton μ_B).

(d) Find the total induced magnetization $M(T = 0)$, and the magnetic susceptibility $\chi(T = 0)$ of the gas.

Hints: in (c), you may want to use a dummy variable $x = \exp[(\epsilon - \mu)/k_B T]$, and:

$$\int \frac{dx}{x(ax + b)} = -\frac{1}{b} \ln\left|\frac{ax + b}{x}\right|$$

Problem 2.

Consider the double pendulum, formed by a disk rotating around the point A in the yz plane, and a string (mathematical) pendulum, rotating around the point B in the plane perpendicular to the disk, shown in Figure. The mass m can oscillate in 3D, while its position is completely described by two angles, α_x and α_y (note: the strings do not compress; neither does the mass slide along the strings).

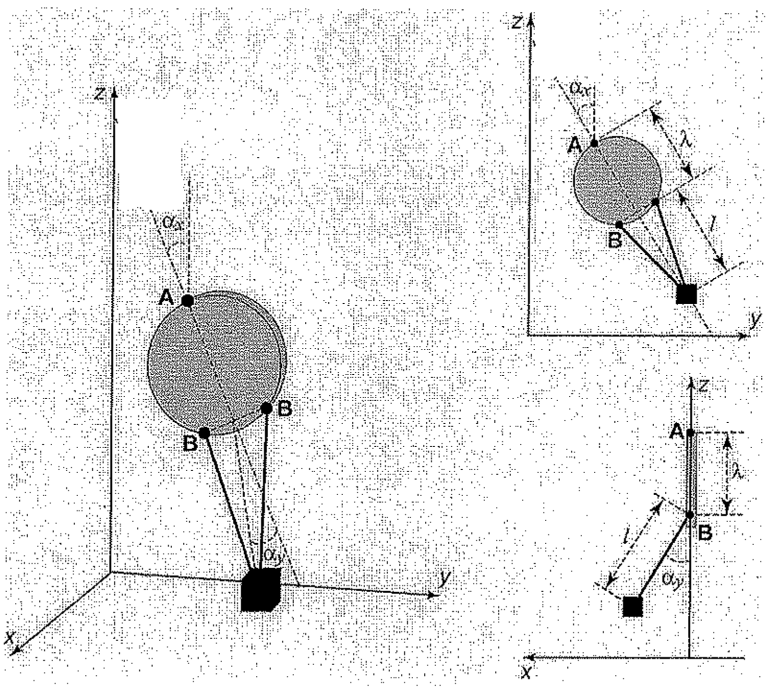


Figure 1: The sketch of a double pendulum described in the text; the left panel shows 3D picture of the system; the 2D projections are shown in right panels; **A** and **B** denote rotation points

Assume that the mass of supporting structures (disk, pivots and the string) can be neglected.
See questions on next page.

- Find the Lagrangian of the system
- Find the equations of motion
- Derive the normal modes and frequencies of small oscillations
- At the time $t = 0$, the mass is released from the point $\{\alpha_x, \alpha_y\} = \{a, -a\}$. Solve the equations of motion (find the position of a mass as a function of time). Sketch the trajectory of the system. Here and below assume that the oscillations remain small.
- In another experiment, the mass is released from the point $\{\alpha_x, \alpha_y\} = \{a, 0\}$. When the mass passed the origin, the length of the string is abruptly changed by $\Delta l \ll l$. Find the linear in Δl correction to the frequency of oscillations
- Assuming that the abrupt change described above conserves the energy of the system, find the change in the magnitude of oscillations after such an adjustment. Find the change in the magnitude of oscillations after 10 such adjustments.
- Assume that the magnitude of oscillations is not changed if the length of the string is abruptly changed at the "turning point" (the position of the maximal displacement). Qualitatively describe the behavior of the system, started from $\{\alpha_x, \alpha_y\} = \{a, 0\}$, where the length the string is changed by Δl each time the system passes the origin, and is changed back (by $-\Delta l$) each time the system passes the turning point. Consider two cases: $\Delta l > 0$ and $\Delta l < 0$. Relate your answer to swing.

Problem 3.

Consider a system of Fermions with orthonormal single particle states $\phi_n(\vec{r})\chi_n(\sigma)$. The spinors χ_n have only two components.

(A) Write down the two particle wave function with Fermions in states i and j.

(B) Using the two particle wave function, calculate the single particle density in terms of single particle states.

(C) What is the probability for two particles to be at the same position, in terms of the single particle states?

(D) In case the two particles have a non-zero probability of being at the same position, what is their total spin? A short answer is not sufficient, you need the mathematical proof.

Problem 4.

The parallel combination of an inductor and a capacitor exhibits oscillatory behavior. Consider the parallel combination of an inductor and a capacitor with a simple switch in between. The inductor of length a consists of N tightly-wound loops of radius R of very thin wire. The inductance is approximately $L = \mu_o \pi R^2 N^2 / a$. The capacitor consists of two circular parallel plates of area A separated by material of dielectric constant ϵ and thickness d. Initially, the switch is open, and the capacitor is charged to potential V_o. When the battery is disconnected and the switch is closed, oscillation ensues.

1. Assuming that the inductor, capacitor and wires are all ideal, solve for both the charge $Q(t)$ on the capacitor and the current $I(t)$ in the inductor. What is the phase relationship between them?

2. This oscillator can lose energy only through radiation, and your task is to find the radiated power. Assume that the lengths of the inductor and the capacitor and the separation between them are small compared to the wavelength of the radiation. The axes of the inductor and the capacitor are parallel. Ignore inductance, capacitance and resistance of the wires, and assume that radiation from the wires is negligible.

   (a) Find the initial radiated $\vec{E}$ and $\vec{B}$ fields.

   (b) Calculate the initial power radiated from the system. Does one object radiate more power than the other?

3. Now consider the case of perpendicular orientation of the capacitor and inductor.

   (a) How does the nature of the radiated electromagnetic field change when the inductor and capacitor are perpendicular?

   (b) How does the initial radiated power change?

Very useful relation:

$$\vec{A}(\vec{r},t) = -i\frac{\omega\mu_o}{4\pi}\frac{e^{ikr}}{r}\vec{p}(t) + \frac{\mu_o}{4\pi}\hat{r} \times \vec{m}(t)\frac{e^{ikr}}{r}\left(ik - \frac{1}{r}\right)$$

for a sinusoidal electric dipole $\vec{p}(t)$ and a sinusoidal magnetic dipole $\vec{m}(t)$.

Problem 5.

The simplest *classical* statistical mechanical model of a solid is a system of N identical non-interacting three-dimensional isotropic classical harmonic oscillators. In the simplest quantum mechanical approach (known as the 'Einstein's solid model'), the solid is represented by N 3-D *quantum* harmonic oscillators.

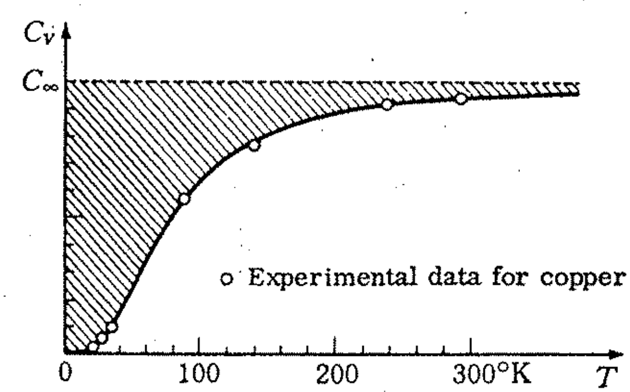
I. Based on the Hamiltonian of a classical 3-D HO, determine the mean thermal energy of a single oscillator, and the heat capacity $C_v^{\mathrm{cl.}}$ of a solid consisting of N such oscillators (assume that the volume does not change with T). It is OK to use the *Equipartition Theorem* – but, if you prefer the 'hard way', you may use the classical partition function formula for a single particle:

$$\mathcal{Z}_1 = \frac{1}{h^3} \int\int\int\int\int\int e^{-\mathcal{H}(x,\, y,\, z,\, p_x,\, p_y,\, p_z)/kT}\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z,$$

and then further process it to obtain $\langle U \rangle$ and C_v.

II. Now, switch to the quantum statistical mechanical formalism. Find the heat capacity C_v^{E} of the Einstein's solid (you may use any method, but canonical approach is recommended).

III. Examine the asymptotic behavior of C_v^{E} for $T \to 0$ and $T \to \infty$. Show that C_v^{E} is a monotonically increasing function of T and that the classical heat capacity is its hight-T limit.



IV. The figure above shows the dependence of C_v of a solid on the temperature. Based on the results you have obtained for I. and II., show that the shaded area above the heat capacity curve corresponds to the zero-point energy of a quantum SHO. The C_∞ symbol denotes the high-T limit of the function.

Problem 6.

In this problem you are asked to find the shape of the wires supporting a suspension bridge. For simplicity, assume that the bridge is supported by a single wire, that the wire has negligible mass, and that the bridge has constant linear density ρ (the mass of any segment of a bridge with length l is $m(l) = \rho l$).

- Draw all forces acting on a segment of a suspension wire located at an arbitrary point $\{x, y\}$ [see Figure]. Find the Cartesian components of the tension force as a function of the x (hint: the components may depend on the "unknown" angle α).
- Using relations for the tension force components, derive the differential equation for the shape of the support wires. (hint: use the relation between the derivative of a function and direction of its tangent)
- Solve the differential equation and find the shape of the support wire. Use the Figure to find the constant of integration. Sketch this shape.
- Analyze the dependence of the shape on the "lateral" (x-direction) wire tension T_x. What happens if $T_x \to 0$; if $T_x \to \infty$? (sketch the resulting shapes)

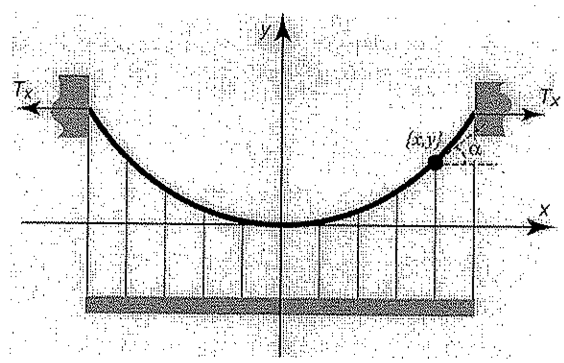


Figure 1: The sketch of a suspension bridge; you have to find the shape of the suspension wire, shown by thick black line

Problem 7.

Consider a quantum system in one dimension, with a time independent potential $V(x)$. The system is described by a wave function $\psi(x,t)$, which does not have to be an eigenstate. Consider the expectation value of the product of position and momentum for this system, i.e. $< xp > (t)$, as a function of time. The quantum virial theorem relates the time derivative of this quantity, $\frac{d}{dt} < xp > (t)$, to expectation values of the kinetic energy and a term which depends on the potential. Derive such a relation.
Consider a potential $V(x) = V_n x^n$, and assume that the system is in an eigenstate j with energy E_j. Show that in this case the expectation value of the potential is given by $\frac{2}{n+2} E_j$. One may assume that n is a positive, even number and that $V_n > 0$.

Problem 8.

One way to attempt nuclear fusion is to use magnetic confinement to raise the pressure of a hot plasma. Consider a modest model of this process, consisting of only a very thin-walled, hollow conducting tube of radius R through which current I is driven. When I is sufficiently large, the tube can be crushed.

1. For a tube oriented along $\hat{z}$, find $\vec{B}(\rho, \phi, z)$ inside and outside the tube.

2. Determine the inward pressure on the tube. One approach to this problem is to orient the tube along $\hat{z}$ and calculate the total force in the $\hat{x}$ direction on one side of the tube.

3. How does the pressure change as the tube collapses?

Problem 8 – solution:

Task (a): The allowed wavevectors for particles in a 2D box are:

$$\vec{k} = (k_x,\ k_y) = \left(n_x\frac{\pi}{L},\ n_y\frac{\pi}{L}\right), \tag{1}$$

where $n_x,\ n_y = 1, 2, 3, \ldots$. Hence, each quantum state 'occupies' a $(\pi/L) \times (\pi/L)$ square in the 2D k-vector space. For a given k value, the number of of states with wavevectors $\leq k$ (i.e., located within a circle of radius k in the first quadrant k_x, k_y coordinate system) is:

$$\mathcal{N}(k) = 2 \times \frac{1}{4} \times \frac{\pi k^2}{\pi^2/L^2} = \frac{L^2 k^2}{2\pi}, \tag{2}$$

The multiplication by 2 reflects the fact that each state corresponding to an allowed $\vec{k}$ vector is doubly degenerate. At zero magnetic field it can accommodate two particles of the same energy, with spin 'up' and spin 'down'.

Since the particle energy is $\epsilon = \hbar^2 k^2/2m$, and conversely: $k^2 = 2m\epsilon/\hbar^2$, from the above equation we obtain that the number of states with energies lower or equal to a certain energy value ϵ is:

$$\mathcal{N}(\epsilon) = \frac{mL^2}{\pi\hbar^2}\epsilon. \tag{3}$$

The number of states with energies between ϵ and $\epsilon + d\epsilon$ can be expressed in terms of the density of states function $D(\epsilon)$:

$$D(\epsilon)d\epsilon = \frac{d\mathcal{N}(\epsilon)}{d\epsilon}d\epsilon = \frac{mL^2}{\pi\hbar^2}d\epsilon \tag{4}$$

So that:

$$D(\epsilon) = \frac{mL^2}{\pi\hbar^2}. \tag{5}$$

Note that in the present case of a 2D fermion gas the density of states function has a constant value, in contrast to the situation in 3D and 1D fermion gases.

Task (b): As is the standard procedure for all gases, the number of particles can be calculated by integrating the $D(\epsilon)$ function multiplied by the state occupancy function $f(\epsilon)$ over all energies:

$$N = \int_0^\infty D(\epsilon) f(\epsilon, T) d\epsilon. \tag{6}$$

In the present case $f(\epsilon, T)$ is the Fermi-Dirac distribution function:

$$f(\epsilon, T) = \frac{1}{\exp[(\epsilon - \mu)/k_B T] + 1}. \tag{7}$$

For $T = 0$, $f(\epsilon) = 1$ for $\epsilon \leq \epsilon_f$, and $f(\epsilon) = 0$ for $\epsilon > E_f$, so that one can write:

$$N = \int_0^{\epsilon_f} D(\epsilon) d\epsilon. \tag{8}$$

By inserting the result for D and carrying out the integration, we get the answer to task (b):

$$N = \frac{mL^2 \epsilon_f}{\pi\hbar^2} \quad \Rightarrow \quad \epsilon_f = \frac{N\pi\hbar^2}{mL^2} = \frac{n\pi\hbar^2}{m}. \tag{9}$$

Task (c): For a finite T, we have to use the full form of $f(\epsilon, T)$ in Eq. (6).

$$N = \int_0^\infty D(\epsilon) f(\epsilon, T) d\epsilon = \frac{mL^2}{\pi\hbar^2} \int_0^\infty \frac{1}{\exp[(\epsilon - \mu)/k_{\mathrm{B}}T] + 1} d\epsilon \tag{10}$$

To calculate the integral, let's use the 'dummy variable' x defined as:

$$x \equiv \exp[(\epsilon - \mu)/k_{\mathrm{B}}T]. \tag{11}$$

By simple manipulations, we get:

$$\epsilon = k_{\mathrm{B}}T \ln(x) + \mu \quad \text{and} \quad d\epsilon = k_{\mathrm{B}}T \frac{dx}{x}. \tag{12}$$

The lower integration limit has to be changed from 0 to:

$$x(\epsilon = 0) = \exp(-\mu/k_{\mathrm{B}}T) \equiv x_0 \tag{13}$$

Accordingly:

$$N = \frac{mL^2 k_{\mathrm{B}}T}{\pi\hbar^2} \int_{x_0}^\infty \frac{dx}{x(x+1)} \tag{14}$$

Now we use the integral given in the *Hint*. Since x is always positive, we can drop the modulus symbol. We get:

$$N = \frac{mL^2 k_{\mathrm{B}}T}{\pi\hbar^2} \left[-\ln\left(\frac{x+1}{x}\right) \right]_{x_0}^\infty \tag{15}$$

For $x \to \infty$, $\ln[(x+1)/x] \to \ln(1) \to 0$. All that has to be done now is a series of rather straightforward manipulations:

$$N = \frac{mL^2 k_{\mathrm{B}}T}{\pi\hbar^2} \ln\left(\frac{x_0+1}{x_0}\right) = \frac{mL^2 k_{\mathrm{B}}T}{\pi\hbar^2} [\ln(x_0+1) - \ln(x_0)].$$

Now, we insert back Eq. (13) for x_0:

$$N = \frac{mL^2 k_{\mathrm{B}}T}{\pi\hbar^2} \left\{ \ln\left[\exp\left(-\frac{\mu}{k_{\mathrm{B}}T}\right) + 1\right] + \frac{\mu}{k_{\mathrm{B}}T} \right\}.$$

After some more algebra and regrouping:

$$\frac{N\pi\hbar^2}{ml^2} = \mu + k_{\mathrm{B}}T \ln\left[\exp\left(-\frac{\mu}{k_{\mathrm{B}}T}\right) + 1\right]$$

Note that the left-hand expression is the Fermi energy ϵ_{f} given by Eq. (9). So:

$$\epsilon_{\mathrm{f}} = \mu + k_{\mathrm{B}}T \ln\left[\exp\left(-\frac{\mu}{k_{\mathrm{B}}T}\right) + 1\right] \tag{16}$$

which is the equation we had to derive.

Task (d): In external magnetic field the state degeneracy is removed. The projection of the electron's magnetic moment on the magnetic field direction μ_z may take the value of $+\mu_{\mathrm{B}}$ or $-\mu_{\mathrm{B}}$. Let's call these states "up" and "down", respectively. For a given wavevector $\vec{k}$ the energy of the "up" and the "down" states is, respectively:

$$\epsilon_+(\vec{k}) = \frac{\hbar^2 k^2}{2m} - \mu_{\mathrm{B}}H \quad \text{and} \quad \epsilon_-(\vec{k}) = \frac{\hbar^2 k^2}{2m} + \mu_{\mathrm{B}}H. \tag{17}$$

Denote the density of states functions for the two spin states as $D_+(\epsilon)$ and $D_-(\epsilon)$, respectively. Each of these functions is now one-half of that given by Eq. (5) for the degenerate states:

$$D_+(\epsilon) = D_-(\epsilon) = \frac{mL^2}{2\pi\hbar^2} \tag{18}$$

Now we find the number of "up" particles (N_+) and "down" particles (N_-) using the familiar $\int D(\epsilon)f(\epsilon)d\epsilon$ recipe. At zero T, all "up" and "down"states with energies lower or equal to the Fermi energy are fully occupied ($f(\epsilon) = 1$), and for $\epsilon > \epsilon_f$ the occupancy of both states is zero ($f(\epsilon) = 0$). The lowest energy for the "up" and "down" states are $-\mu_B H$ and $+\mu_B H$, respectively, so that:

$$N_+ = \int_{-\mu_B}^{\epsilon_f} \frac{mL^2}{2\pi\hbar^2} d\epsilon = \frac{mL^2}{2\pi\hbar^2}(\epsilon_f + \mu_B H); \quad \text{and} \quad N_- = \int_{+\mu_B}^{\epsilon_f} \frac{mL^2}{2\pi\hbar^2} d\epsilon = \frac{mL^2}{2\pi\hbar^2}(\epsilon_f - \mu_B H). \tag{19}$$

The projection of the electron magnetic moment on the magnetic field direction is $\pm\mu_B$. The total magnetic moment of all "up" particles is $M_+ = +\mu_B N_+$, and for the "down" particles it is $M_- = -\mu_B N_-$. The net induced magnetization is thus:

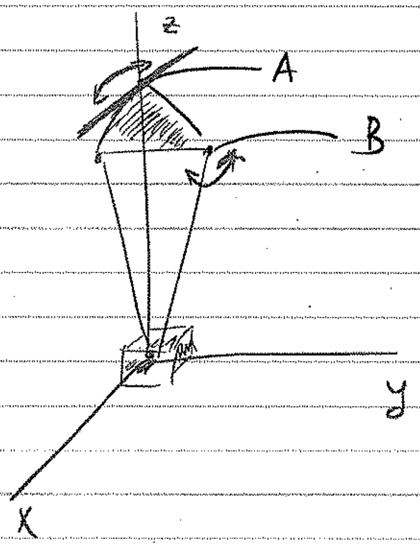
$$M(T=0) = M_+ + M_- = \mu_B \frac{mL^2}{2\pi\hbar^2}(\epsilon_f + \mu_B H) - \mu_B \frac{mL^2}{2\pi\hbar^2}(\epsilon_f - \mu_B H) = \frac{\mu_B^2 mL^2}{\pi\hbar^2} H. \tag{20}$$

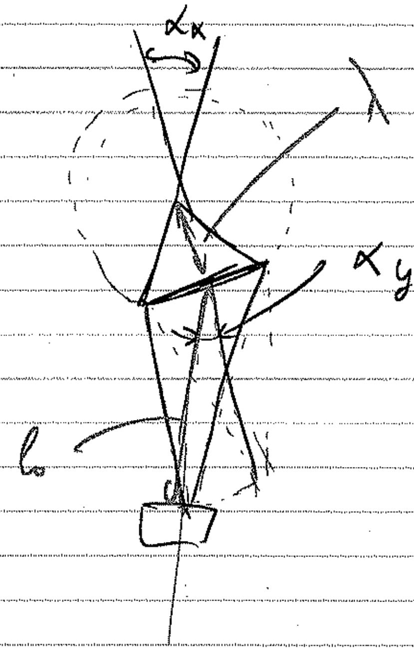
The magnetic susceptibility is the field derivative of the magnetization, $\chi = dM/dH$. Using Eq. (9), one can express the magnetization and the magnetic susceptibility in a simple and elegant form in terms of the Fermi energy:

$$M(T=0) = \frac{\mu_B^2 N}{\epsilon_f} H \quad \text{and} \quad \chi(T=0) = \frac{\mu_B^2 N}{\epsilon_f}. \tag{21}$$

2005 Fall Comp. exam: Mechanics. grad. problem. P2

Note Title 9/7/2005





① Potential energy:

$$U = mg\left(\lambda + l_0 - (\lambda + l_0 \cos\alpha_y)\cos\alpha_x\right)$$

Kinetic energy

$$T = \frac{m(\lambda + l_0 \cdot \cos\alpha_y)^2 \dot{\alpha}_x^2}{2} + \frac{m l_0^2 \dot{\alpha}_y^2}{2}$$

5pts

$$L = T - U = \frac{m(\lambda + l_0 \cos\alpha_y)^2 \dot{\alpha}_x^2}{2} + \frac{m l_0^2 \dot{\alpha}_y^2}{2} - mg\left(\lambda + l_0 - (\lambda + l_0 \cos\alpha_y)\cos\alpha_x\right)$$

(2) Equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0$$

$$\begin{cases} \alpha x: \frac{d}{dt}\left[m(\lambda + l_0 \cos\alpha_y)^2 \dot{\alpha}_x \right] + mg(\lambda + l_0\cos\alpha_y)\cdot \sin\alpha_x = 0 \\ \alpha y: \frac{d}{dt}\left[m l_0^2 \dot{\alpha}_y \right] + mg\, l_0 \cdot \cos\alpha_x \cdot \sin\alpha_y - m(\lambda + l_0\cos\alpha_y) l_0 \dot{\alpha}_x^2 \sin\alpha_y = 0 \end{cases}$$

$$\begin{cases} -2m(\lambda + l_0\cos\alpha_y)\sin\alpha_y \cdot \dot{\alpha}_x \dot{\alpha}_y + m(\lambda + l_0\cos\alpha_y)^2 \ddot{\alpha}_x \\ \qquad + mg(\lambda + l_0\cos\alpha_y)\cdot \sin\alpha_x = 0 \\ m l_0^2 \ddot{\alpha}_y + mg\, l_0 \cos\alpha_x \sin\alpha_y - m(\lambda + l_0\cos\alpha_y) l_0 \dot{\alpha}_x^2 \sin\alpha_y = 0 \end{cases}$$

5pts

$$\begin{cases} (\lambda + l_0\cos\alpha_y)\ddot{\alpha}_x - \sin\alpha_y \cdot \dot{\alpha}_x \dot{\alpha}_y + g\sin\alpha_x = 0 \\ l_0 \ddot{\alpha}_y + g\cos\alpha_x \sin\alpha_y - (\lambda + l_0\cos\alpha_y)\dot{\alpha}_x^2 \sin\alpha_y = 0 \end{cases}$$

(3) Small oscillations: $\alpha_x; \alpha_y \ll 1$

$\Rightarrow \cos\alpha_i \simeq 1 - \frac{\alpha_i^2}{2}$;

3pts $\sin\alpha_i \simeq \alpha_i$;

neglect all terms smaller than $|\alpha_i|$ $(\alpha_i^2, \alpha_i^3, \dot{\alpha}^2, \ldots)$

$$\begin{cases} (\lambda + l_0)\ddot{\alpha}_x + g\alpha_x = 0 \\ l_0 \ddot{\alpha}_y + g\alpha_y \end{cases}$$

From this equation we see that the normal modes are:

$$\rightarrow \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} \alpha_x^0 \cdot \cos(\omega_x t + \varphi_x^0) \\ 0 \end{pmatrix}$$

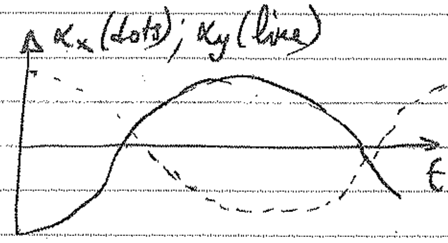
and:

$$\rightarrow \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_y^0 \cos(\omega_y t + \varphi_y^0) \end{pmatrix}$$

$$\boxed{\omega_x = \sqrt{\frac{g}{\lambda + l_0}} \qquad \omega_y = \sqrt{\frac{g}{l_0}}} \qquad \boxed{\omega_y > \omega_x}$$

(4) The motion is given by:

2pts $$\begin{pmatrix} \alpha_x(t) \\ \alpha_y(t) \end{pmatrix} = \begin{pmatrix} \alpha \cos \omega_x t \\ -\alpha \cos \omega_y t \end{pmatrix}$$



(5) The new frequency is:

$$\omega_x(l) = \sqrt{\frac{g}{\lambda + l}} \simeq \sqrt{\frac{g}{\lambda + l_0 + \Delta l}} \simeq \omega_x(l_0)\left[1 - \frac{1}{2}\frac{\Delta l}{\lambda + l_0}\right]$$

2pts Note: the increase in length decreases the frequency

(6) the mass started from $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, its motion is given by:

$$\begin{pmatrix} \alpha_x(t) \\ \alpha_y(t) \end{pmatrix} = \begin{pmatrix} \alpha \cos \omega_x t \\ 0 \end{pmatrix}$$

The energy of this system is given by:

$$W(l) = \frac{mg(\lambda + l_0)}{2} \alpha_{max}^2 = \frac{m(\lambda + l_0)^2 \dot{\alpha}_{max}^2}{2}$$

2pts

If the abrupt change in the length of the system does not change its energy,

$$\alpha_{max}^2(l) = \frac{2W(l)}{mg(\lambda + l)} = \alpha_{max}^2(l_0) \cdot \frac{\lambda + l_0}{\lambda + l}$$

$$\alpha_{max}(l) = \alpha_{max}(l_0) \cdot \sqrt{\frac{\lambda + l_0}{\lambda + l_0 + \Delta l}} \simeq \alpha_{max}(l_0) \left[1 - \frac{\Delta l}{2(\lambda + l_0)} \right]$$

After 10 adjustments:

$$\alpha_{max}(l_0 + 10\Delta l) \simeq \alpha_{max}(l_0) \left[1 - \frac{5\Delta l}{\lambda + l_0} \right]$$

(7) Consider a "cycle" of one length adjustment @ origin by Δl followed by the "counter-adjustment" @ turning point by $(-\Delta l)$

1pt According to the problem, the magnitude is changed only due to first adjustment,

Moreover, $\delta l > 0$ leads to decrease of magnitude

$\delta l < 0 \rightarrow$ increase of magnitude

Qualitatively, the dynamics is similar to "kids" swing — to increase the magnitude of oscillations, kids compress legs @ "minimum" and extend legs @ turning point

Problem 4.3

Consider a system of Fermions with orthonormal single particle states $\phi_n(\vec{r})\chi_n(\sigma)$. The spinors χ_n have only two components.

(A) Write down the two particle wave function with Fermions in states i and j.

$$\psi(\vec{r}_1,\sigma_1,\vec{r}_2,\sigma_2) = \frac{1}{\sqrt{2}}\left(\phi_i(\vec{r}_1)\chi_i(\sigma_1)\phi_j(\vec{r}_2)\chi_j(\sigma_2) - \phi_j(\vec{r}_1)\chi_j(\sigma_1)\phi_i(\vec{r}_2)\chi_i(\sigma_2)\right)$$

(B) Using the two particle wave function, calculate the single particle density.

$$\rho(\vec{r},\sigma) = \int d^3r_2 \sum_{\sigma_2} |\psi(\vec{r},\sigma,\vec{r}_2,\sigma_2)|^2$$

Insert the form for the two particle wave function, and expand in single particle states. These states are orthonormal, and after integration cross terms with i and j become zero. This leaves:

$$\rho(\vec{r},\sigma) = \frac{1}{2}\left(|\phi_i(\vec{r})\chi_i(\sigma)|^2 + |\phi_j(\vec{r})\chi_j(\sigma)|^2\right)$$

(C) What is the probability for two particles to be at the same position, in terms of the single particle states?

$$P(\vec{r},\sigma_1,\sigma_2) = |\psi(\vec{r},\sigma_1,\vec{r},\sigma_2)|^2$$

Insert the expansion again, and collect the spatial terms, which are the same, to get:

$$P(\vec{r},\sigma_1,\sigma_2) = \frac{1}{2}|\phi_i(\vec{r})\phi_j(\vec{r})|^2 \, |\chi_i(\sigma_1)\chi_j(\sigma_2) - \chi_j(\sigma_1)\chi_i(\sigma_2)|^2$$

(D) In case the two particles have a non-zero probability of being at the same position, what is their total spin? A short answer is not sufficient, you need the mathematical proof.

Clearly,if $\chi_i(\sigma) = \chi_j(\sigma)$ (particles have same spin state), this probability is zero. Therefore, the particles must be in opposite spin states. Hence the total spin is zero.

Show this by applying $L_{z1}+L_{z2}$, $L_1^+ + L_2^+$, and $L_1^- + L_2^-$ to $\chi_i(\sigma_1)\chi_j(\sigma_2) - \chi_j(\sigma_1)\chi_i(\sigma_2)$.

Since we now have eigenstates, one of the L_z operators will give plus one half and the other minus on half, so we always get zero for the z-component. Similarly, one of the raising operators will give zero (for the spin up state) and the other will give the up state, and now we have a difference up,up-up,up which is zero. Hence the rasing operator gives zero. Same for the lowering operator. Therefore, L_{tot}^2 will give zero, and the total spin is zero.

Problem 4.

Problem :

System oscillates and radiates

1. Oscillation : $V_L + V_C = 0 \Rightarrow L\frac{dI}{dt} + \frac{Q}{C} = 0 \Rightarrow L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC} \xrightarrow{Q = Q_0 e^{i\omega t}} -\omega^2 Q = -\frac{Q}{LC} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

$$I(t) = I_0 e^{i(\omega t + \beta)} = dQ/dt = i\omega Q = \omega Q e^{i\pi/2} \Rightarrow \beta = \pi/2$$

$$I_0 = \omega Q_0 = \omega V_0 C$$

2. Coil radiates magnetic dipole radiation as $\vec{m} = I_{TOTAL} A \hat{z}$ oscillates (A = area)

Capacitor emits electric dipole radiation as $\vec{p} = Qd\hat{z}$ oscillates

$\vec{m} = I_{TOTAL} A \hat{z}$, where $I_{TOTAL} = NI$ and $A = \pi R^2$, so $\vec{m} = NI\pi R^2 \hat{z}$

From supplied information, $N = \left[\frac{La}{\mu_0 \pi R^2}\right]^{1/2}$

$$\vec{p} = Qd\hat{z} = V_0 C d\hat{z} = V_0 \frac{\epsilon A}{d} d\hat{z} = V_0 \epsilon A \hat{z}$$

E.D. Radiation : $\vec{E}$ is $\hat{z}$ polarized

M.D. radiation : $\vec{E}$ is polarized in xy plane, with phase $\pi/2$

Since the space between the coil and capacitor is $\ll \lambda$,
the total $\vec{E}$ is elliptically polarized.

ED radiation : $\vec{A} = -ik\frac{\mu_0 c}{4\pi} \vec{p}(t) \frac{e^{ikr}}{r}$

$$\vec{B} = \nabla \times \vec{A} = k^2 \frac{\mu_0 c}{4\pi} (\hat{k} \times \vec{p}) \frac{e^{ikr}}{r}$$

$$\vec{E} = c\vec{B} \times \hat{k} = k^2 \frac{\mu_0 c^2}{4\pi} (\hat{k} \times \vec{p}) \times \hat{k} \frac{e^{ikr}}{r}$$

$$\frac{dP}{d\Omega} = \frac{1}{2}(\vec{S} \cdot \hat{k}) r^2 = \frac{k^4 \mu_0 c^2}{32\pi^2} p^2 \sin^2\theta$$ just from $\vec{p}(t)$

$|\hat{k} \times \vec{p}| = p\sin\theta$

M.D. Radiation: The expression for $\frac{dP}{d\Omega}$ will be the same except that p is replaced with m/c

$$\frac{dP}{d\Omega} = \frac{k^4 \mu_0}{32\pi^2} m^2 \sin^2\theta \quad \text{just from } \vec{m}(t)$$

Since the MD and ED fields are orthogonal, the total power per steradian is the sum of these two expressions

$$\frac{dP^{(ED)}/d\Omega}{\frac{dP^{(MD)}}{d\Omega}} = \frac{c^2 p^2}{m^2} = c^2 \frac{V_0^2 C^2}{N^2 I \pi R^2} = c^2 \frac{V_0^2 C^2}{N^2 \omega V_0 C R^2} = \frac{c^2 V_0 C}{N^2 R^2}$$

$$= c^2 \frac{V_0 C}{La} \frac{\mu_0 \pi R^2}{R^2} = \frac{\pi V_0 C}{\epsilon_0 L a}$$

3. For perpendicular coil - capacitor arrangement, there is some destructive interference because the E and B fields are now polarized in the same direction but there is a $\pi/2$ phase difference. So less power will be radiated.

Problem 5 – Solution:

Task I. The classical Hamiltonian of an isotropic 3-D SHO in Cartesian coordinates is:

$$\mathcal{H} = \frac{\kappa}{2}x^2 + \frac{\kappa}{2}y^2 + \frac{\kappa}{2}z^2 + \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_z^2.$$

where κ is the spring constant, and m is the particle mass. It contains six 'quadratic' terms, and, according to the Equipartition Theorem, each contributes a $\frac{1}{2}k_{\rm B}T$ term to the mean thermal energy, and a $\frac{1}{2}k_{\rm B}$ term to the heat capacity. Hence, the mean thermal energy of the entire solid us $\langle U\rangle = 3Nk_{\rm B}T$, and its heat capacity is $3Nk_{\rm B}$. For a mole, N is equal to the Avogadro number $N_{\rm A}$, and since $N_{\rm A}k_{\rm B} = R$, where R is the ideal gas constant, the molar heat capacity of a solid is $c_v = 3R$ (Comment: This is the so-called 'Dulong-Petit's Law'. It works fairly well for most substances at room T and higher temperatures).

One can also find the solution by 'brute force', without evoking the Equipartition Theorem. The partition function for a single oscillator can be rewritten as (note: further on, for convenience, the "B" subscript in the Boltzmann constant will be dropped):

$$\mathcal{Z}_1 = \frac{1}{h^3}\int_{-\infty}^{+\infty} e^{-\kappa x^2/2kT}dx \int_{-\infty}^{+\infty} e^{-\kappa y^2/2kT}dy \int_{-\infty}^{+\infty} e^{-\kappa z^2/2kT}dz$$

$$\times \int_{-\infty}^{+\infty} e^{-p_x^2/2mkT}dp_x \int_{-\infty}^{+\infty} e^{-p_y^2/2mkT}dp_y \int_{-\infty}^{+\infty} e^{-p_z^2/2mkT}dp_z$$

All integrands are Gaussian-like functions. The first three integrals are equal each to:

$$\int_{-\infty}^{+\infty} e^{-\kappa x^2/2kT}dx = \sqrt{\frac{\pi}{\kappa/2kT}} = T^{1/2}\sqrt{\frac{2\pi k}{\kappa}},$$

and the last three equal each to:

$$\int_{-\infty}^{+\infty} e^{-p_x^2/2mkT}dx = \sqrt{\frac{\pi}{1/2mkT}} = T^{1/2}\sqrt{2\pi km},$$

so:

$$\mathcal{Z}_1 = \frac{1}{h^3}T^3\left(\frac{4\pi^2k^2m}{\kappa}\right)^{3/2}$$

Now we use the standard procedure:

$$\mathcal{Z}_{\rm total} = (\mathcal{Z}_1)^N \quad \text{and} \quad \langle U\rangle = kT^2\frac{\partial}{\partial T}\ln(\mathcal{Z}_{\rm total}).$$

All the T-indendent factors disappear upon differentiating the $\ln(\mathcal{Z}_{\rm total})$ function with respect to T, leading to:

$$\langle U\rangle = 3NkT \quad \text{and} \quad C_v^{\rm cl} = \frac{\partial}{\partial T}\langle U\rangle = 3Nk,$$

which are the same results as those obtained before directly from the equipartition theorem.

Task II. The discrete energies of a 3-D SHO are:

$$\epsilon(n_x, n_y, n_z) = \hbar\omega_0\left(\frac{3}{2} + n_x + n_y + n_z\right) \quad \text{with} \quad n_x, n_y, n_z = 0, 1, 2, 3, \ldots$$

The partition sum of a single oscillator is then:

$$\mathcal{Z}_1 = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty} e^{-\epsilon(n_x,n_y,n_z)/kT} = e^{-3\hbar\omega_0/2kT}\left[\sum_{n=1}^{\infty}\left(e^{\hbar\omega_0/kT}\right)^n\right]^3.$$

The expression in the brackets [...] is the sum of a geometric progression, so one can use the well-known formula: $\sum_{n=0}^{\infty} q^n = 1/(1-q)$. Hence:

$$\mathcal{Z}_1 = e^{-3\hbar\omega_0/2kT}\left(\frac{1}{1-e^{-\hbar\omega_0/kT}}\right)^3 = \left(\frac{e^{-\hbar\omega_0/2kT}}{1-e^{-\hbar\omega_0/kT}}\right)^3 \quad \text{and} \quad \mathcal{Z}_{\text{tot}} = (\mathcal{Z}_1)^N = \left(\frac{e^{-\hbar\omega_0/2kT}}{1-e^{-\hbar\omega_0/kT}}\right)^{3N}$$

where $\mathcal{Z}_{\text{tot}}$ is the partition sum for the total system of N oscillators. Now, everything that needs to be done is to perform some tedious but rather straightforward differentiations:

$$\langle U\rangle = kT^2\frac{\partial}{\partial T}\ln(\mathcal{Z}_{\text{tot}}) = 3NkT^2\frac{\partial}{\partial T}\left[-\frac{\hbar\omega_0}{2kT} - \ln\left(1-e^{-\hbar\omega_0/kT}\right)\right]$$

$$= 3NkT^2\frac{\hbar\omega_0}{2kT^2} - 3NkT^2\,\frac{-\frac{\hbar\omega_0}{kT^2}e^{-\hbar\omega_0/kT}}{1-e^{-\hbar\omega_0/kT}} \;=\; 3N\hbar\omega_0\left(\frac{1}{2}+\frac{1}{e^{\hbar\omega_0/kT}-1}\right),$$

and

$$C_v^{\text{E}} = \frac{\partial}{\partial T}\langle U\rangle = 3N\hbar\omega_0\left(-\frac{\hbar\omega_0}{kT^2}\right)\left[\frac{-e^{\hbar\omega_0/kT}}{(e^{\hbar\omega_0/kT}-1)^2}\right] \;=\; 3Nk\left(\frac{\hbar\omega_0}{kT}\right)^2\frac{e^{\hbar\omega_0/kT}}{(e^{\hbar\omega_0/kT}-1)^2}.$$

Task III. To examine the behavior of C_v^{E} at different temperatures, let's introduce $(\hbar\omega_0/kT)\equiv x$. For $T\to 0$, x becomes very large and we can write:

$$C_v^{\text{E}} = \frac{3Nkx^2e^x}{(e^x-1)^2} \to \frac{3Nke^x}{e^{2x}} = 3Nkx^2e^{-x} \to 0.$$

This is consistent with the experimental fact that the heat capacity of solids falls to zero at low temperatures. In contrast, the classical model wrongly predicts that C_v^{E} has a constant value of $3Nk$ at all temperatures.

If $x \ll 1$, we expand the e^x function. But here one has to act with caution. Neglecting x^2 and higher-order terms leads to:

$$C_v^{\text{E}} \approx \frac{3Nkx^2(1+x)}{(1+x-1)^2} = 3Nk(1+x).$$

At the first glance this expression looks OK, because it leads to the proper result $C_v^{\text{E}} = 3Nk$ at $T=\infty$. Actually, it is <u>**incorrect**</u> because it implies that the function reaches the limit *from above* – meaning that for some temperatures the Einstein's model heat capacity *is higher* than the classical value of $3Nk$. This is not true, because the Einstein's C_v^{E} function is lower than 3Nk *for any finite T*. Neglecting the $x^2/2$ term in the expansion is OK only for the numerator, because it is small compared to 1. But in the denominator 1 and -1 cancel out and with respect to x that stays $x^2/2$ is a first-order term, so it cannot be neglected. The proper procedure is therefore:

$$C_v^{\text{E}} \to 3Nk\frac{x^2(1+x)}{(1+x+x^2/2-1)^2} == 3Nk\frac{x^2(1+x)}{x^2(1+x/2)^2} \to 3Nk\frac{(1+x)}{(1+x/2)^2}$$

Here we use the l'Hospital's recipe, and continue:

$$\to 3Nk\frac{1}{1+x} \to 3Nk(1-x) \to 3Nk \quad \text{(from below)}.$$

But a more elegant way and less prone to errors, because one does not need to decide which terms can be neglected and which have to stay, is the following:

$$C_v^{\text{E}} = 3Nk\frac{x^2e^x}{(e^x-1)^2} = 3Nk\left(\frac{x}{e^{-x/2}(e^x-1)}\right)^2 = 3Nk\left(\frac{x}{e^{x/2}-e^{-x/2}}\right)^2$$

Now we expand the exponential functions in the denominator:

$$e^{x/2} - e^{-x/2} = 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} \cdots - 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} \cdots$$

$$= x(1 + x^2/24 + x^4/1920 + \ldots + \frac{x^{n+1}}{2^{n-1}\, n!} + \ldots) \qquad \text{with odd } n \text{ values,}$$

and we obtain:

$$C_v^{\mathrm{E}} = 3Nk \left(\frac{1}{1 + x^2/24 + x^4/1920 + \ldots} \right)^2$$

This is not an approximation, but an exact expression with an infinite sum in the denominator. For $x \to 0$, the limit of this expression is $3Nk$, of course, and since all sum terms are positive, it is obvious that the functions is monotonically increasing with decreasing x and reaches the limit from below.

Task IV: The area of the shaded contour in the figure – let's call it A – is equal:

$$A = \int_0^\infty C_v^{\mathrm{cl}} dT - \int_0^\infty C_v^{\mathrm{E}} dT$$

Since the C_v^{E} function was obtained as the result of differentiation, calculating the second integral is a "non-brainer" – it's just the T-dependent term in the expression for $\langle U \rangle$:

$$A = \left[3NkT - \frac{3N\hbar\omega_0}{e^{\hbar\omega_0/kT} - 1} \right]_{T=0}^{T=\infty} = 3N\hbar\omega_0 \left[\frac{kT}{\hbar\omega_0} - \frac{1}{e^{\hbar\omega_0/kT} - 1} \right]_{T=0}^{T=\infty}$$

It can be immediately seen that the lower limit value of the expression in the brackets is zero, but some more work is needed for finding its $T = \infty$ limit. Again, let's use $x \equiv (\hbar\omega_0/kT)$, and look for the limit value of the $[1/x + 1/(e^x - 1)]$ expression for $x \to 0$:

$$\lim_{x\to 0} \left(\frac{1}{x} + \frac{1}{e^x - 1} \right) = \lim_{x\to 0} \frac{e^x - 1 + x}{(x(e^x - 1)} = \lim_{x\to 0} \frac{e^x + 1}{(xe^x + e^x - 1)} = \lim_{x\to 0} \frac{e^x}{(e^x + xe^x + e^x)} = \lim_{x\to 0} \frac{1}{2 + x} = \frac{1}{2}$$

In the above, we twice used the de l'Hospital's recipe:

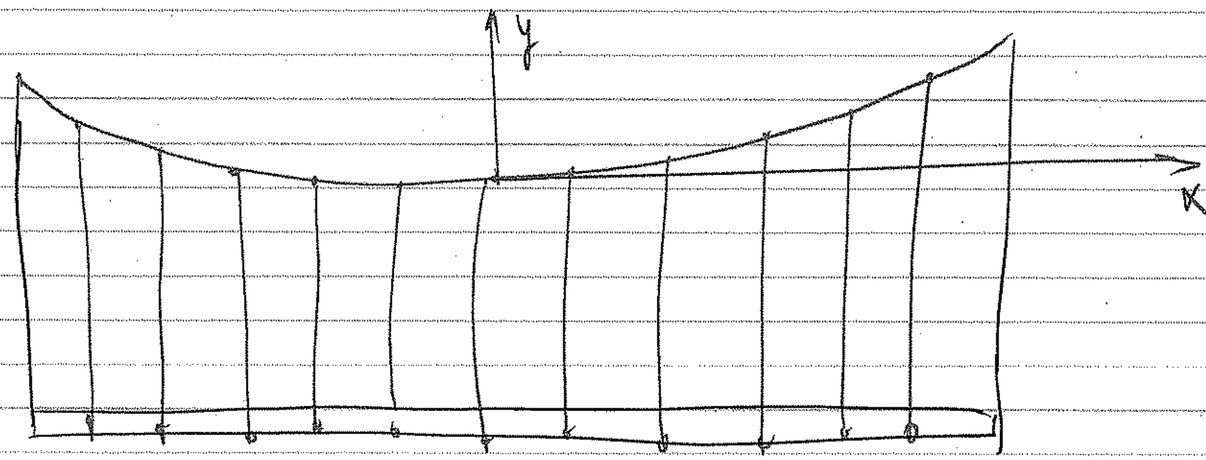
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)}$$

The area of the shaded contour in the figure is then $A = \frac{3}{2} N\hbar\omega_0$, which indeed is the zero-energy of the quantum oscillator system considered.

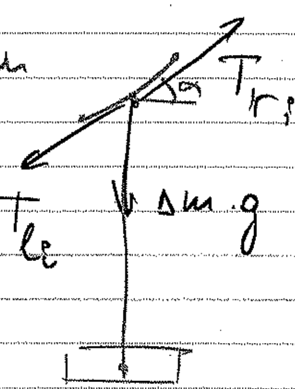
2005 Fall term Comp. Exam – Mechanics UG,

Note Title 8/21/2005

Start with analyzing the forces on the bridge element:



① The force force diagram is shown on the right



$$y: \begin{cases} T_{r_i} \sin\alpha = \Delta m g + T_{l_i} \cdot \sin\alpha \\ T_{r_i} \cos\alpha = T_{l_i} \cdot \cos\alpha \end{cases} :x$$

$$\text{or: } \begin{cases} \Delta T_y = T_{r_i} \sin\alpha - T_{l_i} \sin\alpha = \Delta m g = \rho g \Delta x \\ \Delta T_x = 0 \end{cases}$$

$$\text{thus: } \begin{cases} T_y(x) = \int_0^x \rho g \cdot d\xi = \frac{mg}{2} \cdot x \\ T_x(x) = \text{const} \end{cases}$$

Thus, the suspension @ point $\{x, y_0\}$ supports a portion of a bridge extending $0 < x < x_0$:

$$\text{(Sols)} \begin{cases} T_y(x) = T_0(x) \cdot \sin\alpha = \frac{m(x)}{2} g \\ T_x(x) = T_0(x) \cos\alpha = \text{const} = C \end{cases}$$

(2)

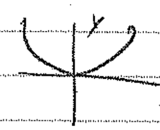
Assume that $y = y(x)$; then:

(5pts) $\tan \alpha = \boxed{y'(x) = \frac{m(x) \cdot g}{2c} = \frac{\rho \cdot x g}{2c}}$

(3)

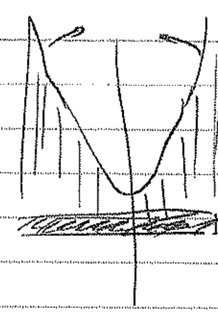
(5pts) $\boxed{y(x) = \frac{\rho g}{c} \cdot \frac{x^2}{4} + C_2}$; from Fig: $\boxed{C_2 = 0}$

The resulting shape is a parabola:

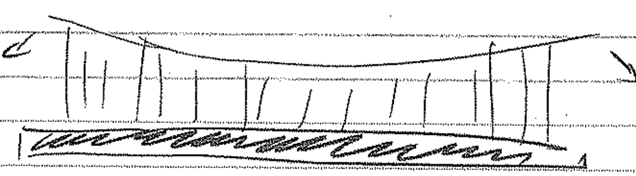


(4) T_x defines the 'curvature' of a parabola:

(5pts) $T_x \to 0$



$T_x \to +\infty$



Problem 7.

Consider a quantum system in one dimension, with a time independent potential $V(x)$. The system is described by a wave function $\psi(x,t)$, which does not have to be an eigenstate. Consider the expectation value of the product of position and momentum for this system, i.e. $< xp > (t)$, as a function of time. The quantum virial theorem relates the time derivative of this quantity, $\frac{d}{dt} < xp > (t)$, to expectation values of the kinetic energy and a term which depends on the potential. Derive such a relation.
Consider a potential $V(x) = V_n x^n$, and assume that the system is in an eigenstate j with energy E_j. Show that in this case the expectation value of the potential is given by $\frac{2}{n+2}E_j$.

$$< xp >= \int \left(\psi^*(x,t) xp \psi(x,t)\right) dx$$

$$\frac{d}{dt} < xp >= \int \left(\dot{\psi}^*(x,t) xp \psi(x,t) + \psi^*(x,t) xp \dot{\psi}(x,t)\right) dx$$

Schrödinger's equation:

$$\dot{\psi} = -\frac{\imath}{\hbar} H \psi$$

Yields:

$$\frac{d}{dt} < xp >= \frac{\imath}{\hbar} \int \left((H\psi)^*(x,t) xp \psi(x,t) - \psi^*(x,t) xpH\psi(x,t)\right) dx$$

or

$$\frac{d}{dt} < xp >= \frac{\imath}{\hbar} \int \psi^*(x,t) \left(Hxp - xpH\right) \psi(x,t) dx$$

Commutators, using $p = \frac{\hbar}{\imath}\frac{d}{dx}$:

$$[H, x] = \frac{\hbar}{\imath}\frac{p}{m}$$

$$[H, p] = -\frac{\hbar}{\imath}\frac{dV}{dx}$$

$$[H, xp] = [H, x]p + x[H, p] = \frac{p^2}{m} - \frac{\hbar}{\imath} x \frac{dV}{dx}$$

and hence

$$\frac{d}{dt} < xp >=< \frac{p^2}{m} > - < x\frac{dV}{dx} >$$

is our requested relation.

If the system is in an eigenstate we have $\psi(x,t) = \phi_j(x)e^{-i\frac{E_j}{\hbar}t}$ and $< xp >$ becomes time independent. If $V(x) = V_n x^n$ we have $x\frac{dV}{dx} = nV$, and hence we get for the expectation values $V_j =< V >_j$ and $T_j =< T >_j$

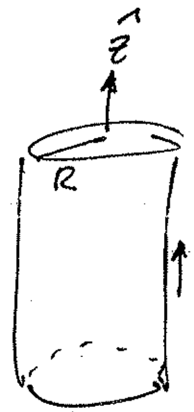
$$0 = 2T_j - nV_j$$

But, of course, $E_j = T_j + V_j$ and hence

$$V_j = \frac{2}{n+2}E_j$$

Problem 8

Picture

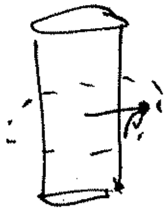


cylinder (hollow) of radius R

$\vec{I} = I\hat{z}$ is the total current.

$$\vec{J} = \frac{I}{2\pi R}\,\delta(\rho - R)\,\hat{z}$$

a) $\vec{B}$ outside: $\int \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.}$ and $\vec{B} = B\hat{\varphi}$ only by symmetry.

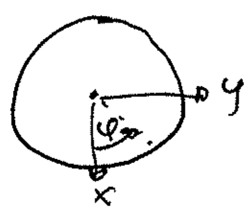


$$\int_0^{2\pi} B(\rho)\,\rho\, d\varphi = B(\rho)2\pi\rho = \mu_0 I$$

$$\vec{B}(\rho > R) = \frac{\mu_0 I}{2\pi\rho}\,\hat{\varphi}$$

b) $\vec{F} = q\vec{v}\times\vec{B} \Rightarrow \vec{F} = \int \vec{J}\times\vec{B}\,dV = \int \frac{I}{2\pi R}\,\delta(\rho - R)\frac{\mu_0 I}{2\pi\rho}\,(\hat{z}\times\hat{\varphi})\,\rho\, d\rho\, d\varphi\, dz$

so $\vec{F} = \frac{\mu_0 I^2}{4\pi^2 R} L \int \hat{z}\times\hat{\varphi}\,d\varphi$, where L is a length in z direction



The total $F_x = \frac{\mu_0 I^2 L}{4\pi^2 R}\int_{-\pi/2}^{\pi/2} -\cos\varphi\, d\varphi$

$$F_x = -\frac{2\mu_0 I^2 L}{4\pi^2 R}$$

Inward pressure $= \frac{|F_x|}{\text{area}} = \frac{|F_x|}{\pi R L} = \frac{\mu_0 I^2}{2\pi^2 R^2}$

c) Pressure increases as $\frac{1}{R^2}$ as tube collapses