Department of Physics Comprehensive Examination # 92

Part I

1 April 2002

This Comprehensive Examination for Spring 2002 consists of eight Problems each worth 20 points. The Problems are grouped into four sessions:

Session 1	Problems 1, 2	9-12 AM	Monday 1 April
Session 2	Problems 3, 4	1:30-4:30 PM	Monday 1 April
Session 3	Problems 5, 6	9-12 AM	Tuesday 2 April
Session 4	Problems 7, 8	1:30-4:30 PM	Tuesday 2 April

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators will be provided if necessary. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

1. A piece of coaxial cable is available for inspection at the front of the examination room. Rulers and a bar magnet are also supplied.

   Calculate the total inductance of the cable, when the conductors are short-circuited at one end. Make reasonable assumptions and estimates.

2. A relativistic rocket is moving through space with velocity v and mass m. Its exhaust gasses leave the rocket with velocity a *relative to the rocket.* The rocket is moving in a straight line and there are no external forces acting on it.

   (a) Derive a differential equation relating the change in the rocket's mass with respect to its velocity, dm/dv. Do not make any approximations but be sure to check that it reduces to the correct non-relativistic result when $v << c$.

   (b) Is mass conserved? If not, what happens to it?

   (c) Assume the rocket starts out with mass $m_\circ$ and $v = 0$. Derive a formula for m as a function of v. What is the maximum speed the rocket can reach?

   Be sure to justify each step in your calculation so that we can assign partial credit in the event of algebraic error in your derivation.

Problem #1

PROBLEM EM UNDERGRAD

A piece of coaxial cable is available for inspection at the front of the examination room. Rulers and a bar magnet are also supplied.

Calculate the total inductance of the cable, when the conductors are short-circuited at one end. Make reasonable assumptions and estimates.

SOLUTION

Length about $l = 1$ m, inner diameter $d = 0.001$ m, outer diameter $D = 0.005$ m

Insulation doesn't respond to magnet, so assume $\mu \approx \mu_o = 4\pi \times 10^{-7}$ mks.

Assume current I on inner conductor; short circuit $\Rightarrow$ $I_2 = -I$ in outer conductor.

Total energy in magnetic fields $= \frac{1}{2} \int d\mathit{volume}\, \boldsymbol{B} \bullet \boldsymbol{H} = \frac{\mu_o}{2} \int d\mathit{volume}\, \boldsymbol{H}^2 = \frac{1}{2} L I^2$

Use cylindrical coordinate system.
All fields are azimuthal, wind around wire; need only magnitudes
Magnitudes depend on distance from center of wire ρ .

Field due to wire 1 = H_1 , Ampere's law $\Rightarrow I_1 = \oint \boldsymbol{H}_1 \bullet dl = 2\pi\rho\, H_1(\rho)$

So
$$H_1(\rho) = \frac{I_1}{2\pi\rho} \text{ for } \rho > \frac{d}{2},$$

Similarly
$$H_2(\rho) = \frac{I_2}{2\pi\rho} \text{ for } \rho > \frac{D}{2} \Rightarrow \text{ fields cancel outside wires}$$

Ignore volume of conductors which are very thin, then

energy for inductance comes from region between them.

$$L = \frac{\mu_o}{I^2} \int_0^l dz \int_0^{2\pi} d\phi \int_{d/2}^{D/2} \rho\, d\rho\ \boldsymbol{H}_1^2 = \mu_o\,(l)\,(2\pi) \frac{1}{(2\pi)^2} \int_{d/2}^{D/2} \rho\, d\rho\, \frac{1}{\rho^2}$$

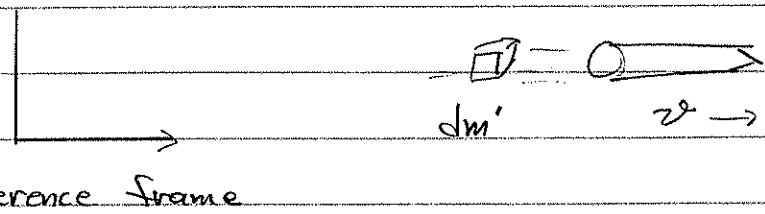
$$L = \frac{\mu_o l}{2\pi} \ln \frac{D}{d}$$

numerical(mks) $L = \dfrac{4\pi \times 10^{-7}}{2\pi}\,(1\text{ m})\,\ln\left(\dfrac{0.005}{0.001}\right) = 0.2 \ln\ 5$ microhenries

Problem # 2

The rocket problem

(a)



The speed of the exhaust dm' is $-a$ in rocket frame and v' in the reference frame. m is rocket mass.

use $\gamma = 1/\sqrt{1-v^2/c^2}$ $\qquad \gamma' = 1/\sqrt{1-v'^2/c^2}$

Each component of the 4-momentum $(\gamma mc, \gamma m\vec{v})$ must be conserved so $(P_\mu)_{\text{before}} = (P_\mu)_{\text{after}}$

$$m\gamma c = (m - dm)\,\gamma_{(v+dv)}\, c + \gamma' dm' c$$

$$\underline{m\gamma v = (m-dm)\,\gamma_{(v+dv)}\,(v+dv) + \gamma' v' dm'}$$

$$\gamma_{(v+dv)} = \gamma_v + \frac{d\gamma}{dv}dv = \gamma + \frac{\gamma^3 v}{c^2}dv$$

Substitute above and subtract to eleminate $\gamma' dm'$.

$$0 = \gamma(v - v') + \gamma\left[\frac{\gamma^2 v}{c^2}(v' - v) - 1\right] m\frac{dv}{dm}$$

The usual addition of velocities formula gives

$$v' = \frac{v - a}{1 - av/c^2} \qquad \text{or} \qquad v' - v = \frac{-a\gamma^{-2}}{1 - av/c^2}$$

after various cancellations we get

$$0 = a\left(1 - v^2/c^2\right) + m\frac{dv}{dm}$$

(b) Mass is not conserved. It has been partly converted to K.E.

(c) $$\int_{m_0}^{m} \frac{dm'}{m'} = -\frac{1}{a}\int_0^v \frac{dv'}{1 - v'^2/c^2}$$

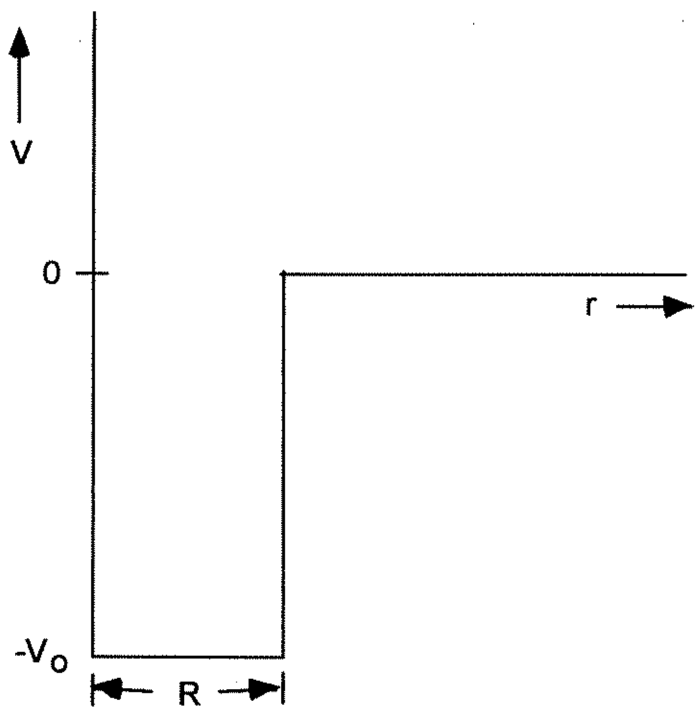
$$\ln\left(m/m_0\right) = \frac{c}{2a}\ln\left(\frac{c-v}{c+v}\right)$$

$$\frac{m}{m_0} = \left[\frac{1-\beta}{1+\beta}\right]^{c/2a}$$

$\beta_{max} = 1$ but only when $m = 0$!

3. A flat frictionless plane is tangent to the earth at some colatitude θ. Assume that it is attached to the earth at some definite point (Corvallis for example) so that it rotates with the earth. A mass m is placed on the plane some distance ρ from the point of contact. It is placed in such a way that *relative to the plane* it has no initial velocity. Describe its subsequent motion relative to the plane in detail and give rough numeric estimates of the quantities in your calculations. One significant figure accuracy will do. You don't really need a calculator. You may need the radius of the earth $R = 6.4 \times 10^6$ meters. You may assume that $\rho << R$ and make any reasonable approximation based on this inequality. Carefully justify your conclusions – in other words, don't just guess.

4. Consider the square well potential shown immediately below.



The differential scattering cross section, $d\sigma/d\Omega$, for a particle of reduced mass μ in the first Born approximation can be written as

$$\frac{d\sigma}{d\Omega} = \left(\frac{\mu}{2\pi\hbar^2}\right)^2 |V_{\boldsymbol{k},\boldsymbol{k}'}(\boldsymbol{r})|^2$$

where $V_{\boldsymbol{k},\boldsymbol{k}'}$ is the matrix element associated with elastic scattering by the square well potential of a particle of reduced mass μ from the initial plane wave state $\boldsymbol{k}$ to a final plane wave state $\boldsymbol{k}'$.

Show that in the Born approximation, the differential scattering cross section has its first zero at an angle $\Theta_\circ$, approximated by

$$\Theta_\circ \approx 2\sin^{-1}\left(\frac{2.247}{kR}\right)$$

Problem # 3

~~1~~. Mechanics - UG

A flat frictionless plane is tangent to the earth at some cola~~t~~itude θ. Assume that it is attached to the earth at some definite point (Corvallis for example) so that it rotates with the earth. A mass m is placed on the plane some distance ρ from the point of contact. It is placed in such a way that *relative to the plane* it has no initial velocity. Describe its subsequent motion relative to the plane in detail and give rough numeric estimates of the quantities in your calculations. One significant figure accuracy will do. You don't really need a calculator. You may need the radius of the earth $R = 6.4 \times 10^6$ meters. You may assume that $\rho << R$ and make any reasonable approximation based on this inequality. Carefully justify your conclusions – in other words, don't just guess.

Solution: First ignore the rotation of the earth. Gravity produces a force $F = -mg\rho/R$. This is a restoring force, so the mass will execute simple harmonic motion with a frequency $\omega = Sqrtg/R$ Evidentally, this is a Foulcot pendulum of length R. The period works out to 1.4 hours.

If the tangent point were at the north pole, the earth would turn under the mass, so in the rotating coordinate system the plane of oscillation would precess with angular frequency opposite to that of the earth, $-\Omega$.

The real question is what happens at some other latitude. In general

$$\frac{d\boldsymbol{r}}{dt} = \frac{d^*\boldsymbol{r}}{dt} + \boldsymbol{\omega} \times \boldsymbol{r}$$

The starred derivative refers to the rotating coordinate system. Differentiating a second time gives

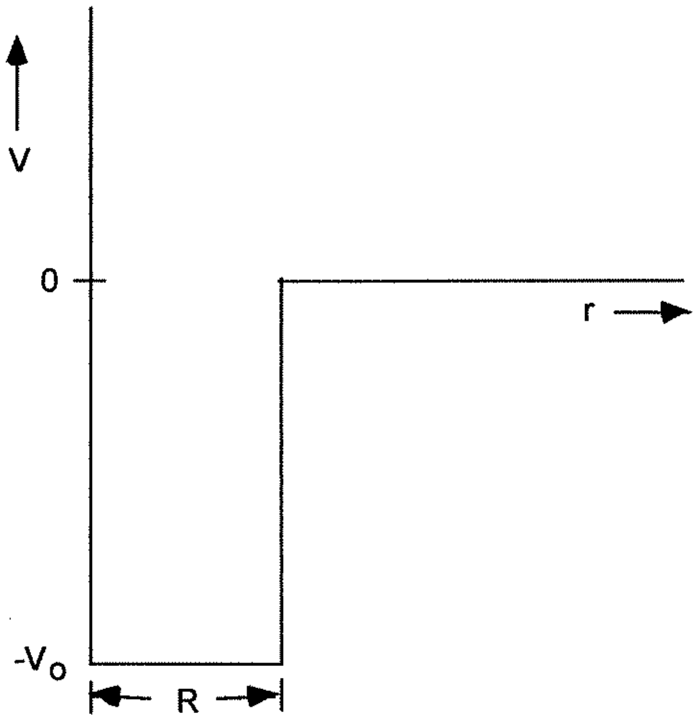
$$m\frac{d^{*2}}{dt^2} = m\boldsymbol{g}_e - 2m\boldsymbol{\omega} \times \frac{d^*\boldsymbol{r}}{dt}$$

The last term, the Coriollis force, is responsible for the precession. At the north pole where $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are perpendicular it must produce the precession frequency $-\Omega$. At lower latitudes the force is reduced by a factor $\cos\theta$ and the precession frequency is reduced accordingly.

Problem #4

GRADUATE QUANTUM MECHANICS QUESTION FOR COMP EXAM - SPRING TERM 2002

Consider the square well potential shown immediately below



The differential scattering cross-section, $\frac{d\sigma}{d\Omega}$, for a particle of reduced mass, μ, can be written as $\frac{d\sigma}{d\Omega} = \left(\frac{\mu}{2\pi \hbar^2}\right)^2 \left| V_{\vec{K}',\vec{K}} \right|^2$, where $V_{\vec{K}',\vec{K}}$ is the matrix element associated with elastic scattering by the square well potential of a particle of reduced mass μ from initial plane wave state $\vec{K}$, to final plane wave state $\vec{K}'$.

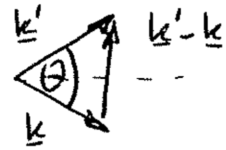
Show that, in the Born approximation, the differential scattering cross-section has its first zero at an angle, θ_0, approximated by

$$\theta_0 \approx 2 \sin^{-1}\left(\frac{2.247}{KR}\right)$$

$$V_{\underline{k}'\underline{k}} = \int V(r)\, e^{i(\underline{k}-\underline{k}')\cdot\vec{r}}\, d\tau$$

$$V_{\underline{k}'\underline{k}} = -V_0 \int_0^R \int_0^\pi \int_0^{2\pi} e^{i\chi r\cos\Theta}\, r^2 \sin\Theta\, dr\, d\Theta\, d\Phi$$

where $\chi = 2K\sin\left(\frac{\theta}{2}\right)$, from



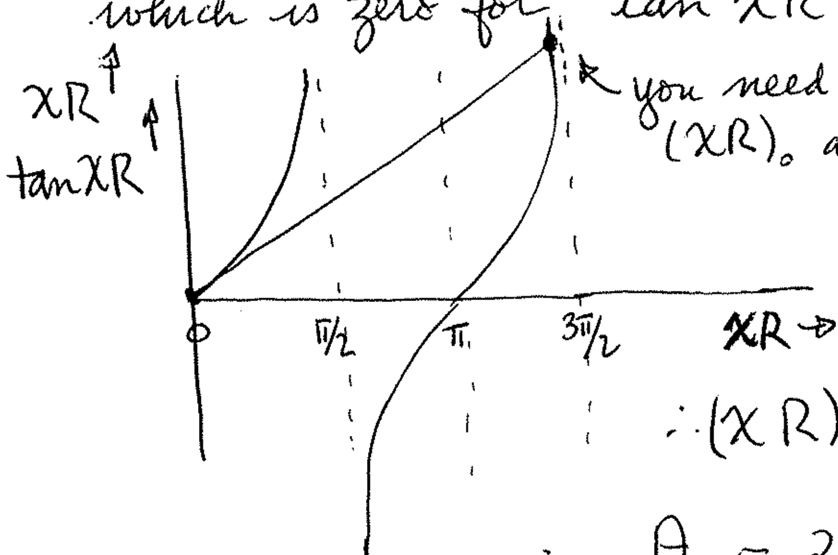
Putting $z = i\chi r\cos\Theta$, and integrating over Θ and Φ,

$$V_{\underline{k}'\underline{k}} = -4\pi V_0 \int_0^R \frac{\sin\chi r}{\chi r}\, r^2\, dr.$$ This gets integrated by parts to yield

$$V_{\underline{k}'\underline{k}} = \frac{4\pi V_0 R^3}{(\chi R)^3}\left[\chi R\cos\chi R - \sin\chi R\right]$$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{(4\pi)^2 V_0^2 R^6}{(\chi R)^6}\left(\frac{\mu}{2\pi\hbar^2}\right)^2 (\chi R\cos\chi R - \sin\chi R)^2.$$

which is zero for $\tan\chi R = \chi R$.



you need to "guess" a value for $(\chi R)_0$ at this point.

$(\chi R)_0 = 4.493$ (my gues

$$\therefore (\chi R)_0 = 2KR\sin\frac{\theta_0}{2} = 4.493$$

$$\therefore \theta_0 = 2\sin^{-1}\left(\frac{2.247}{KR}\right)$$

5. Consider an elemental, diatomic gas of molecular weight M at equilibrium at temperature T.

   (a) Apply Boltzmann statistics to the vibrational energy levels to obtain the average vibrational energy per molecule. You may assume Hooke's Law holds for the forces required to stretch or compress the molecular bond along the axis of the molecule.

   The following identity should be helpful:

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$$

   (b) Show that it reduces to the expectation from classical partition of energy in the limit of high temperatures. State explicitly your criterion for "high temperature."

   (c) Show that your results from part (a) reduces to the zero point energy as $T \to 0$.

   (d) Calculate the molar heat capacity of the gas.

   (e) Show that your result in part (d) yields the vibrational contribution to the Dulong-Petit law at high temperatures.

   (f) Using your result from part (d), find the functional for form the heat capacity as $T \to 0$.

Problem 5

SM Problem #1

Consider an elemental, diatomic gas of molecular weight M at equilibrium at temperature T.

(a) Apply Boltzmann statistics to the vibrational energy levels to obtain the average vibrational energy per molecule. You may assume that Hooke's Law holds for the forces required to stretch or compress the molecular bond along the axis of the molecule.

The following identity should be helpful:

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$$

(b) Show that reduces to the expectation from classical partition of energy in the limit of high temperatures. State explicitly your criterion for "high temperature".

(c) Show that your result from part (a) reduces to the zero point energy as $T \rightarrow 0$.

(d) Calculate the molar heat capacity of the gas.

(e) Show that your result in part (d) yields the vibrational contribution to the Dulong-Petit Law at high temperatures.

(f) Using your result from part (d), find the functional form of the heat capacity as $T \rightarrow 0$.

SM problem #1

If Hooke's law applies to the molecular bond, we can view the molecules as a simple harmonic oscillator with frequency $\omega = \sqrt{K/\mu}$ where K is the effective spring constant of the bond and μ is the reduced mass:

$$\mu = m_{atom}/2 = M/4N_0$$

The energy levels are given by $E_n = (n+\frac{1}{2})\hbar\omega$

(a) Average vibrational energy:

$$\overline{E} = \frac{\sum_{n=0}^{\infty} e^{-\beta E_n} E_n}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{Z}\frac{\partial Z}{\partial\beta}$$

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta(n+1/2)\hbar\omega} \qquad \beta \equiv 1/kT$$

$$= e^{-\frac{1}{2}\beta\hbar\omega}\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega}$$

Use identity: $\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$

$$Z = e^{-\frac{1}{2}\beta\hbar\omega}\cdot\frac{1}{1-e^{-\beta\hbar\omega}}$$

$$\frac{\partial Z}{\partial \beta} = -\frac{1}{2}\hbar\omega \cdot Z$$

$$+ e^{-\frac{1}{2}\beta\hbar\omega} \cdot (-\hbar\omega)(-e^{-\beta\hbar\omega}) \cdot \frac{-1}{(1-e^{-\beta\hbar\omega})^2}$$

$$= -\frac{1}{2}\hbar\omega Z$$

$$- \frac{(\hbar\omega)(e^{-\beta\hbar\omega})}{1-e^{-\beta\hbar\omega}} \cdot Z = Z\left(-\frac{1}{2}\hbar\omega - \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right)$$

$$\boxed{\overline{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right)}$$

(b) @ <u>high</u> <u>temperature</u>: $\underline{\underline{\hbar\omega << kT}}$ $(\beta\hbar\omega << 1$

$$e^{\beta\hbar\omega} - 1 \simeq 1 + \beta\hbar\omega - 1 = \beta\hbar\omega$$

$$\overline{E} \simeq \hbar\omega\left(\frac{1}{2} + \frac{1}{\beta\hbar\omega}\right) \simeq \frac{\hbar\omega}{\beta\hbar\omega} = kT$$

Classical partition of energy for a one-dimensional harmonic oscillator $\Rightarrow$ $\frac{1}{2}kT$ for kinetic energy and $1/2\, kT$ for potential energy

$$\overline{E}_{class.} = \frac{1}{2}kT + \frac{1}{2}kT = kT \quad \text{(vibrational energy onl}$$

(c) @ low temprature $\beta\hbar\omega >> 1$

$$\bar{E} \simeq \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}}\right) = \hbar\omega\left(\frac{1}{2} + e^{-\beta\hbar\omega}\right)$$

$\bar{E} \longrightarrow \frac{1}{2}\hbar\omega$ as $T \rightarrow 0$ (zero point energy)

(d) molar heat capacity

$$C_v = \left(\frac{\partial E_m}{\partial T}\right)_v \qquad \bar{E}_{molar} = N_0 \bar{E} = N_0\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right)$$

$$\frac{\partial E_m}{\partial T} = \frac{\partial E_m}{\partial \beta}\frac{\partial \beta}{\partial T} = -\frac{(\hbar\omega)e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} \cdot N_0\hbar\omega\left(\frac{-k_B}{k_B^2 T^2}\right)$$

$$C_v = \frac{N_0 k_B (\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2 (kT)^2} = N_0 k \frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2}$$

(e) @ high temperatures ($\beta\hbar\omega \ll 1$)

$$C_v = N_0 k \frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} \simeq \frac{N_0 k (\beta\hbar\omega)^2}{(\beta\hbar\omega)^2}$$

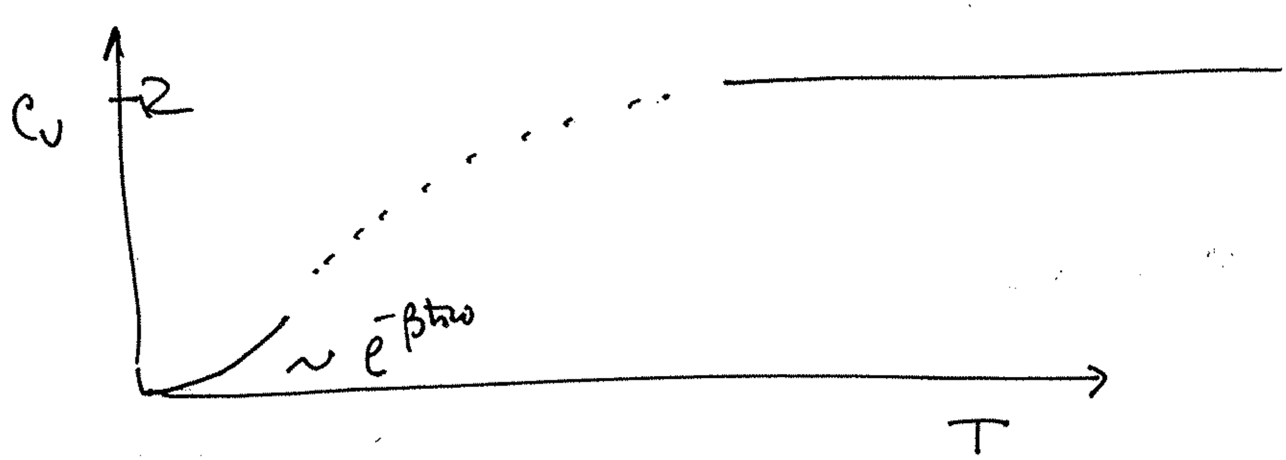
$$= N_0 k = R$$

= vibrational contribution to Dulong-Petit law

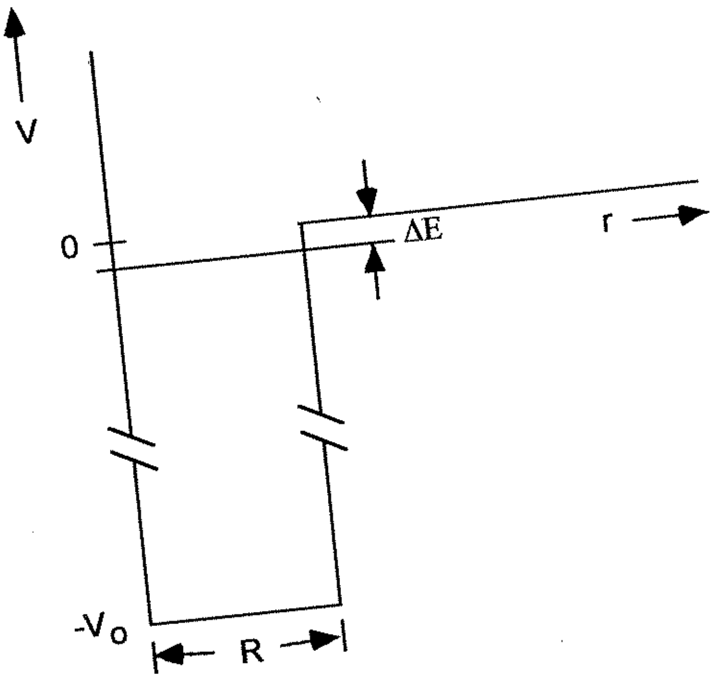
(f) @ low temperatures ($\beta\hbar\omega \gg 1$)

$$C_v \simeq \frac{N_0 k (\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{e^{2\beta\hbar\omega}} = N_0 k (\beta\hbar\omega)^2 e^{-\beta\hbar\omega}$$

$C_v \longrightarrow 0$ exponentially as $T \to 0$



6. Consider that the deuteron consists of a proton and a neutron bound in a s state in an attractive square well potential.



There is only a single bound state of this system, and its binding energy is much less than the depth of the attractive square well (*i.e.* $|\Delta E| \ll |V_\circ|$). Show that under these circumstances the magnitue of the depth of the square well can be approximated by

$$V_\circ \approx \frac{\pi^2 \hbar^2}{4mR^2}$$

where m is the mass of either the proton or the neutron (assume they have the same mass), and where R is the "width" of the well.

Problem #6

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) + \frac{2\mu}{\hbar^2}[E - V(r)]R(r) = 0.$$

where $\mu = \frac{m}{2}$.

Try solving for $u(r) = rR(r)$

i.e. $$\frac{d^2u(r)}{dr^2} + \frac{2\mu}{\hbar^2}[E - V(r)]u(r) = 0.$$

I: $0 < r < R$: $$\frac{d^2u}{dr^2} + k_I^2 u = 0$$

where $k_I = \sqrt{2\mu(V_0 - \Delta E)}/\hbar$.

II: $R < r$: $$\frac{d^2u}{dr^2} - k_{II}^2 u = 0$$

where $k_{II} = \sqrt{2\mu\Delta E}/\hbar$.

Solutions : I : $u_I = A\sin k_I r + B\cos k_I r$

But with $u=0$ for $r=0$ ($R(r)$ never diverging)

Then, $u_I = A\sin k_I r$, $r < R$

In region II : $u = Ce^{-k_{II}r} + De^{k_{II}r}$

To prevent u from diverging as $r \to \infty$, we need $D = 0$

thus, $u_{II} = Ce^{-k_{II}r}$, $r > R$

Matching u and $\frac{du}{dr}$ at $r = R$ produces

$$A \sin k_{I} R = C e^{-k_{II} r}, \text{ and}$$

$$A k_{I} \cos k_{I} R = -C k_{II} e^{-k_{II} r}$$

Thus, $k_{I} \cot k_{I} R = -k_{II}$, or

$$\frac{\sqrt{2\mu(V_0 - \Delta E)}}{\hbar} \cot\left(\frac{2\mu(V_0 - \Delta E)}{\hbar} R\right) = -\frac{\sqrt{2\mu \Delta E}}{\hbar}$$

Under the condition that $|\Delta E| \ll |V_0|$, this becomes

$$\cot\left(\frac{\sqrt{2\mu V_0}}{\hbar} R\right) \simeq \sqrt{\frac{\Delta E}{V_0}} \simeq 0.$$

which is satisfied by $\frac{\sqrt{2\mu V_0}}{\hbar} R = \frac{\pi}{2}$

$$\therefore \quad V_0 \approx \frac{\pi^2 \hbar^2}{4(2\mu) R^2} \approx \frac{\pi^2 \hbar^2}{4 m R^2}.$$

7. A long straight wire of radius $R = 1$ cm hangs horizontally in midair. It hold no net charge.

   A thunderstorm causes a uniform charge density to accumulate on a layer of clouds above the wire, creating a uniform vertical electric field of strength $E_\circ$.

   Find the distribution of charges induced on and in the wire. Give the charge densities σ and ρ. How do they depend on position?

   HINT: What set of orthogonal functions will be useful?

8. Glasses and other amorphous material contain localized structural features ("centers") whose energetic configurations can be described by two energy levels separated by an energy difference Δ.

   (a) The centers in glasses can be probed with ultrasound. The ultrasound absorption at a given temperature T is proportional to the difference in the population N_l of the lower energy level and the population N_u of the upper level, *i.e.* $\delta N = N_l - N_u$.
   Derive an expression for $\delta N(T)$ for glass containing N centers of splitting Δ.

   (b) Show that $\delta N(T)$ varies as $1/T$ at sufficiently high temperature.

   (c) Write an expression for the entropy of a system of N centers at a temperature such that $k_B T \gg \Delta$.

   (d) The centers in glasses were first detected by a linear temperature dependence of the heat capacity at low temperatures, *i.e.* $C \propto T$. Explain briefly why this observation was unexpected in materials that are electrically insulating (non-metallic).

   (e) Because glasses are inherently disordered, the actual centers in glasses do not all have the same energy Δ.
   By deriving an expression for the low temperature heat capacity, show that a linear temperature dependence for the heat capacity can be explained if there is a **distribution** of energy splittings – assume that all values of Δ are equally probable up to some maximum value $\Delta_\circ$, and that $\Delta_\circ \gg k_B T$. There are no centers with spitting greater than $\Delta_\circ$.
   The following integral may be useful

$$\int x^2 e^{-x}\, dx = -e^{-x}(x+1)^2$$

Problem # 7

PROBLEM EM GRAD

A long straight round wire of radius R = 1 cm hangs horizontally in midair. It holds no net charge.

A thunderstorm causes a uniform charge density to accumulate on a layer of clouds above the wire, creating a uniform vertical electric field of strength E_0.

Find the diestribution of charges induced on and in the wire. Give the charge densities σ and ρ. How do they depend on position?

HINT: What set of orthogonal functions will be useful?

SOLUTION

The symmetries of the wire are manifest in a cylindrical coordinate system with preferred axis along the center of the wire. Then there is translational invariance along the axis and rotational invariance around the axis. The rotational invariance is broken by the applied field, while the translational invariance remains intact.

It will be convenient to find the electrostatic potenial $V(z, \rho, \phi)$. The electric field is its gradient, and the charge densities on the surface will be found from the discontinuity in the field. Inside the conductor V is constant, we can choose $V = 0$. The volume charge density is $\rho = 0$. All charge resides on the surface.

To exploit the translational symmetry we can use harmonic (plane-wave) functions of the axial coordinate z. Since this symmetry is unbroken we need only solutions which respect the translational invariance, *i.e.* are independent of z.

To exploit the rotational symmetry we use harmonic functions of the azimuthal coordinate ϕ. The periodicity of the coordinates requires integer angular frequencies: outside the conductor

$$V(z, \rho, \phi) = \sum_{m=0}^{\infty} \{ f_m(\rho) \sin m\phi + g_m(\rho) \cos m\phi \} \ .$$

Since there are no charges outside the conductor,

$$0 = \nabla \cdot \boldsymbol{E} = -\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \sum_{m=0}^{\infty} \left\{ \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial f_m}{\partial\rho}\right) - m^2 f_m \right] \sin m\phi + \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial g_m}{\partial\rho}\right) - m^2 g_m \right] \cos m\phi \right\}$$

The left-right symmetry is unbroken. If we choose $\phi = 0$ direction as vertical, then potential won't change on left-right reflection $\phi \to -\phi$, so the $\sin \phi$ terms go away;

Guess $g_m = a_m \rho^\alpha$, then $\alpha^2 = m^2$ makes coefficients vanish, $\alpha = \pm m$

(except when $m = 0$, can also have $g_0 = b_0 \ln \rho$), so

$$V(z, \rho, \phi) = a_0 + b_0 \ln \rho + \sum_{m=1}^{\infty} [a_m \rho^m + b_m \rho^{-m}] \cos m\phi \; .$$

(This formula is given in the hand-out collection from 1991).

Far from wire, $\rho^{-m} \to 0$ so only a_m and b_0 terms contribute,

Given uniform electric field E_0 in x-direction $\phi = 0$,

For large ρ, $\quad V \to V_0 - E_0 x = V_0 - E_0 \rho \cos \phi$

Conclude $a_0 = V_0$, $b_0 = 0$, $a_1 = -E_0$, all other a_m's $= 0$.

For $\rho = R$, $V = 0 = V_0 - E_0 R \cos \phi + b_1 R^{-1} \cos \phi + \sum_{m=2}^{\infty} b_m \rho^{-m} \cos m\phi$

Conclude $b_1 = R^2 E_0$, $V_0 = 0 = b_m$ for all $m > 1$.

$$V(z, \rho, \phi) = -\left(\rho - \frac{R^2}{\rho}\right) E_0 \cos \phi.$$

To find $\boldsymbol{E}$, take gradient:

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z}$$

$$= E_0 \left[\hat{\rho} \cos \left(1 + \frac{R^2}{\rho^2}\right) + \hat{\phi} \sin \phi \left(1 - \frac{R^2}{\rho^2}\right) \right]$$

To find charge density use $\Delta E = \sigma/\varepsilon_0$.

At surface $\rho = R$, $\quad \Delta E = E = 2 E_0 \hat{\rho} \cos \phi$, $\quad$ charge density $\sigma = 2E_0 \varepsilon_0 \cos \phi$

Problem #8

SM Problem ~~#2~~

Glasses and other amorphous materials contain localized structural features ("centers") whose energetic configurations can be described by two energy levels separated by an energy difference Δ.

(a) The centers in glasses can be probed with ultrasound. The ultrasonic absorption at a given temperature T is proportional to the difference in the population N_l of the lower energy level and the population N_u of the upper level, i.e. $\delta N = N_l - N_u$.

Derive an expression for $\delta N(T)$ for a glass containing N centers of splitting Δ.

(b) Show that $\delta N(T)$ varies as $1/T$ at sufficiently high temperature.

(c) Write an expression for the entropy of a system of N centers at a temperature such that $k_B T >> \Delta$.

(d) The centers in glasses were first detected by a linear temperature dependence of the heat capacity at low temperatures, i.e. $C \propto T$. Explain briefly why this observation was unexpected in materials that are electrically insulating (non-metallic).

(e) Because glasses are inherently disordered, the actual centers in glasses do not all have the same energy Δ.

By deriving an expression for the low temperature heat capacity, show that a linear temperature dependence for the heat capacity can be explained if there is a distribution of energy splittings -- assume that all values of Δ are equally probable up to some maximum value Δ_0, and that $\Delta_0 >> k_B T$. There are no centers with splitting greater than Δ_0.

The following integral may be useful: $\int x^2 e^{-x} dx = -e^{-x}(x+1)^2$

SM problem ~~#~~ #8

(a) Let $E_u = +\frac{\Delta}{2}$ and $E_\ell = -\frac{\Delta}{2}$:

$$Z = \sum_{i=1}^{2} e^{-\beta E_i} = e^{+\beta\frac{\Delta}{2}} + e^{-\beta\frac{\Delta}{2}}$$

$$N_u = \frac{N e^{-\beta E_u}}{Z} = N e^{-\beta\frac{\Delta}{2}} \qquad N_\ell = N e^{+\beta\frac{\Delta}{2}}$$

$$\delta N(T) = N(N_\ell - N_u) = \frac{N\left(e^{\beta\frac{\Delta}{2}} - e^{-\beta\frac{\Delta}{2}}\right)}{e^{\beta\frac{\Delta}{2}} + e^{\beta\Delta/2}}$$

$$= \underline{\underline{N\tanh\left(\frac{\beta\Delta}{2}\right)}}$$

Note: This problem is equivalent to a spin ½ in a magnetic field.

(b) At high temperature, $\beta\Delta \ll 1$,

$$\delta N(T) \simeq N\frac{\left(1+\frac{\beta\Delta}{2}\right) - \left(1-\frac{\beta\Delta}{2}\right)}{\left(1+\frac{\beta\Delta}{2}\right) + \left(1-\frac{\beta\Delta}{2}\right)} = N\frac{\beta\Delta}{2} = \frac{N\Delta}{2k_BT}$$

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Litho in U.S.A.

(c) At high temperature where population of both levels is essentially equally probable, the entropy is determined by the number of ways of distributing N centers on 2 levels:

$$S = k_B \ln(2^N) = \underline{\underline{N k_B \ln 2}}$$

(d) Metals contain high concentrations of fermions (electrons) whose statistics are degenerate @ low temperature. Degenerate fermions give a linear heat capacity at low temperatures.

The glasses in question are electrically insulating and therefore would not be expected to exhibit characteristics of a degenerate Fermi system.

(e) First, calculate the low temperature heat capacity of uniform level splittings Δ:

$$E = N \cdot \frac{\sum_i e^{-\beta E_i} E_i}{Z} = N \frac{\left(e^{\beta\frac{\Delta}{2}}\left(-\frac{\Delta}{2}\right) + e^{-\beta\frac{\Delta}{2}}\left(\frac{\Delta}{2}\right)\right)}{e^{\beta\frac{\Delta}{2}} + e^{-\beta\frac{\Delta}{2}}}$$

$$= -N\left(\frac{\Delta}{2}\right)\tanh\left(\frac{\beta\Delta}{2}\right)$$

$$C = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial \beta}\frac{\partial \beta}{\partial T}$$

$$= \left(-N\frac{\Delta}{2}\right)\frac{\partial}{\partial\beta}\tanh\left(\frac{\beta\Delta}{2}\right)\cdot\left(-k\beta^2\right)$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{4}{(e^x + e^{-x})^2}$$

$$C = \left(-N\frac{\Delta}{2}\right)\left(-k\beta^2\right)\cdot\frac{4}{\left(e^{\beta\Delta/2} + e^{-\beta\Delta/2}\right)^2}\cdot\left(\frac{\Delta}{2}\right)$$

@ low T $\quad \beta\Delta/2 \gg 1$

$$C = N\left(\frac{\Delta\beta}{2}\right)^2 k \cdot \frac{4}{\left(e^{\beta\Delta/2} + e^{-\beta\Delta/2}\right)^2} \Rightarrow 4Nk\left(\frac{\Delta\beta}{2}\right)^2 e^{-\beta\Delta}$$

Now, for a distribution $f(\Delta) = \frac{1}{\Delta_0}$ $\quad 0 < \Delta < \Delta_0$

$$C = \frac{4Nk}{\Delta_0}\int_0^{\Delta_0} d\Delta \left(\frac{\Delta\beta}{2}\right)^2 e^{-\beta\Delta}$$

$$= \frac{Nk}{\Delta_0}\frac{1}{\beta}\int_0^{\Delta_0\beta} dx\, x^2 e^{-x} \qquad x = \Delta\beta$$

$$= -\frac{Nk}{\Delta_0}\frac{1}{\beta}\cdot\left[e^{-\beta\Delta}(\beta\Delta+1)^2\right]_0^{\beta\Delta_0} \qquad \text{integral given}$$

$$= -\frac{Nk}{\Delta_0}\frac{1}{\beta}\left[e^{-\beta\Delta_0}(\beta\Delta_0+1) - 1\right]$$

At temperatures such that $\beta\Delta_0 >> 1$

$$C_v \simeq \frac{Nk}{\Delta_0}\frac{1}{\beta} = \frac{Nk_B^2 T}{\Delta_0}$$

linear T-dependence.