

OSU PHYSICS DEPARTMENT  
COMPREHENSIVE EXAMINATION #98

March 28 and 29, 2005

Comprehensive examination for Spring 2005

PART I, Monday March 28, 9:00 am

General Instructions

This Comprehensive Examination for Spring 2005 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 28, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 29, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

### Problem 1.

For a simple quantum harmonic oscillator with a Hamiltonian:

$$\mathcal{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}Kx^2 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}\omega^2 mx^2,$$

the allowed energies are  $E_n = (n + \frac{1}{2})\hbar\omega$  ( $n = 0, 1, 2, \dots$ ). Now suppose the spring constant increases slightly:  $K \rightarrow (1 + \epsilon)K$  (perhaps we cool the spring, so it becomes more rigid).

- Find the *exact* new energies. Expand your formula as a power series in  $\epsilon$ , up to second order.
- Now calculate the first-order perturbation in energy. What is the perturbation Hamiltonian  $\mathcal{H}'$  here? Compare your result with part (a). *Hint:* It is not necessary to calculate a single integral for obtaining the solution in this part, for an arbitrary  $n$  value. But if you do solve this part by calculating integrals, such a solution will also be accepted (all "tools" you will need for that, i.e., equation for normalized SHO wavefunctions, and some integrals, are provided below; calculate the energy perturbation for several  $n$  values and try to find a "pattern" and then write the solution for an arbitrary  $n$  value).
- Calculate the second-order perturbation in energy for the  $n = 2$  state, and again compare the result with that from Part (a). *Hint:* Now you have to calculate integrals. You are allowed to accept without proof that for SHO states  $\psi_n, \psi_m$  with  $|n - m| > 2$ , all  $\langle \psi_n | x^2 | \psi_m \rangle$  matrix elements equal zero.

Formulae you may need:

$$\psi_n(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x\sqrt{\alpha}) e^{-\alpha x^2} \quad \text{where } \alpha = \frac{m\omega}{\hbar}$$

The first few  $H_n(\xi)$  polynomials:

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$H_0 = 1,$	$H_1 = 2\xi,$	$H_2 = 4\xi^2 - 2,$
$H_3 = 8\xi^3 - 12\xi,$	$H_4 = 16\xi^4 - 48\xi^2 + 12,$	$H_5 = 32\xi^5 - 160\xi^3 + 120\xi.$

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$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^{2n} n!} \sqrt{\frac{\pi}{\alpha^{2n+1}}}$$

## Problem 2.

A small car went off the road and a small crane was called up to rescue the car. However, the rescue was not very successful (see Fig.1, next page). The interesting fact about this double-accident is that the crane was able to lift the car on the substantial height before two of them turned over. This means that there should be some mechanism other than the Archimedes buoyant force for such an overturn. The set-up below describe a possible model yielding the observed results.

### The "lever effect".

Assume that both the car and the crane are cubes with two rows of wheels, and that the "lever" of the crane is weightless and is attached to the middle of a platform (see Fig.2, next page). Assume further that the crane has a string of a fixed length and lifts the car only via "lever action". As you can see, the effective torque, applied by the car on the crane changes as the angle  $\alpha$  is changed. Given the width of the crane  $W$ , height of the crane  $H$ , mass of the crane  $M$ , mass of the car  $m$ , and "effective" length of the crane lever  $L$ :

- (i) write down the torque equation
- (ii) find the weight imposed onto the surface by the right (farthest from the car) row of crane's wheels
- (iii) find the critical angle  $\alpha_0$  when the crane tips over
- (iv) find the "stability condition" - the combination of  $L$ ,  $W$ ,  $m$ ,  $H$ , and  $M$  so that the crane would be able to safely lift the car.
- (v) Find the maximum lever length of a crane lifting the car of its own weight
- (vi) What is the dependence of crane stability on a crane height  $H$ ? The "height" of SUVs is typically mentioned as main concern for their stability on the road. Explain the agreement/disagreement of the criterion derived in (v) and the above concern.

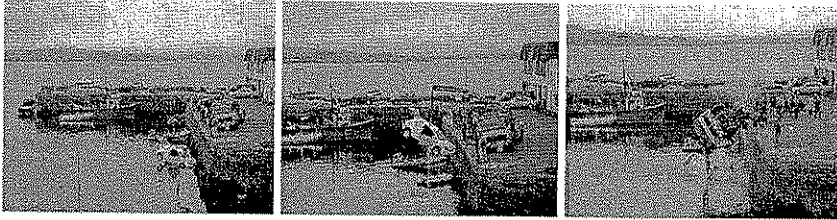


Figure 1: The accident

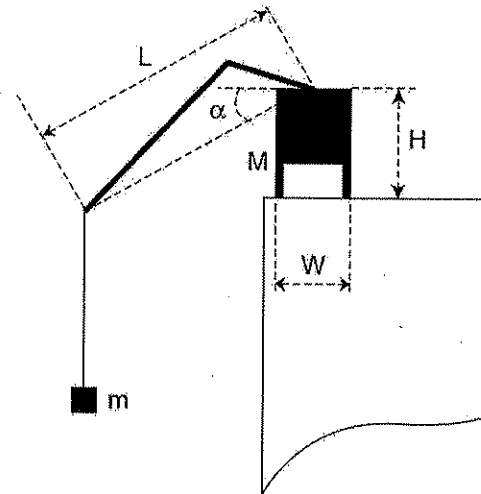


Figure 2: Schematics of the crane

### Problem 3.

An electronic circuit board consists of thin conductive traces on one side of a dielectric board and a ground plane on the other. To determine the impedance of a single path on such a board one must know the capacitance and inductance per unit length. It is reasonable to begin by assuming that the wire is very thin compared to the thickness of the dielectric.

1. Determine the capacitance per unit length of a system consisting of a very long thin wire oriented parallel to the conducting ground plane and lying on top of a material of thickness  $a$ , permittivity  $\epsilon$  and permeability  $\mu_0$ .
2. Consider a time-dependent current applied to the wire and determine the inductance of the same structure.

### Problem 4.

An air conditioning system takes in air at 1 atm, 40°C, and 70 percent relative humidity. It delivers air at 20°C and 50 percent relative humidity. The air flows first over cooling coils, where it is cooled to a low temperature  $T_l$  and dehumidified. The liquid is taken out at a temperature  $T_l$ . Next, the air flows over a heating element where it is heated to the desired temperature.

- (I) What is the temperature  $T_l$  of the air when it exits the cooler?
- (II) Find the amount of heat removed in the cooling section.
- (III) Find the amount of heat transferred in the heating section.

Express your results for parts (II) and (III) in kJ/kg dry air. Assume dry air has  $C_p = 1$  kJ/(kg C). Treat both dry air and water vapor as an ideal gas, with gas constant  $R_a = 0.287$  kJ/(kg C) for dry air and  $R_v = 0.461$  kJ/(kg C) for water vapor. Note that 1 atm is about 100 kPa.

Use data from the following table for saturated air:

$T$ (C)	$P_{sat}$ (kPa)	$h_g$ (kJ/kg)
10	1.23	2520
20	2.34	2538
30	4.25	2556
40	7.38	2574
50	12.35	2592

where the last column is the enthalpy for the saturated water vapor.

### Problem 5.

Part a. Consider a pair of two identical spins  $S_1 = S_2$  coupled by the Heisenberg exchange interaction. The interaction Hamiltonian in zero magnetic field is:  $\mathcal{H} = -2J_{12}\hat{S}_1 \cdot \hat{S}_2$ , where  $J_{12}$  is the exchange interaction constant. Let's denote  $\hat{S}_T = \hat{S}_1 + \hat{S}_2$ , and the total spin number as  $S_T$ . What values can  $S_T$  take? Show that  $S_T$  is a good quantum number, so that the eigenvalues of the Hamiltonian can be expressed as  $E(S_T)$ . Find the equation for  $E(S_T)$ . Assuming that  $S_1 = S_2 = \frac{3}{2}$  and  $J < 0$  (antiferromagnetic coupling), sketch the excitation level scheme and label each level with its energy value, taking the lowest energy level (ground state) as zero. Also, show the degeneracy of each level on the graph.

Part b. Now consider a linear chain of *three* identical spins  $S_1 = S_2 = S_3$  (with  $S_2$  in the middle). The spin pairs  $S_1, S_2$  and  $S_2, S_3$  are exchange-coupled, with the interaction constants  $J_{12}$ , and  $J_{23}$ , respectively. There is no interaction between the spins at the chain ends ( $J_{13} = 0$ ). Write the Hamiltonian for this chain. Again, we denote the total spin number as  $S_T$ , and now  $\hat{S}_T = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$ . However, in the present situation of three spins  $S_T$  cannot unambiguously express the orientation of individual spins  $\hat{S}_i$ . Therefore, it is necessary to introduce an additional spin quantum number  $S_{13}$  defined by  $\hat{S}_{13} = \hat{S}_1 + \hat{S}_3$ . Are  $S_T$  and  $S_{13}$  good quantum numbers? For  $S_1 = S_2 = S_3 \equiv S = \frac{3}{2}$ , what values can  $S_T$  and  $S_{13}$  take? List all possible states  $|S_T, S_{13}\rangle$  (a table would probably be the best form). Note that not all combinations of the two quantum numbers are possible! Derive the equation for the energy eigenvalues  $E(S_T, S_{13})$  of a "symmetric" chain (i.e., for  $J_{12} = J_{23} = J$ ). Compute the energy value (in the units of  $J$ ) for each state you have listed and write it in the same table box. Determine which state is the ground state (for an antiferromagnetic chain, i.e.,  $J < 0$ ) and sketch a scheme of the first few excitation levels (there are too many levels to plot them all).

Part c. A slightly more complicated situation is a spin triad usually called "an open triangle": in such a system, there is an additional interaction between the spins  $S_1$  and  $S_3$ , of different strength than between the two other pairs. Denote the interaction constant for these two spins as  $J_{13} \equiv J'$ . Write the interaction Hamiltonian, and derive the expression for the open triangle energy eigenvalues  $E(S_T, S_{13})$ .

Part d. Finally, based on the (c) result, find the expression for energy eigenvalues of a system usually referred to as a "closed triangle", i.e., a spin trio with all interaction constants equal ( $J_{12} = J_{23} = J_{13} \equiv J$ ). Is it possible to use a single quantum number now? Again, draw the excitation level scheme.

### Problem 6.

We consider the same accident as yesterday. A small car went off the road and a small crane was called up to rescue the car. However, the rescue was not very successful (see Fig.1 next page ).

#### The "pendulum effect"

Assume now that a crane works by fixing its lever in some position and pulling the string (Fig.2, next page). The lever change is not an issue now. However, if the initial position of the car had some offset with respect to the crane, the will start to oscillate and the crane may tip over. Here you should formally consider this phenomenon.

- (i) Assume that the car of a mass  $m$  on a string with length  $l$  represents a pendulum; assuming the motion in both angular coordinate  $\phi$  and radial coordinate  $r$  is possible, and using the equation  $r=l$  as a constraint, write down Lagrangian for this pendulum. Derive the Lagrange equations of motion.
- (ii) Calculate the frequency and energy of oscillations
- (iii) Use the constrained Lagrangian to calculate the force exerted on a string when the car is at bottom-most position; relate this force to centripetal force and to the energy of oscillations
- (iv) Assume that the length of a string is changed slowly (with respect to period of oscillations). Use the adiabatic invariant  $J = \oint p d\phi = \text{const}$  to show that the product of the period of the oscillations and the energy remains constant
- (v) Use the results of (iii)-(iv) to calculate the force exerted on the string as a function of a string length  $l$ , initial string length  $l_0$ , and initial displacement  $d$ .
- (vi) Find the relation between the  $M$ ,  $m$ ,  $L$ ,  $l_0$ ,  $\alpha$ ,  $h$ , and  $d$  under which the crane in configuration shown in Fig.2 would be able to lift the car. Assume that the car exerts a maximum torque when it is in bottom-most position.

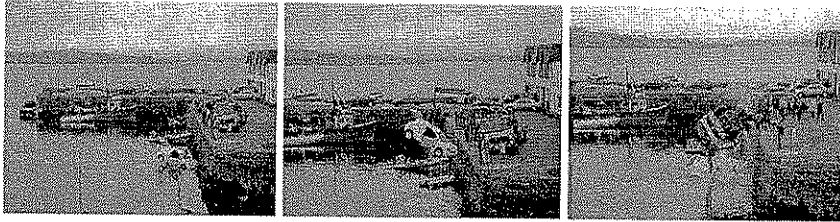


Figure 1: The accident

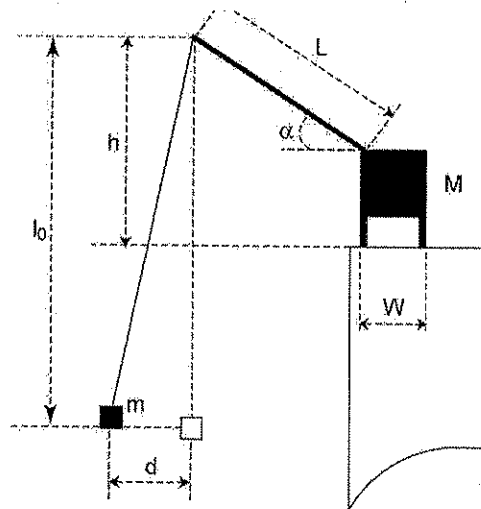


Figure 2: Schematics of the crane

Problem 7.

Determine the electrostatic potential inside a hollow metal wedge when the potential is  $V_0$  on the top surface and zero on the other four sides. The wedge is a slice of a cylinder of radius  $R$  and length  $L$ , and its angular width is  $\alpha = \pi/2$ . Useful relations:

$$\int_0^{\pi/2} \sin m' \phi \sin m \phi d\phi = \frac{\sin((m' - m)\pi/2)}{2(m' - m)} - \frac{\sin((m' + m)\pi/2)}{2(m' + m)}$$

$$\int_0^R J_m(k_{mn'} \rho) J_m(k_{mn} \rho) \rho d\rho = \frac{R^2}{2} J_{m+1}^2(k_{mn} R) \delta_{nn'}$$

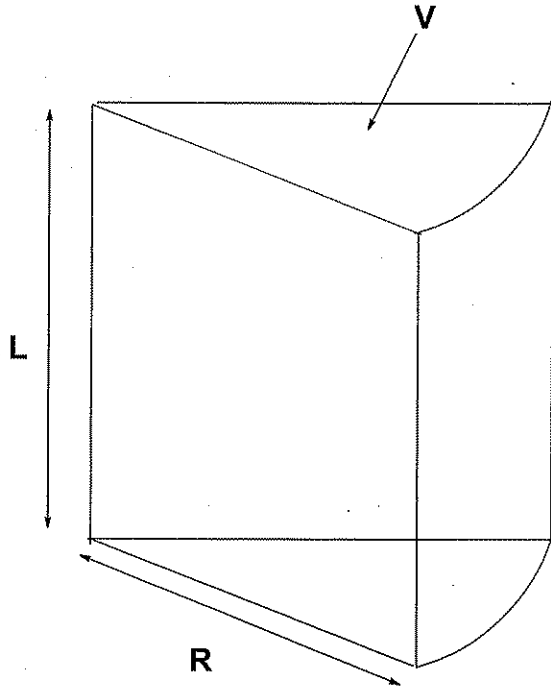


Figure 1: The Wedge

### Problem 8.

Consider the following situation. We put  $N$  molecules H-D (hydrogen-deuterium) in a bottle, and keep the bottle at temperature  $T$ . The atoms collide and can exchange components, so after a while we are in equilibrium and have  $n_{eq}$  molecules H-D, with the rest H-H and D-D. You are asked to calculate  $n_{eq}$ , the number of molecules H-D, as a function of temperature when equilibrium is established. You are allowed to make the following assumptions. First, the temperature is only around room temperature, which means that all translational and rotational degrees of freedom can be treated classically. It also means that the molecules are always in the vibrational ground state. Second, the electronic bonds do not depend on the mass of the nucleus, hence the spring constants are the same for all three molecules. Assume that the mass of an H-atom is  $m_H$  and of a D-atom  $m_D$ . Finally, the Helmholtz free energy for an ideal classical mono-atomic gas is

$$F_{ideal} = -Nk_B T \left( \log\left(\frac{n}{n_Q(T)}\right) - 1 \right) \quad (1)$$

with

$$n_Q(T) = \left( \frac{Mk_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} \quad (2)$$

$$K \rightarrow K(1+\epsilon) \quad \text{or} \quad \frac{1}{2}m\omega^2 \rightarrow \frac{1}{2}m\omega^2(1+\epsilon) = \text{SOL. 1}$$

So:  $\rightarrow \frac{1}{2}m(\omega')^2$  with  $\omega' = \omega\sqrt{1+\epsilon}$  Perturbation potential is simply  $V' = V \cdot \epsilon$

$$E' = \hbar\omega' \left(\frac{1}{2} + n\right) = \hbar\omega \left(\frac{1}{2} + n\right) \sqrt{1+\epsilon}$$

$$\sqrt{1+\epsilon} = 1 + \left(\frac{1}{2\sqrt{1+\epsilon}}\right)\epsilon \approx \frac{1}{2} \left(\frac{1}{2 \cdot 2\sqrt{1+\epsilon}^3}\right)\epsilon^2 + \dots$$

$$\approx 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \dots$$

Exact solution, expanded in  $\epsilon$ :

$$E' \approx \hbar\omega \left(\frac{1}{2} + n\right) \left(1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \dots\right)$$

First-order perturbation (the way we expect students to do that)

$$\Delta E'_n = \langle \psi_n | V' | \psi_n \rangle = \langle \psi_n | V \cdot \epsilon | \psi_n \rangle$$

$$= \epsilon \langle \psi_n | V | \psi_n \rangle$$

This is ~~the~~ simply  $\langle V \rangle$ , and student should know that for SHO  $\langle V \rangle = \langle V \rangle = \frac{1}{2} \hbar\omega \left(n + \frac{1}{2}\right)$

$$\text{So, } \Delta E'_n = \frac{\epsilon}{2} \cdot \hbar\omega \left(n + \frac{1}{2}\right)$$

Total energy:  $E' = \hbar\omega \left(\frac{1}{2} + n\right) \left(1 + \frac{\epsilon}{2}\right)$  agrees to the 1st order.

It can also be done by integrating, or by the algebraic method, using the  $\hat{a}, \hat{a}^\dagger$  operators

We first do integrating (including the second order correction) then

By integrating:

Sol. #2

$$\Delta E'_n = \langle \psi_n | V' | \psi_n \rangle = \frac{E}{2} \omega^2 m \langle \psi_n | x^2 | \psi_n \rangle$$

$$\langle \psi_0 | x^2 | \psi_0 \rangle = \int_{-\infty}^{\infty} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\alpha}}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \frac{\sqrt{\pi}}{\alpha^{\frac{3}{2}}} = \frac{1}{2\alpha}$$

$$\begin{aligned} \Delta E'_1 &= \langle \psi_1 | x^2 | \psi_1 \rangle = \int \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{(\sqrt{2})^2} \cdot x^2 \cdot 4\alpha x^2 e^{-\alpha x^2} dx \\ &= \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \frac{\alpha}{2} \cdot 4 \int x^4 e^{-\alpha x^2} dx = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{\alpha}{2} \cdot 4 \cdot \frac{3}{4} \cdot \frac{\sqrt{\pi}}{\sqrt{\alpha^5}} = \frac{3}{2\alpha} \end{aligned}$$

$$\begin{aligned} \langle \psi_2 | x^2 | \psi_2 \rangle &= \int \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \frac{1}{(\sqrt{8})^2} \cdot 4(2\alpha x^2 - 1)^2 e^{-\alpha x^2} dx \\ &= \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \frac{1}{8} \cdot 4 \int (4\alpha^2 x^4 - 4\alpha x^2 + 1) e^{-\alpha x^2} dx \\ &= \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \frac{1}{2} \int (4\alpha^2 x^4 - 4\alpha x^2 + 1) e^{-\alpha x^2} dx \\ &= \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{2} \left( 4\alpha^2 \frac{15}{8} \frac{\sqrt{\pi}}{\sqrt{\alpha^7}} - 4\alpha \frac{3}{4} \frac{\sqrt{\pi}}{\sqrt{\alpha^5}} + \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\alpha^3}} \right) = \frac{1}{2} \left( \frac{60}{8\alpha} - \frac{24}{8\alpha} + \frac{4}{8\alpha} \right) \\ &= \frac{1}{2} \frac{40}{8\alpha} = \frac{5}{2\alpha} \end{aligned}$$

$$\Delta E'_0 = \frac{E}{2} \omega^2 m \cdot \frac{1}{2} \frac{m\omega}{E} = \frac{E}{4} \hbar\omega$$

$$\Delta E'_1 = \frac{3E}{4} \hbar\omega$$

$$\Delta E'_2 = \frac{5E}{4} \hbar\omega$$

$$\Delta E'_n = \frac{2n+1}{4} E \hbar\omega =$$

$$= \frac{E}{2} \hbar\omega \left(n + \frac{1}{2}\right) \text{ Same as by the "easy way"}$$

Second-order correction  $\Delta E_2''$  by integration

$$\Delta E_2'' = \frac{|\langle \psi_0^0 | V' | \psi_2^0 \rangle|^2}{E_2^0 - E_0^0} + \frac{|\langle \psi_4^0 | V' | \psi_2^0 \rangle|^2}{E_2^0 - E_4^0}$$

$$E_2^0 - E_0^0 = 2\hbar\omega; \quad E_2^0 - E_4^0 = -2\hbar\omega$$

$$\begin{aligned} \langle \psi_0^0 | V' | \psi_2^0 \rangle &= \langle \psi_0^0 | \frac{\delta}{2} m \omega^2 x^2 | \psi_2^0 \rangle = \\ &= \frac{\delta}{2} m \omega^2 \langle \psi_0^0 | x^2 | \psi_2^0 \rangle = \frac{\delta}{2} m \omega^2 \cdot \frac{\sqrt{2}}{2\alpha} \end{aligned}$$

$$\langle \psi_0^0 | x^2 | \psi_2^0 \rangle = \int_{-\infty}^{+\infty} \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{1}{\sqrt{2}} x^2 (2\alpha x^2 - 1) e^{-\frac{\alpha x^2}{2}} dx$$

$$= \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{1}{\sqrt{2}} \left[ 2 \int_{-\infty}^{+\infty} x^4 e^{-\alpha x^2} dx - \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx \right] =$$

$$= \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{1}{\sqrt{2}} \left[ 2\alpha \frac{3}{4} \frac{\sqrt{\pi}}{\alpha^{3/2}} - \frac{1}{2} \frac{\sqrt{\pi}}{\alpha^{3/2}} \right] = \frac{\sqrt{\alpha}}{\sqrt{2}} \left[ \frac{3}{2} \frac{1}{\alpha^{3/2}} - \frac{1}{2} \frac{1}{\alpha^{3/2}} \right] =$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{1}{\alpha} \right] = \frac{\sqrt{2}}{2\alpha}$$

$$\langle \psi_2 | V | \psi_4 \rangle = \langle \psi_2 | \frac{1}{2} m \omega^2 x^2 | \psi_4 \rangle$$

$$= \frac{\hbar}{2} m \omega^2 \langle \psi_2 | x^2 | \psi_4 \rangle = \frac{\hbar}{2} m \omega^2 \frac{\sqrt{12}}{2} \frac{1}{\alpha} \leftarrow$$

$$\langle \psi_2 | x^2 | \psi_4 \rangle = \int \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{2}{2\sqrt{2}} x^2 (2\alpha x - 1) \cdot \frac{4}{4\sqrt{24}} (16x^4 - 48x^2 + 12) e^{-\alpha x^2} dx$$

$$x^2 (2\alpha x - 1) (4x^4 - 12x^2 + 3) =$$

$$= x^2 (8\alpha^3 x^8 - 24\alpha^2 x^6 + 6\alpha x^4 - 4x^4 + 12x^2 - 3)$$

$$= 8\alpha^3 x^8 - 28\alpha^2 x^6 + 18\alpha x^4 - 3x^2$$

$$= \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{24}} \int (8\alpha^3 x^8 - 28\alpha^2 x^6 + 18\alpha x^4 - 3x^2) dx$$

$$= \frac{\sqrt{\alpha}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{24}} \cdot \frac{24}{2} \frac{\sqrt{\pi}}{\alpha^3} = \frac{\sqrt{24}}{2\sqrt{2}} \frac{1}{\alpha} = \frac{2\sqrt{6}}{2\sqrt{2}} \frac{1}{\alpha} = \frac{\sqrt{12}}{2} \frac{1}{\alpha}$$

$$\Delta E_2 = \frac{\hbar^2}{4} m \omega^2 \left( \frac{2}{4\alpha^2} - \frac{12}{4\alpha^3} \right)$$

$$= \frac{5}{4} \hbar^2 m \omega^4 \frac{1}{8\hbar \omega \alpha^2}$$

$$= \frac{-5}{16} \frac{\hbar^2 m \omega^4}{\hbar \omega \alpha^2} = -\frac{5}{16} \hbar \omega = -\left(2 + \frac{1}{2}\right) \hbar \omega \cdot \frac{\hbar^2}{8}$$

Same as from expansion!!

Operator method:

An alternative method of calculating the matrix elements needed for obtaining the perturbed energies is by using the creation/annihilation operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p}_x \right); \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i}{m\omega} \hat{p}_x \right)$$

Then  $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$

short-hand notation  
↓ ↓

$$\Delta E'_n = \langle \psi_n | V' | \psi_n \rangle = \frac{1}{2} m \omega^2 \langle \psi_n | \hat{x}^2 | \psi_n \rangle = \frac{1}{2} m \omega^2 \langle n | \hat{x}^2 | n \rangle$$

$$\langle n | \hat{x}^2 | n \rangle = \frac{\hbar}{2ma} \langle n | \hat{a}^2 + \hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}^{\dagger 2} | n \rangle$$

$$\hat{a} | n \rangle = \sqrt{n} | n-1 \rangle; \quad \hat{a}^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle$$

so that  $\langle n | \hat{a} \hat{a}^\dagger | n \rangle = (\sqrt{n+1})^2 \langle n | n \rangle = n+1$

$$\langle n | \hat{a}^\dagger \hat{a} | n \rangle = (\sqrt{n})^2 \langle n | n \rangle = n$$

Hence:  $\Delta E'_n = \frac{\hbar\omega E}{4} (n+1+n) = \hbar\omega (2n+1) \cdot \frac{E}{2}$

Agrees with the first term of exact solution!

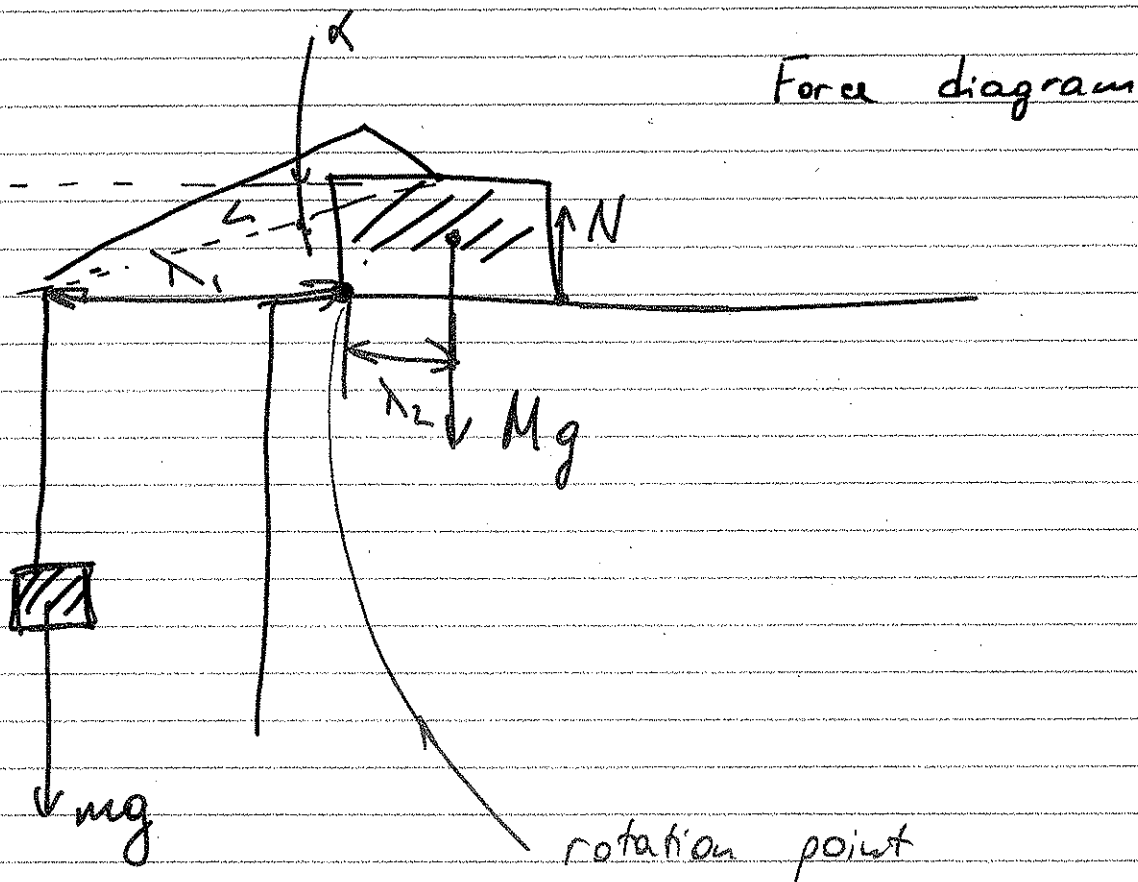
$$\Delta E''_2 = \frac{|\langle 2 | V' | 0 \rangle|^2}{E_2 - E_0} + \frac{|\langle 2 | V' | 4 \rangle|^2}{E_2 - E_4}$$

(it's evident that  $\langle 2 | V' | 1 \rangle = 0$  and  $\langle 2 | V' | 3 \rangle = 0$ )

Using the same method as above, we get that

$$\Delta E''_2 = -\frac{1}{8} E^2 \cdot \hbar\omega \cdot \frac{5}{2} = -\frac{1}{8} E^2 \hbar\omega \left( 2 + \frac{1}{2} \right)$$

so it agrees with the second-order term in the exact solution expansion.



①

$$\lambda_2 = w/2$$

$$\lambda_1 = L \cos \alpha - w/2$$

③

$$mg \left( L \cos \alpha - \frac{w}{2} \right) + N \cdot w = Mg \cdot \frac{w}{2}$$

②

$$N = \frac{Mg}{2} - mg \left( \frac{L}{w} \cos \alpha - \frac{1}{2} \right)$$

②

$$N = g \left[ \frac{M}{2} + m \left( \frac{1}{2} - \frac{L}{w} \cos \alpha \right) \right]$$

3) if such an angle exists, it would be given by:

$$N = 0$$

$$\frac{M}{2} = m \left( \frac{L}{w} \cos \alpha_0 - \frac{1}{2} \right)$$

$$\frac{L}{w} \cos \alpha_0 = \frac{1}{2} \left( \frac{M}{m} + 1 \right)$$

$$\alpha_0 = \cos^{-1} \left[ \frac{w}{L} \left( \frac{M+m}{2m} \right) \right]$$

4) the stability requirement is:

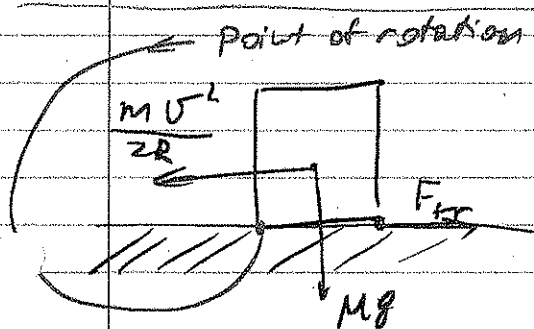
$$(\cos \alpha_0 > 1) \quad \text{or} \quad \left[ N(\alpha = \frac{\pi}{2}) > 0 \right]$$

$$\frac{w}{L} \cdot \frac{M+m}{2m} > 1 \quad \text{or} \quad \frac{M+m}{2m} > \frac{L}{w}$$

5) if  $M = m$ , we obtain:

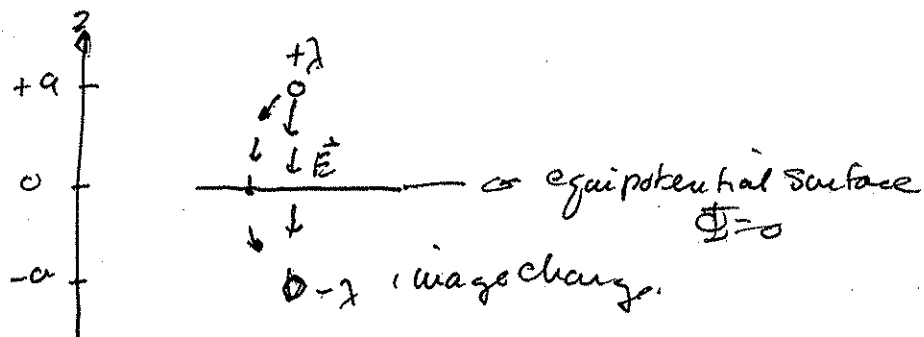
$$\frac{L}{w} < 1; \quad \text{or} \quad L < w$$

6) As we see, the height does not affect the stability of a crane. The situation with stability of an SUV is quite different (see force diagram below). Namely, the torque of the centrifugal force (in SUV ref. frame) is proportional to height. Hence, the higher the SUV, the more likely it will tip in the turn.



### PROBLEM 3

3a. Concept



$$\lambda = C (\Phi(z=a) - \Phi(z=0)) \quad C \equiv \text{capacitance/unit length}$$

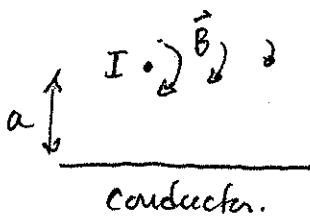
From Gauss' law,  $\Phi = \frac{-2\lambda \ln(\rho/\rho_0)}{4\pi\epsilon_0}$  in general, with  $\rho_0 = \text{radius of wire}$

So,  $\Phi$  at  $z = a - b$ , where  $b = \text{radius of wire}$  is

$$4\pi\epsilon_0 \Phi_{\text{wire}} = -2\lambda \ln 1 + 2\lambda \ln\left(\frac{2a-b}{b}\right)$$

$$\text{So } C = \frac{4\pi\epsilon_0}{2 \ln\left(\frac{2a-b}{b}\right)}$$

3b. Concept



From this current,

$$B_\phi = \frac{\mu_0 I}{2\pi\rho}$$

Basic definition of inductance is energy  $U = \frac{1}{2} LI^2 = \frac{1}{2\mu_0} \int B^2 dV$

At low frequency,  $B$  extends everywhere so the energy per unit length is  $\frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi}\right)^2 2\pi \int_{\rho_0}^{\infty} \frac{\rho d\rho}{\rho^2}$  where  $\rho_0 = \text{radius of wire}$

So,  $L \approx \frac{\mu_0}{2\pi} \ln 1/\rho_0$  is the inductance per unit length

At high frequencies,  $B$  does not extend into the conductor, so the integral over  $\rho$  terminates at distance  $a$ .

Then

$L \approx \frac{\mu_0}{2\pi} \ln a/\rho_0$  is an approximation since the field in the vicinity of the conductor is difficult to describe.

## Problem 4

An air conditioning system takes in air at 1 atm, 40°C, and 70 percent relative humidity. It delivers air at 20°C and 50 percent relative humidity. The air flows first over cooling coils, where it is cooled to a low temperature  $T_l$  and dehumidified. The liquid is taken out at a temperature  $T_l$ . Next, the air flows over a heating element where it is heated to the desired temperature.

- (I) What is the temperature  $T_l$  of the air when it exits the cooler?
- (II) Find the amount of heat removed in the cooling section.
- (III) Find the amount of heat transferred in the heating section.

Express your results for parts (II) and (III) in kJ/kg dry air. Assume dry air has  $C_p = 1$  kJ/(kg C). Treat both dry air and water vapor as an ideal gas, with gas constant  $R_a = 0.287$  kJ/(kg C) for dry air and  $R_v = 0.461$  kJ/(kg C) for water vapor. Note that 1 atm is about 100 kPa.

Use data from the following table for saturated air:

$T$ (C)	$P_{sat}$ (kPa)	$h_g$ (kJ/kg)
10	1.23	2520
20	2.34	2538
30	4.25	2556
40	7.38	2574
50	12.35	2592

where the last column is the enthalpy for the saturated water vapor.

The partial pressure of the water vapor at 20°C is 1.17 kPa ( 50 % of the saturated pressure). At the end of the cooling part of the problem the air is completely saturated, and this must be at a pressure of 1.17 kPa, which gives us a temperature of around 9°C, using the table provided.

We are dealing with a process at constant pressure, and hence we need to use the enthalpy. The total enthalpy  $H$  is the sum of the enthalpy of the dry air and the water vapor. The enthalpy per kg dry air is the sum of the enthalpy of dry air per kg and the enthalpy of the vapor  $h_g$ , multiplied by the mass ratio of the dry air and water vapor:

$$h = h_a + \frac{m_v}{m_a} h_g$$

Because both follow the ideal gas law, we have:

$$\frac{m_v}{m_a} = \frac{P_v V / R_v T}{P_a V / R_a T} = 0.62 \frac{P_v}{P_a} = 0.62 \frac{P_v}{P - P_v}$$

where  $P$  is 1 atm. Therefore, the mass fraction of water vapor is initially:

$$\frac{m_v}{m_a} = 0.62 \frac{7.38}{100 - 7.38} = 0.049$$

Hence at  $40^\circ C$  we have in step 1:

$$h_1 = h_a(T = 0) + 40 \times 1 + 0.049 \times 2574$$

for the same at  $9^\circ C$  we have

$$h_2 = h_a(T = 0) + 9 \times 1 + 0.049 \times 2518$$

Therefore the heat taken out is  $31 + 0.049 \times (2574 - 2518) = 33.7$  kJ/(kg dry air).

The air leaving the cooler now has  $P_v = 1.17$  kPa, and hence the mass fraction of the water vapor going into the heater is

$$\frac{m_v}{m_a} = 0.62 \frac{1.17}{100 - 1.17} = 0.0073$$

The amount of heat needed to in the heating cycle is therefore:

$$(20 - 9) \times 1 + 0.0073 \times (2538 - 2518) = 11.1$$
 kJ/(kg dry air)

# Problem 5 - solution

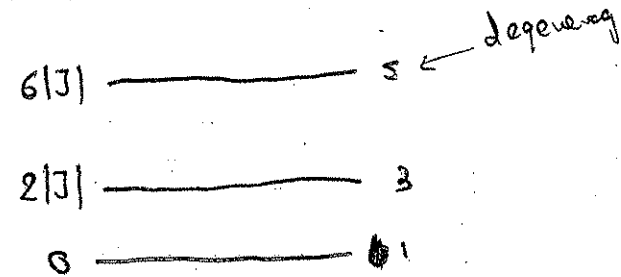
(1)

$$(a) \quad (\hat{S}_1 + \hat{S}_2)^2 = S_1^2 + S_2^2 + 2\hat{S}_1\hat{S}_2 \Rightarrow -2\hat{S}_1\hat{S}_2 = -(\hat{S}_1 + \hat{S}_2)^2 + S_1^2 + S_2^2$$

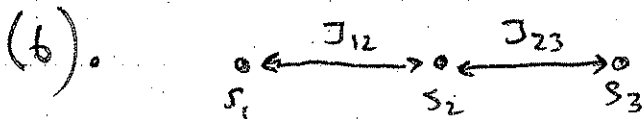
$$= -\hat{S}_T^2 + S_1^2 + S_2^2$$

$$E(S_T) = J[-S_T(S_T+1) + 2S(S+1)] \quad \left\{ \begin{array}{l} \text{(we take } \hbar=1 \text{ for} \\ \text{simplicity).} \end{array} \right.$$

~~12~~  $0 \leq S_T \leq 2S$  so for  $S = \frac{3}{2}$ ,  $S_T = 0, 1, \dots, 3$ .



Essentially, the right side should be multiplied by  $\hbar^2$



$$\hat{\mathcal{H}} = -2J_{12}\hat{S}_1\hat{S}_2 - 2J_{23}\hat{S}_2\hat{S}_3$$

Showing that  $\hat{S}_T$  and  $\hat{S}_{13}$  commute with  $\hat{\mathcal{H}}$  is trivial.

$$\hat{S}_T^2 = (\hat{S}_1 + \hat{S}_2 + \hat{S}_3)^2 = (\hat{S}_1 + \hat{S}_3)^2 + 2(\hat{S}_1 + \hat{S}_2)\hat{S}_3 + \hat{S}_3^2$$

$$= (\hat{S}_1 + \hat{S}_3)^2 + 2\hat{S}_1\hat{S}_2 + 2\hat{S}_2\hat{S}_3 + \hat{S}_3^2$$

so:  $-2\hat{S}_1\hat{S}_2 - 2\hat{S}_2\hat{S}_3 = -\hat{S}_T^2 + \hat{S}_{13}^2 + \hat{S}_3^2$ . for  $J_{12} = J_{23} = J$ ,

$$E(S_T, S_{13}) = J[-S_T(S_T+1) + S_{13}(S_{13}+1) + S(S+1)]$$

$S_T$  can take values  $0 \leq S_T \leq 2S$ , i.e., for  $S = \frac{3}{2}$ ,  ~~$S_T = 0, 1, 2, 3$~~   $S_T = \frac{1}{2}, \frac{3}{2}, \dots, \frac{9}{2}$ .

$S_{13}$  " " "  $0 \leq S_{13} \leq 2S$  " " " "  $S_{13} = 0, 1, 2, 3$ .

But not all combinations of these numbers are possible!

$S_+(S_+ + 1)$		$\frac{3}{4}$	$\frac{15}{4} =$	$\frac{35}{4}$	$\frac{63}{4}$	$\frac{99}{4}$
$S_+$		$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
$S_{13}(S_{13} + 1)$	$S_{13}$					
0	0	impossible	$10, \frac{3}{2}$			
2	1	$11, \frac{1}{2}$ $E = -1\frac{1}{4}$	$11, \frac{3}{2}$ $+1\frac{3}{4}$	etc. $+6\frac{3}{4}$		
6	2	$5\frac{1}{4}$	$2\frac{1}{4}$	$+2\frac{3}{4}$	$+9\frac{3}{4}$	
12	3		$-8\frac{1}{4}$	$-3\frac{1}{4}$	$+3\frac{3}{4}$	$12\frac{3}{4}$

(for  $J < 0$ ) ground state

Energies are in  $|J|$  units, constant term  $S(S+1)$  is omitted.

(c). For open triangle, the Hamiltonian is

$$\hat{H} = -2J \hat{S}_1 \hat{S}_2 - 2J \hat{S}_2 \hat{S}_3 - \cancel{2J \hat{S}_1 \hat{S}_3} \quad 2J' \hat{S}_1 \hat{S}_3$$

The interaction between  $S_1$  and  $S_3$  does not depend on  $S_T$ , only on  $S_{13}$

The eigenvalue for  $\hat{S}_1 \hat{S}_3$  we obtain from an operation on  $\hat{S}_{13} = \hat{S}_1 + \hat{S}_3$ :

$$(\hat{S}_1 + \hat{S}_3)^2 = \hat{S}_1^2 + \hat{S}_3^2 + 2\hat{S}_1 \hat{S}_3 \Rightarrow -2\hat{S}_1 \hat{S}_3 = -\hat{S}_{13}^2 + \hat{S}_1^2 + \hat{S}_3^2$$

$$E(S_{13}) = J' [-S_{13}(S_{13} + 1) + 2S(S + 1)]$$

Putting together:

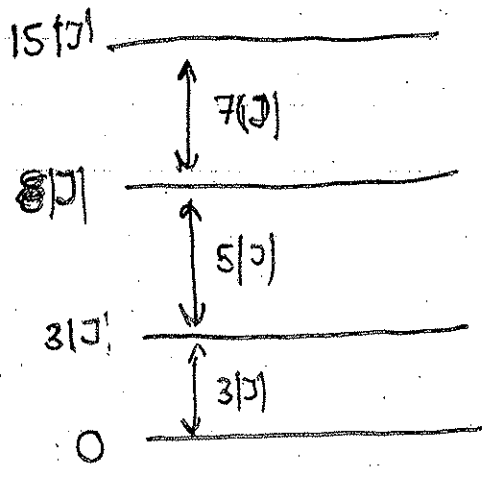
$$E(S_T, S_{13}) = J [-S_T(S_T+1) + S_{13}(S_{13}+1) + S(S+1)] + J' [S_{13}(S_{13}+1) + 2S(S+1)]$$

(d) For  $J' = J^{\equiv}$ , i.e., for a closed triangle:

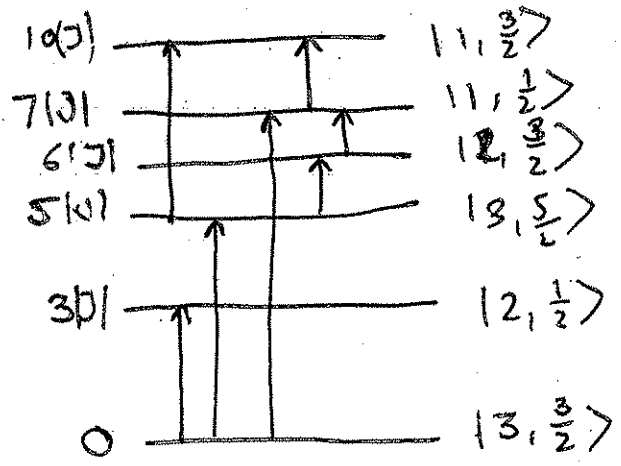
$$E(S_T, S_{13}) = -J S_T(S_T+1) + 3S(S+1)$$

$S_{13}$  no longer is relevant

For  $J < 0$ , ground state (ignoring the constant term)  $\bar{s} + \frac{3}{4}|J|$ , then  $\frac{15}{4}|J|$

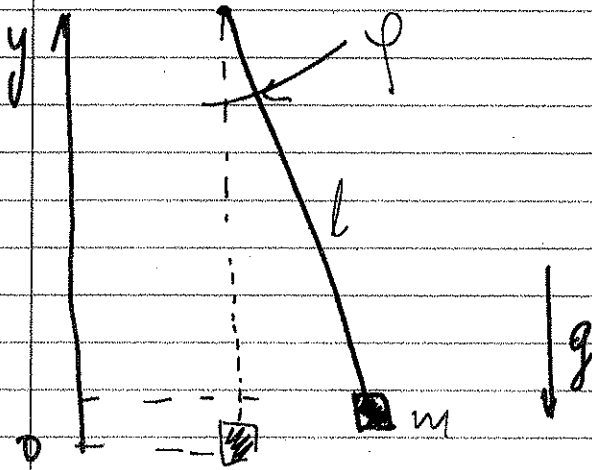


For (b) the level diagram:



(I marked the allowed transitions in neutron scattering).

It was not required in the test!



$$\textcircled{1} \quad U = mg(l - r \cos \varphi)$$

$$T = \frac{m}{2} (r^2 \dot{\varphi}^2 + \dot{r}^2)$$

constraint:  $r - l = 0$

$$h = \frac{m}{2} (r^2 \dot{\varphi}^2 + \dot{r}^2) - mg(l - r \cos \varphi) - \lambda(r - l)$$

Eq. of motion:

$$\begin{cases} m r^2 \ddot{\varphi} + mg \sin \varphi = 0 \\ m \ddot{r} - m g \cos \varphi - m r \dot{\varphi}^2 + \lambda = 0 \\ r - l = 0 \end{cases}$$

$$\lambda = mg \cos \varphi + m r \dot{\varphi}^2$$

$$\textcircled{2} \quad E_i = mgl(1 - \cos \varphi) + \frac{m}{2} l^2 \dot{\varphi}^2 = \text{const}$$

$$\textcircled{a} \quad \varphi = 0: \quad E = \frac{m}{2} l^2 \dot{\varphi}_{\text{max}}^2$$

Small oscillations:  $\sin \varphi \approx \varphi$

3pts

$$l\ddot{\varphi} = -g\varphi$$

$$\omega^2 = \frac{g}{l}$$

$$\varphi \approx a \cos \omega t \quad ; \quad \dot{\varphi} = -a \cdot \omega \sin \omega t$$

$$E = mgl \frac{\varphi^2}{2} + ml^2 \dot{\varphi}^2 = mgl \frac{a^2 \cos^2 \omega t}{2} + \frac{ml^2 a^2 \omega^2 \sin^2 \omega t}{2} = \\ = \frac{ma^2 gl}{2}$$

3

$$F = \lambda = mg \cos \varphi + ml\dot{\varphi}^2$$

2pts

$$\text{@ } \varphi = 0:$$

$$F = mg + \frac{2E}{l}$$

4

$$p = \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \dot{\varphi}$$

2pts

$$J = \oint p d\varphi = \oint ml^2 \dot{\varphi}^2 dt = \oint ml^2 a^2 \omega^2 \sin^2 \omega t dt = \frac{ml^2 \omega^2 a^2}{2} \cdot T =$$

$$= \pi m \omega a^2 l^2 = \text{const}$$

$$\pi m l^2 \omega a^2 = 2\pi \frac{ml^2 \omega E}{mgl} = 2\pi \frac{E}{\omega} = E \cdot T = \text{const}$$

5) Use 4) to find  $E(l)$ :

$$E(l_0) \cdot \sqrt{\frac{l_0}{g}} = E(l) \cdot \sqrt{\frac{l}{g}}$$

3pts

$$E(l) = E(l_0) \cdot \sqrt{\frac{l_0}{l}} = \frac{m d^2 g}{2 l_0} \sqrt{\frac{l_0}{l}} = \frac{m g}{2} \frac{d^2}{\sqrt{l_0 l}}$$

$$F = mg + \frac{2E(l)}{l} = mg \left( 1 + \frac{d^2}{l_0 l^2} \right)$$

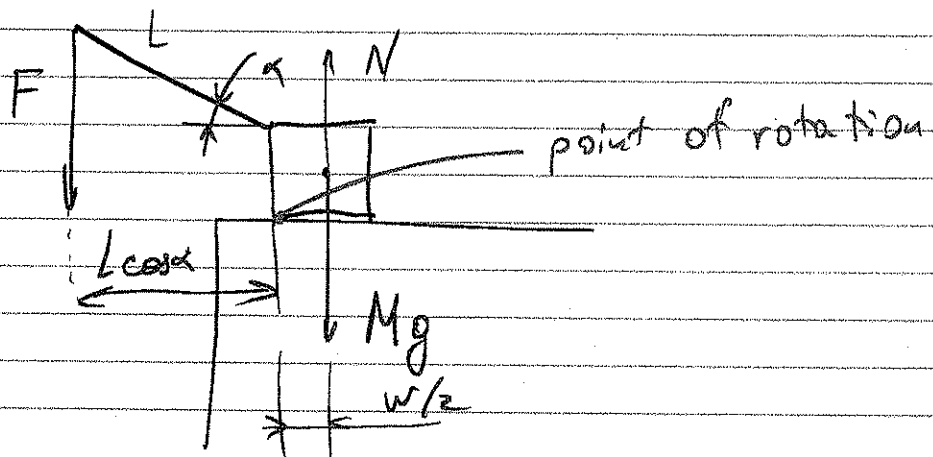
6) Stability condition:

3pts

$$\frac{M W}{2} > F(h) \cdot L \cos \alpha$$

$$\frac{M W}{2} - mg l \cos \alpha \left( 1 + \frac{d^2}{l_0 h^3} \right) > 0$$

$$\frac{m}{M} < \frac{W}{2 l \cos \alpha} \cdot \frac{\sqrt{l_0 h^3}}{d^2 + \sqrt{l_0 h^3}}$$



## PROBLEM 7

1) First B.C.  $\Phi(z=L, \rho < R, 0 \leq \varphi \leq \pi/2) = V$  and  $\Phi(z=0) = 0$

So use  $\mathcal{F}(z) = \sinh kz$

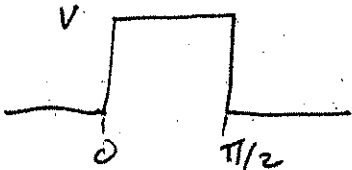
Hence  $\Phi(\rho, \varphi, z) = \sum_{\substack{n, m \\ \neq 0}} J_m(k\rho) \sinh(kz) [A_m \sin m\varphi + B_m \cos m\varphi]$

2) BC at  $\rho=0, R$ :  $\Phi(\rho=0) = 0$  and  $\Phi(\rho=R) = 0$

So  $m \neq 0$  and  $J_m(k\rho) = 0$  when  $\rho = R$

So  $k_{mn} = \frac{x_{mn}}{R}$  with  $J_m(x_{mn}) = 0$

$\Phi = \sum_{n, m} J_m(k_{mn}\rho) \sinh(k_{mn}z) [A_{mn} \sin m\varphi + B_{mn} \cos m\varphi]$

3)  $V(\varphi) =$   So  $V(\varphi) = \sum_{\substack{m=-\infty \\ m \neq 0 \\ m=2,6,10, \dots}}^{\infty} A_m \sin m\varphi$  only.

$$V \int_0^{\pi/2} \sin m'\varphi d\varphi = \sum_{m=-\infty}^{\infty} A_m \int_0^{\pi/2} \sin m'\varphi \sin m\varphi d\varphi$$

$$\int_0^{\pi/2} \sin m'\varphi d\varphi = \frac{1}{m'} (1 - \cos m'\pi/2) = \frac{2}{m'} \text{ for } m' = 2, 6, 10$$

$$\int_0^{\pi/2} \sin m'\varphi \sin m\varphi d\varphi = \frac{\sin \frac{(m'-m)\pi/2}{2}}{2(m'-m)} - \frac{\sin \frac{(m'+m)\pi/2}{2}}{2(m'+m)}$$

PROBLEM 7 CONTINUED

4) Now find  $\{A_{nm}\}$  using the BC at  $z=L$ .

$$\Phi(\rho, \varphi, z=L) = V = \sum_{n,m} J_m(k_{mn}\rho) \sinh(k_{mn}L) A_{nm} \sin m\varphi$$

Find  $\{A_{nm}\}$  by multiplying by  $\rho J_{m'}(k_{m'n'}\rho) \sin m'\varphi$  and integrating  $\int_0^R \rho d\rho \int_0^{\pi/2} d\varphi$

$$V \int_0^R \rho J_{m'}(k_{m'n'}\rho) d\rho \int_0^{\pi/2} \sin m'\varphi d\varphi$$

$$= \sum_{n,m} A_{nm} \sinh(k_{mn}L) \int_0^R \rho J_{m'}(k_{mn}\rho) J_{m'}(k_{m'n'}\rho) d\rho \cdot \int_0^{\pi/2} \sin m'\varphi \sin m\varphi d\varphi$$

Use  $\int_0^{\pi/2} \sin m'\varphi d\varphi = 2/m'$ ,  $\int_0^{\pi/2} \sin m'\varphi \sin m\varphi d\varphi = \frac{\pi}{4} \delta_{mm'}$

and  $\int_0^R \rho J_m(k_{mn}\rho) J_m(k_{m'n'}\rho) d\rho = \frac{R^2}{2} J_{m+1}^2(k_{mn}R) \delta_{nn'}$

So  $A_{nm} = \frac{2V}{m} \int_0^R \rho J_m(k_{mn}\rho) d\rho \frac{4}{\pi} \frac{2}{R^2 J_{m+1}^2(k_{mn}R)} \frac{1}{\sinh(k_{mn}L)}$

Finally

$$\Phi(\rho, \varphi, z) = \frac{16V}{\pi R^2} \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{\int_0^R \rho J_m(k_{mn}\rho) d\rho}{m J_{m+1}^2(k_{mn}R) \sinh(k_{mn}L)} J_m(k_{mn}\rho) \sinh(k_{mn}z) \sin m\varphi$$

where  $m = 2k$

## Problem 8

Consider the following situation. We put  $N$  molecules H-D (hydrogen-deuterium) in a bottle, and keep the bottle at temperature  $T$ . The atoms collide and can exchange components, so after a while we are in equilibrium and have  $n_{eq}$  molecules H-D, with the rest H-H and D-D. You are asked to calculate  $n_{eq}$ , the number of molecules H-D, as a function of temperature when equilibrium is established. You are allowed to make the following assumptions. First, the temperature is only around room temperature, which means that all translational and rotational degrees of freedom can be treated classically. It also means that the molecules are always in the vibrational ground state. Second, the electronic bonds do not depend on the mass of the nucleus, hence the spring constants are the same for all three molecules. Assume that the mass of an H-atom is  $m_H$  and of a D-atom  $m_D$ . Finally, the Helmholtz free energy for an ideal classical mono-atomic gas is

$$F_{ideal} = +Nk_B T \left( \log\left(\frac{n}{n_Q(T)}\right) - 1 \right) \quad (1)$$

(Corrected for minus sign error)  
with

$$n_Q(T) = \left( \frac{Mk_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} \quad (2)$$

Suppose we have  $L$  molecules H-D, and hence  $\frac{1}{2}(N-L)$  molecules H-H and D-D. This gives for the free energies:

$$F(L, T, V) = F_{HH}\left(\frac{1}{2}(N-L), T, V\right) + F_{HD}(L, T, V) + F_{DD}\left(\frac{1}{2}(N-L), T, V\right)$$

where the free energy for a species is

$$F_{HD}(L, T, V) = F_{ideal}(L, T, V) + F_{rot}(L, T, V) + \frac{1}{2}L\hbar\omega_0$$

and where we use  $M = m_H + m_D$  in all three terms. Similar for HH and DD.

We also need to include the entropy of mixing:

$$S_{mix} = k_B \log \frac{N!}{L! \left(\frac{1}{2}(N-L)\right)! \left(\frac{1}{2}(N-L)\right)!} \approx k_B (N \log N - L \log L - (N-L) \log\left(\frac{1}{2}(N-L)\right))$$

If we include this in the free energy, we have

$$F_{HD}(L, T, V) = F_{ideal}(L, T, V) + F_{rot}(L, T, V) + \frac{1}{2}L\hbar\omega_0 + k_B T L \log\left(\frac{L}{N}\right)$$

Equilibrium follows from  $\frac{\partial}{\partial L} F = 0$ , which gives

$$0 = -\frac{1}{2}\mu_{HH}\left(\frac{1}{2}(N-L), T, V\right) + \mu_{HD}(L, T, V) - \frac{1}{2}\mu_{DD}\left(\frac{1}{2}(N-L), T, V\right)$$

We have

$$\mu_{ideal} = k_B T \log\left(\frac{n}{n_Q(T)}\right)$$

and since the rotational degrees of freedom are classical, too, we have (note that these are independent)

$$\mu_{rot} = -k_B T \log(Z_{rot})$$

Hence

$$\mu = k_B T \log\left(\frac{L^2}{V N n_Q(T) Z_{rot}}\right) + \frac{1}{2}\hbar\omega_0 + k_B T$$

where at all places we need to use the appropriate mass (and moment of inertia, and frequency) to evaluate the answer. From

$$\mu_{HH}\left(\frac{1}{2}(N-L), T, V\right) + \mu_{DD}\left(\frac{1}{2}(N-L), T, V\right) = 2\mu_{HD}(L, T, V)$$

we get

$$k_B T \log\left(\frac{(\frac{1}{2}(N-L))^2}{V N n_Q^{HH}(T) Z_{rot}^{HH}}\right) + k_B T \log\left(\frac{(\frac{1}{2}(N-L))^2}{V N n_Q^{DD}(T) Z_{rot}^{DD}}\right) + \frac{1}{2}\hbar(\omega_0(HH) + \omega_0(DD)) =$$

$$2k_B T \log\left(\frac{L^2}{V N n_Q^{HD}(T) Z_{rot}^{HD}}\right) + \hbar\omega_0(HD)$$

or

$$\log\left(\frac{(N-L)^4}{2^4 L^4} \frac{[n_Q^{HD}(T)]^2}{n_Q^{HH}(T) n_Q^{DD}(T)} \frac{[Z_{rot}^{HD}]^2}{Z_{rot}^{HH} Z_{rot}^{DD}}\right) =$$

$$\frac{\hbar}{2k_B T} (2\omega_0(HD) - \omega_0(HH) - \omega_0(DD))$$

We know that  $n_Q \propto m^{\frac{3}{2}}$ . For rotations we only have two degrees of freedom, and since the length of the molecules is the same, we have  $Z_{rot} \propto m$ . Finally, we have  $\omega_0 = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$ , and hence:

$$\log \left( \frac{(N-L)^4 (m_H + m_D)^5}{2^4 L^4 (2m_H)^{\frac{5}{2}} (2m_D)^{\frac{5}{2}}} \right) =$$

$$\frac{\hbar\sqrt{k}}{2k_B T} \left( 2\sqrt{\frac{1}{m_H} + \frac{1}{m_D}} - \sqrt{\frac{2}{m_H}} - \sqrt{\frac{2}{m_D}} \right)$$

which can easily be solved for L.

