

Department of Physics Comprehensive Examination # 89

Part I

25 September 2000

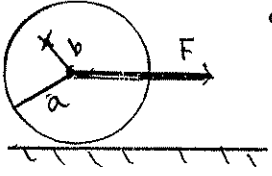
This Comprehensive Examination for Fall 2000 consists of eight problems each worth 20 points. The problems are grouped into four sessions:

Session 1	problems 1, 2	9-12 AM	Mon 25 September
Session 2	problems 3, 4	1-4 PM	Mon 25 September
Session 3	problems 5, 6	9-12 AM	Tues 26 September
Session 4	problems 7, 8	1-4 PM	Tues 26 September

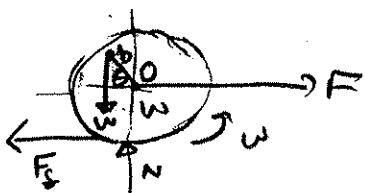
Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination. If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

1. A nonuniform, right circular cylinder of radius a and total mass M has its center of mass a distance b away from its geometrical axis. The moment of inertia about a line passing through the center of mass and parallel to the cylinder axis is I . The cylinder rolls without slipping along a horizontal plane at a constant angular velocity ω under the action of a horizontal force F as shown in the figure.



- What is the value of ω at which the cylinder is just about to leave the horizontal surface at some time during its motion?
 - What is the magnitude of the force F as a function of time?
 - What is the force of friction as a function of time?
 - What is the normal force as a function of time?
 - For angular velocity less than the critical value in part (a), what is the minimum required coefficient of friction such that slipping does not occur?
 - How much work does F do when the center of mass moves from its lowest to its highest position?
2. A parallel-plate capacitor has plates of area A separated by a small distance d . The plates hold equal and opposite charges of magnitude Q . Neglecting edge effects:
- Calculate the force between the plates assuming that the charge is held constant.
 - Calculate the force assuming the the potential between the plates is held constant.
 - Why does the following approach give the wrong answer?
Since force and electric field are related by the simple equation $\vec{F} = Q\vec{E}$, one could calculate the force on one plate just by multiplying the charge on the plate by the electric field between the plates.



leaves surface when $N \rightarrow 0$
 $\omega = \text{const} \Rightarrow$ uniform motion

$$\sum \vec{F} = M \vec{a}_{\text{com}}$$

$$F_x \quad 1) \quad F + F_f = m a_x = m b \omega^2 \cos \theta$$

$$F_y \quad 2) \quad N - mg = m a_y = m b \omega^2 \sin \theta$$

3) $\sum \tau = I \frac{d\omega}{dt} = 0$ (since $\omega = \text{constant}$)
 about center O
 $m g b \cos \theta - a F_f = 0$

a) 2) $N = mg - m b \omega^2 \sin \theta$
 leave surface $\Rightarrow N = 0 \Rightarrow mg = m b \omega^2 \sin \theta$
 $\Rightarrow \omega^2 = g / (b \sin \theta)$
 $\omega^2 \text{ max, } \sin \theta = 1 \Rightarrow \omega_{\text{max}} = \sqrt{g/b}$

b) 1) $F = F_f + m b^2 \omega^2 \cos \theta$ (write not right)
 3) $F_f = \frac{m g b \cos \theta}{a}$
 $\Rightarrow F = \cos \theta \left[\frac{m g b}{a} + m b^2 \omega^2 \right] = \left[\frac{m g b}{a} + m b^2 \omega^2 \right] \cos(\omega t)$

c) $F_f = \frac{m g b}{a} \cos \omega t = \frac{m g b}{a} \cos(\omega t)$

d) $N = mg - m b \omega^2 \sin \omega t$

e) Slip: least max friction when $F_f = \mu_s N$ (min μ)

$\Rightarrow \mu_s (mg - m b \omega^2 \sin \omega t) = \frac{m g b}{a} \cos \omega t$
 must be min for all time $\frac{m g b}{a} = \text{max}$

$$\mu_s = \frac{mgb/a}{mg - mb\omega^2}$$

f) Work done = $\Delta PE = mg\Delta h = \underline{2mgb}$



Problem # 2

aws

Work in MKS units: ignore signs (like charges repel)

a) Gauss's law $\int_S \vec{E} \cdot d\vec{s} = Q/\epsilon_0$

area of plates = A $EA = Q/\epsilon_0$ or $E = \sigma/\epsilon_0$

$\sigma =$ charge per unit area

Energy in field $W = \frac{\epsilon_0}{2} \int_V |\vec{E}|^2 dV$

$$= \frac{\epsilon_0}{2} \left(\frac{\sigma}{\epsilon_0}\right)^2 Ad = \frac{\sigma^2 Ad}{2\epsilon_0} = \frac{\sigma^2 Cd^2}{2\epsilon_0}$$

In general $F_i = - \left(\frac{\partial W}{\partial x_i} \right)_Q$

In this case $F = \sigma^2 A / 2\epsilon_0 = \frac{\sigma^2 C d}{2\epsilon_0}$

b) If potential = $V = \text{const}$

$$E = - \frac{dV}{dx} = - \frac{V}{d}$$

$$W = \frac{\epsilon_0}{2} E^2 Ad = \frac{\epsilon_0}{2} \left(\frac{V}{d}\right)^2 Ad = \frac{\epsilon_0 V^2 A}{2d}$$

$$F_i = + \left(\frac{\partial W}{\partial x_i} \right)_V \Rightarrow \frac{\epsilon_0 V^2 A}{2d^2} = \frac{CV^2}{2d}$$

c) The formula $F = qE$ assumes \vec{E} is a given external field, i.e. not produced by q .

3. A particle of mass m is in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$, where ω is the classical angular frequency. For a quantum particle, the allowed energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad (n = 0, 1, 2, \dots).$$

The corresponding energy eigenfunctions are

$$\psi(x) = H_n(\sqrt{m\omega\hbar} x) \exp(-m\omega x^2/2\hbar),$$

where $H_n(q)$ is a polynomial of degree n [i.e. the highest-power term in $H_n(q)$ is q^n].

- Consider an isotropic *three-dimensional* harmonic oscillator for which $V(r) = \frac{1}{2}m\omega^2 r^2$. Explain how the allowed energy eigenvalues and eigenfunctions can be constructed from the solutions of the one-dimensional problem.
 - What are the three lowest allowed energies for the three-dimensional system? What is the degeneracy of each level?
 - What can you say about the energy of the state $\psi(\vec{r}) = x \exp(-m\omega r^2/2\hbar)$? (e.g. is this a stationary state? If so, what is its energy?)
 - Is it possible for this three-dimensional oscillator to be in a simultaneous eigenstate of the orbital angular momentum operators L_x , L_y , and L_z ? Explain.
 - Suppose that the orbital angular momentum component L_z is measured for the state of part (c). What measured values are possible? What are the probabilities of measuring these values?
 - Again consider the measurement of L_z . Prior to the measurement, what is ΔL_z , the root-mean-square deviation of L_z ? What is ΔL_z immediately after the measurement?
4. The equation of state of one mole of a van der Waals gas is given by

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT.$$

Since the terms distinguishing the van der Waals gas from an ideal gas are small, one can consider the internal energy E to be written as

$$E = E_1(T) + E_2(V),$$

where $E_1(T)$ depends only on temperature and $E_2(V)$ only on volume. Show that the difference in the molar specific heats at constant pressure and at constant volume can be given by

$$C_P - C_V \simeq R \left(1 + \frac{2a}{VRT}\right),$$

under the assumption that $b \ll V$.

QM - (a) Since $V(r) = \frac{1}{2} m \omega^2 (x^2 + y^2 + z^2)$, the problem immediately separates into 3 separate ^{1D} equations in x, y, z , where $\Psi(x, y, z) = \Psi_{n_x}(x) \Psi_{n_y}(y) \Psi_{n_z}(z)$ and $E = E_{n_x} + E_{n_y} + E_{n_z}$. Then $H \Psi = E \Psi$.

(b) $E = (n_x + \frac{1}{2}) \hbar \omega + (n_y + \frac{1}{2}) \hbar \omega + (n_z + \frac{1}{2}) \hbar \omega$
 $= (n_x + n_y + n_z + \frac{3}{2}) \hbar \omega$.

Ground state: $n_x = 0; n_y = 0; n_z = 0 \rightarrow E = \frac{3}{2} \hbar \omega$.
 Nondegenerate.

1ST excited state has $(n_x, n_y, n_z) = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$
 $\rightarrow E = \frac{5}{2} \hbar \omega$ triply degenerate.

2ND excited state: $(2, 0, 0)$ or $(0, 2, 0)$ or $(0, 0, 2)$
 or $(0, 1, 1)$ or $(1, 0, 1)$ or $(1, 1, 0) \rightarrow E = \frac{7}{2} \hbar \omega$
 6-fold degeneracy.

(c) $\Psi(x, y, z) = N x e^{-\frac{m\omega}{2\hbar} x^2} = \Psi_{n_x=1}(x) \cdot \Psi_{n_y=0}(y) \cdot \Psi_{n_z=0}(z)$
 This is a stationary energy eigenstate with $n_x = 1, n_y = 0, n_z = 0$
 $\rightarrow E = \frac{5}{2} \hbar \omega$

(d) $\Psi(\vec{r}) = r \cdot f(r)$ form. $Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$

$\therefore \Psi(r) = \underbrace{F(r)}_{\substack{\uparrow \\ \text{spherically symmetric function}}} \cdot [Y_{1,1} - Y_{1,-1}] \leftarrow \because Y_{1,1} - Y_{1,-1} = -\sqrt{\frac{3}{8\pi}} \cdot \frac{2x}{r}$

Thus Ψ contains both $(l=1, m_l=1)$ and $(l=1, m_l=-1)$ orbital angular momentum states.

$\Psi = \frac{1}{\sqrt{2}} (Y_{1,1} - Y_{1,-1})$ • Spherically symmetric function.

Measured values of L_z are eigenvalues $m_l \hbar$ or $\pm \hbar$.

$P_{+\hbar} = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$; $P_{-\hbar} = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$.

(e) $\Delta L_z = \sqrt{\langle L_z^2 \rangle - \langle L_z \rangle^2}$

$\langle L_z^2 \rangle = \langle \Psi | L_z^2 | \Psi \rangle = \hbar^2$

$\langle L_z \rangle = \langle \Psi | L_z | \Psi \rangle = \frac{1}{2} \left(\underbrace{\langle 1,1 | L_z | 1,1 \rangle}_{\hbar} + \underbrace{\langle 1,-1 | L_z | 1,-1 \rangle}_{-\hbar} \right) = 0$

$\Delta L_z = \hbar$ before measurement

After the measurement, the system will be in a reduced state $\sim Y_{1,1}$ or $Y_{1,-1}$, for which $\langle L_z \rangle = \pm \hbar \Rightarrow \underline{\Delta L_z = 0}$

(f) Since $[L_x, L_y] = i\hbar L_z$ et cycl., simultaneous eigenstates of $L_x, L_y,$ and L_z are precluded except for S-states ($l=0$).

Then $L_x \Psi = L_y \Psi = L_z \Psi = 0$. Answer: yes

$$dQ = dE + dW$$

$$dQ = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV + \underline{P dV}$$

$$\left(\frac{\partial Q}{\partial T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_V + \left[\left(\frac{\partial E}{\partial V}\right)_T + P\right] \left(\frac{\partial V}{\partial T}\right)_P$$

$$\therefore C_P - C_V = \left[\left(\frac{\partial E}{\partial V}\right)_T + P\right] \left(\frac{\partial V}{\partial T}\right)_P$$

From the form of $E = E_1(T) + E_2(V)$,
work done on system at constant T

$$dW = dE_2(V) = \frac{a}{V^2} dV$$

$$\text{i.e. } E_2(V) = -\frac{a}{V}$$

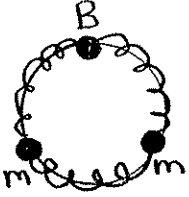
$$\therefore \left(\frac{\partial E}{\partial V}\right)_T = \frac{a}{V^2}$$

$$\text{From EOS, } \left(\frac{\partial T}{\partial V}\right)_P = \frac{1}{R} \left\{ P + \frac{a}{V^2} - \frac{2a(V-b)}{V^3} \right\}$$

$$\therefore C_p - C_v = \left(P + \frac{a}{V^2} \right) \left(\frac{R}{P + \frac{a}{V^2} - \frac{2a(V-b)}{V^3}} \right)$$

$$\text{or } C_p - C_v \approx R \left(1 + \frac{2a}{VRT} \right).$$

5. Three masses B , m , and m are connected by identical, massless springs with spring constants k . The masses and springs are constrained to a smooth, *fixed*, horizontal circle.



- Deduce, but do not solve, the equation whose solutions yield the frequencies of the normal modes of small oscillations about equilibrium.
- Determine if uniform rotational motion is a solution to your equation.
- Determine the frequencies of the normal modes.
- Describe the motion of the masses for each mode.
- Indicate where the assumption of “small oscillations” has been made in your solution.

6. A three-dimensional solid is represented as a collection of N quantum oscillators.

- Assuming the Debye model, show that at sufficiently high temperatures, the internal energy of this system can be approximated by

$$E \simeq 3NkT \left(1 - \frac{3\Theta_D}{8T} \right),$$

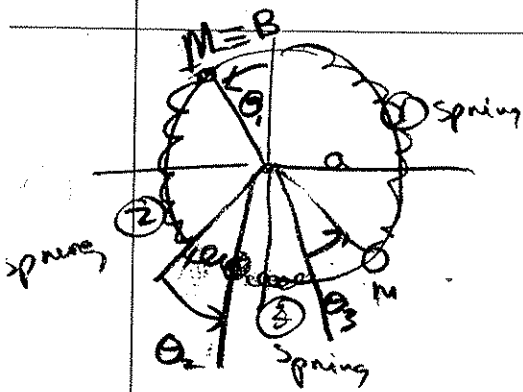
where Θ_D is the Debye temperature, and is related to the cutoff frequency ω_D for the Debye density of states by

$$\Theta_D = \frac{\hbar\omega_D}{k}.$$

- Suppose the solid to be subjected to an isothermal compression. Show that the increase in the solid's internal energy, ΔE , is given by

$$\Delta E = \frac{3}{8}N \left(\frac{\Delta V}{V} \right) k\Theta_D.$$

Solution 5



The constraint is that hoop remains in place and masses on hoop.

$$T = \frac{1}{2} m (\dot{\theta}_2^2 + \dot{\theta}_3^2) + \frac{1}{2} B a^2 \dot{\theta}_1^2$$

Assume masses initially in equilateral triangle

$$V_1 = \frac{1}{2} k a^2 \left(\frac{2}{3} \pi + \theta_1 - \theta_2 - s \right)^2 = \frac{1}{2} k a^2 (\theta_1 - \theta_2)^2$$

↑ spring 1

$$V_2 = \frac{1}{2} k a^2 \left(\frac{2}{3} \pi + \theta_2 - \theta_1 - s \right)^2 = \frac{1}{2} k a^2 (\theta_2 - \theta_1)^2$$

$$V_3 = \frac{1}{2} k a^2 \left(\frac{2}{3} \pi + \theta_3 - \theta_2 - s \right)^2 = \frac{1}{2} k a^2 (\theta_3 - \theta_2)^2$$

$$V = \frac{1}{2} k a^2 \left[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 \right] \quad \text{take } s = \frac{2}{3} \pi$$

$$L = T - V = \frac{m a^2}{2} (\dot{\theta}_2^2 + \dot{\theta}_3^2) + \frac{B a^2}{2} \dot{\theta}_1^2 - \frac{k a^2}{2} \left[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 \right]$$

$$q_i = \theta_1, \theta_2, \theta_3 \quad \rightarrow \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = B a^2 \dot{\theta}_1 \quad , \quad \frac{\partial L}{\partial \dot{\theta}_2} = m a^2 \dot{\theta}_2 \quad , \quad \frac{\partial L}{\partial \dot{\theta}_3} = m a^2 \dot{\theta}_3$$

$$\frac{\partial L}{\partial \theta_1} = -k a^2 [2(\theta_1 - \theta_2) - (\theta_2 - \theta_1)] = -k a^2 (\theta_1 - \theta_2 - \theta_2 + \theta_1) \\ = -k a^2 (2\theta_1 - \theta_2 - \theta_3)$$

$$\frac{\partial L}{\partial \theta_2} = -k a^2 [\theta_2 - \theta_1 - \theta_3 + \theta_2] = -k a^2 (2\theta_2 - \theta_1 - \theta_3)$$

$$\frac{\partial L}{\partial \theta_3} = -k a^2 [-\theta_1 + \theta_3 + \theta_3 - \theta_2] = -k a^2 (2\theta_3 - \theta_1 - \theta_2)$$

$$1) \quad -B a^2 \ddot{\theta}_1 + k a^2 (2\theta_1 - \theta_2 - \theta_3) = 0$$

$$\ddot{\theta}_1 + \frac{k}{B} (2\theta_1 - \theta_2 - \theta_3) = 0$$

$$2) \quad \ddot{\theta}_2 + \frac{k}{m} (2\theta_2 - \theta_1 - \theta_3) = 0$$

$$3) \quad \ddot{\theta}_3 + \frac{k}{m} (2\theta_3 - \theta_1 - \theta_2) = 0$$

For normal modes, assume $\theta_i = A_i e^{i\omega t}$

1) $-\omega^2 A_1 + \frac{k}{B} [2A_1 - A_2 - A_3] = 0$

2) $-\omega^2 A_2 + \frac{k}{M} [2A_2 - A_1 - A_3] = 0$

3) $-\omega^2 A_3 + \frac{k}{M} [2A_3 - A_2 - A_1] = 0$

$$\begin{pmatrix} \frac{2k}{B} - \omega^2 & -\frac{k}{B} & -\frac{k}{B} \\ -\frac{k}{M} & \frac{2k}{M} - \omega^2 & -\frac{k}{M} \\ -\frac{k}{M} & -\frac{k}{M} & \frac{2k}{M} - \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

$\alpha = k/B \rightarrow \beta = k/M$

$\det \begin{vmatrix} 2\alpha - \omega^2 & -\alpha & -\alpha \\ -\beta & 2\beta - \omega^2 & -\beta \\ -\beta & -\beta & 2\beta - \omega^2 \end{vmatrix} = 0$

$\lambda = 2\beta - \omega^2$
 $\omega^2 = 2\beta - \lambda$
 $2\beta - \omega^2 = 2\beta - 2\beta + \lambda = \lambda$

~~$\det \begin{vmatrix} \lambda + \mu & -\alpha & -\alpha \\ -\beta & \lambda & -\beta \\ \beta & \beta & \lambda \end{vmatrix} = 0$~~

$\lambda + \mu = -\omega^2 + 2\alpha$

$\lambda^2(\lambda + \mu) - 2\alpha\beta^2 - 2\alpha\lambda\beta - \beta^2(\lambda + \mu) = 0$

Easy to guess a mode with B at rest & M_1, M_2 symmetric

$\theta_2 = -\theta_3, \theta_1 = 0$

$\ddot{\theta}_3 + k/M [3\theta_3 - \theta_1] = 0$

$\ddot{\theta}_2 + \frac{k}{M} [3\theta_2 - \theta_1] = 0$

$\Rightarrow \ddot{\theta}_3 + \frac{k}{M} 3\theta_3 = 0$

$\Rightarrow \omega_1 = \sqrt{3k/M}$

For others, we can find by brute force,

$$\frac{4k}{m} \omega^2 = \left\{ \begin{array}{l} -\frac{2k}{B} + \frac{3k}{m} + \frac{2k}{B} + \frac{k}{m} = \frac{4k}{m} \\ -\frac{2k}{B} + \frac{3k}{m} - \frac{2k}{B} - \frac{k}{m} = \frac{4k}{B} + \frac{2k}{m} \end{array} \right.$$

$$2\omega^2 = \left\{ \begin{array}{l} \frac{4k}{m} - \frac{4k}{m} = 0 \\ \frac{4k}{m} - \frac{2k}{m} + \frac{4k}{B} = \frac{2k}{m} + \frac{4k}{B} \end{array} \right.$$

$$\omega = 0, \quad \omega = \sqrt{\frac{k}{m} + \frac{2k}{B}} = \frac{k(B+m)}{mB}$$

Uniform Translation
(we did not fix com of mass)

So $\omega = 0 \Rightarrow \lambda = 2\beta = \text{root?}$ (give as a hint)

$$(\beta - \omega^2)^2 (2\alpha - \omega^2) - 2\alpha\beta^2 - 2\alpha\beta(2\beta - \omega^2) - \beta^2(2\alpha - \omega^2) = 0$$

$$\omega = 0: \quad 4\beta^2 2\alpha - 2\alpha\beta^2 - 2\alpha\beta 2\beta - 2\alpha\beta^2 = 0$$

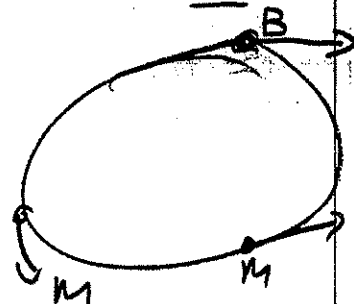
$$8\alpha\beta^2 - 2\alpha\beta^2 - 4\alpha\beta^2 - 2\alpha\beta^2 = 0 \quad \checkmark$$

— ii —

Since $\omega = \sqrt{\frac{k}{m} + \frac{2k}{B}}$ one spring with ak is not stretching

so normal mode must be opposing motion

(Can substitute for ω and then determine A_1, A_2, A_3 .)



$$\lambda^2(\lambda + \mu) - 2\alpha\beta^2 - 2\alpha\beta\lambda - \beta^2(\lambda + \mu) = 0$$

one root: $\omega^2 = 3k/m$, $\lambda = 2\beta - \omega^2 = \frac{2k}{B} - \frac{3k}{M} = -\frac{k}{M}$

$$\mu = 2\alpha - 2\beta = \frac{2k}{B} - \frac{2k}{M}$$

so: $\lambda^2(\lambda + \mu) - 2\alpha\beta^2 - 2\alpha\beta\lambda - \beta^2(\lambda + \mu) = (1 + \frac{k}{M})[A\lambda^2 + B'\lambda + C]$

determine A, B, C, then use quadratic formula

$$RHS = A\lambda^3 + \lambda^2[B + \frac{A\beta}{M}] + \lambda[C + \frac{B'\beta}{M}] + \frac{k}{M}C$$

$$\Rightarrow \underline{A=1}$$

$$\lambda^2: B + \frac{k}{M} = \frac{2k}{B} - \frac{2k}{M} \Rightarrow \underline{B' = \frac{2k}{B} - \frac{3k}{M}}$$

$$\lambda^0: \frac{k}{M}C = -2\frac{k}{B}\frac{k^2}{M^2} - \frac{k^2}{M^2}\left(\frac{2k}{B} - \frac{2k}{M}\right) = -\frac{2k^3}{BM^2} - \frac{2k^3}{BM^2} + \frac{2k^3}{M^3} = \frac{-4k^3}{BM^2} + \frac{2k^3}{M^3}$$

$$\underline{C = \frac{2k^2}{M^2} - \frac{7k^2}{BM}}$$

check: $\lambda: C + \frac{B'\beta}{M} = \frac{2k^2}{M^2} - \frac{4k^2}{BM} + \frac{2k^2}{BM} - \frac{3k^2}{M^2} = \frac{-k^2}{M^2} - \frac{2k^2}{BM}$

λ LHS: $-2\alpha\beta - \beta^2 = -\frac{2k^2}{BM} - \frac{k^2}{M^2} \checkmark$ check \checkmark

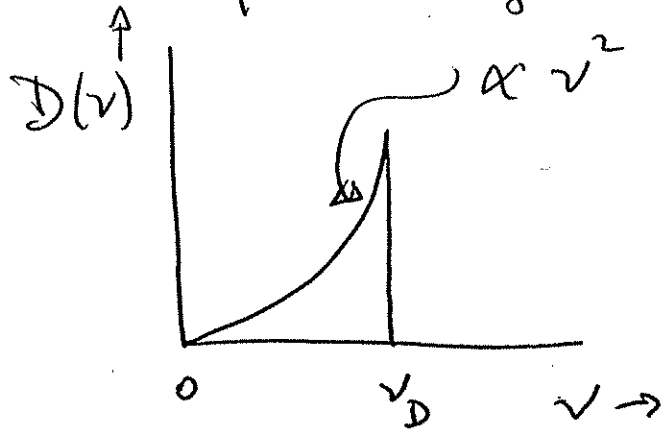
other Modes, roots of $A\lambda^2 + B'\lambda + C = 0$

$$\lambda = \frac{-B' \pm \sqrt{B'^2 - 4C}}{2} = \frac{-\frac{2k}{B} + \frac{3k}{M} \pm \sqrt{\frac{4k^2}{B^2} + \frac{9k^2}{M^2} - \frac{12k^2}{MB} - \frac{8k^2}{M^2}}}{2}$$

$$2\lambda = \frac{4k}{M} \omega^2 = -\frac{2k}{B} + \frac{3k}{M} \pm \sqrt{\frac{4k^2}{B^2} + \frac{k^2}{M^2} + \frac{4k^2}{MB}}$$

$$= -\frac{2k}{B} + \frac{3k}{M} \pm \left(\frac{2k}{B} + \frac{k}{M}\right)$$

(a) Debye Density of States



$$D(v)dv = 4\pi v^2 \left(\frac{1}{c_l^3} + \frac{2}{c_t^3} \right) dv$$

where the polarizations and wave speeds have been incorporated into the expression for the number of states per unit volume with v between v and $v+dv$.

$$\int_0^{v_D} 4\pi V v^2 \left(\frac{1}{c_l^3} + \frac{2}{c_t^3} \right) dv = 3N$$

and so

$$v_D = \left[\frac{9N}{4\pi V} \frac{1}{\left(\frac{1}{c_l^3} + \frac{2}{c_t^3} \right)} \right]^{1/3}$$

$$E = \int_0^{\nu_D} \frac{V h \nu D(\nu) d\nu}{e^{h\nu/kT} - 1}$$

$$= V 4\pi \left(\frac{1}{c^3} + \frac{2}{c^3 t} \right) h \int_0^{\nu_D} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

$$E = \frac{9N h}{\nu_D^3} \int_0^{\nu_D} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

Let $x = \frac{h\nu}{kT}$

$$E = \frac{9N kT}{x_D^3} \int_0^{x_D} \frac{x^3 dx}{e^x - 1}$$

At high T , x is small, and so

$$\int_0^{x_D} \frac{x^3 dx}{e^x - 1} \approx \int_0^{x_D} x^2 \left(1 - \frac{x}{2} \right) dx$$

\therefore

$$\therefore E \approx \frac{9NkT}{x_D^3} \left(\frac{x_D^3}{3} - \frac{x_D^4}{8} \right)$$

$$E \approx 3NkT - \frac{9}{8} NkT x_D$$

But since $x_D = \frac{h\nu_D}{kT} = \frac{k\omega_D}{kT} = \frac{\omega_D}{T}$

and so, $E \approx 3NkT - \frac{9}{8} Nk\omega_D$

(b) To first order, $\Delta E = \left(\frac{\partial E}{\partial V} \right)_T \Delta V$

$$\left(\frac{\partial E}{\partial V} \right)_T = -\frac{9}{8} Nk \left(\frac{\partial \omega_D}{\partial V} \right)_T$$

$$= -\frac{9}{8} Nk \left(\frac{\partial \nu_D}{\partial V} \right)_T$$

Given the earlier expression for ν_D , we get

$$\left(\frac{\partial \nu_D}{\partial V} \right)_T = -\frac{1}{3} \frac{\nu_D}{V}$$

$$\therefore \Delta E = -\frac{9}{8} Nk \left(-\frac{1}{3} \frac{\nu_D}{V} \right) \Delta V = \frac{3}{8} N \left(\frac{\Delta V}{V} \right) k\omega_D$$

7. The Fourier transform $\hat{\phi}(\vec{x}, k)$ of any electromagnetic potential $\phi(\vec{x}, t)$ satisfies the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\hat{\phi}(\vec{x}, k) = -4\pi f(\vec{x}, \omega)$$

for each value of the frequency ω . Here $k = \omega/c$ is the wave number associated with ω , and f is the source function, which is assumed to be known. Such an equation can be solved using a Green's function.

- What is a Green's function? How is it defined? How is it used to solve an equation such as the Helmholtz equation above?
- Derive the Green's function for this equation using any method you like.

8. A particle of mass m is in an infinitely deep one-dimensional square well potential

$$V(x) = 0 \text{ for } 0 \leq x \leq a; \text{ and } V(x) = \infty \text{ elsewhere.}$$

At time $t = 0$ the normalized wave function $\Psi(x, t)$ is given by

$$\Psi(x, 0) = \frac{2}{\sqrt{3a}} \left[1 - \cos \frac{4\pi x}{a} \right] \text{ for } 0 \leq x \leq \frac{a}{2};$$

and $\Psi(x, 0) = 0$ elsewhere, including the right half of the well.

- Is this a bound state? Is it stationary? Does it have a well-defined parity?
- What is the probability that at time $t = 0$ the particle will be found between $x = 0$ and $x = a/4$?
- What is the expectation value of the momentum $\langle p_x \rangle$ at $t = 0$?
- At time $t = 0$, what is the probability P that a measurement of the total energy E will yield a value *smaller* than $\pi^2 \hbar^2 / ma^2$?
- What is the wave function $\Psi(x, t)$ for $t > 0$? [A clear explanation is more important than calculational details.]
- What is the expectation value $\langle H \rangle$ of the Hamiltonian for $t > 0$?

The following integral may be useful: $\int_0^{\pi/2} \sin y \cos 4y \, dy = -\frac{1}{15}$

A-Styl - Problem # 7

Given $(\nabla^2 + \kappa^2)\phi = -4\pi f$

where $\phi = \phi(\vec{x}, \omega)$ and $f = f(\vec{x}, \omega)$

The Green's fn. has the property:

$$(\nabla^2 + \kappa^2)G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

So that if $\phi(\vec{x}) = \int G(\vec{x}, \vec{x}') f(\vec{x}') d\vec{x}'$,

$$\begin{aligned} \text{then } (\nabla^2 + \kappa^2)\phi(\vec{x}) &= -4\pi \int \delta(\vec{x} - \vec{x}') f(\vec{x}') d\vec{x}' \\ &= -4\pi f(\vec{x}) \end{aligned}$$

We can also "build in" boundary conditions.

In this case we assume there are no spatial boundaries.

Define the Fourier transform as follows.

$$G(\vec{x}, \omega) = \frac{1}{(2\pi)^3} \int \tilde{G}(\vec{\kappa}, \omega) e^{i\vec{\kappa} \cdot \vec{x}} d\vec{\kappa}$$

Here we have temporarily redefined $\vec{x} - \vec{x}' \rightarrow \vec{x}$.

$$\begin{aligned} (\nabla^2 + \frac{\omega^2}{c^2})G &= \frac{1}{(2\pi)^3} \int \tilde{G}(\vec{\kappa}, \omega) \left(-\kappa^2 + \frac{\omega^2}{c^2}\right) e^{i\vec{\kappa} \cdot \vec{x}} d\vec{\kappa} \\ &= -4\pi \delta(\vec{x}) \end{aligned}$$

Take the inverse Fourier transform.

$$\tilde{G}(\vec{\kappa}, \omega) \left(\frac{\omega^2}{c^2} - \kappa^2\right) = -4\pi$$

$$\vec{a} = \frac{4\pi}{(\kappa - w/c)(\kappa + w/c)}$$

Now take the inverse transform

$$G(\vec{x}, w) = \frac{1}{(2\pi)^3} \int \frac{4\pi}{(\kappa - w/c)(\kappa + w/c)} e^{i\vec{\kappa} \cdot \vec{x}} d\Omega \kappa^2 d\kappa$$

The integration is simpler if we take $\vec{\kappa}$ as the

polar axis. Then $\int e^{i\vec{\kappa} \cdot \vec{x}} d\Omega$

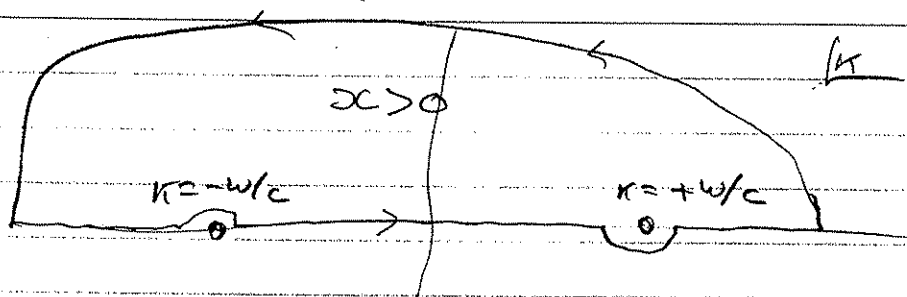
$$= \frac{2\pi}{i\kappa x} (e^{i\kappa x} - e^{-i\kappa x})$$

$$G(\vec{x}, w) = \frac{2}{(2\pi)^3 i x} \int_0^{\infty} \frac{1}{(\kappa^2 - w^2/c^2)} (e^{i\kappa x} - e^{-i\kappa x}) \kappa d\kappa$$

Replace $\kappa \rightarrow -\kappa$ in the second term.

$$= \frac{2}{(2\pi i) x} \int_{-\infty}^{+\infty} \frac{e^{i\kappa x} \kappa d\kappa}{(\kappa - w/c)(\kappa + w/c)}$$

Use the following contour in the complex κ plane.



$$G(\bar{x}, w) = \frac{2(2\pi i) \lim_{\kappa \rightarrow w/c} (\kappa - w/c) \times \frac{\kappa e^{i\kappa x}}{(\kappa - w/c)(\kappa + w/c)}}{(2\pi i)x}$$

$$\frac{2}{x} \frac{w/c e^{iwx/c}}{2w/c} = \frac{e^{iwx/c}}{x}$$

QM- (a) Bound; not stationary; parity not well-defined

$$(b) P = \int_0^{a/4} |\psi(x,0)|^2 dx$$

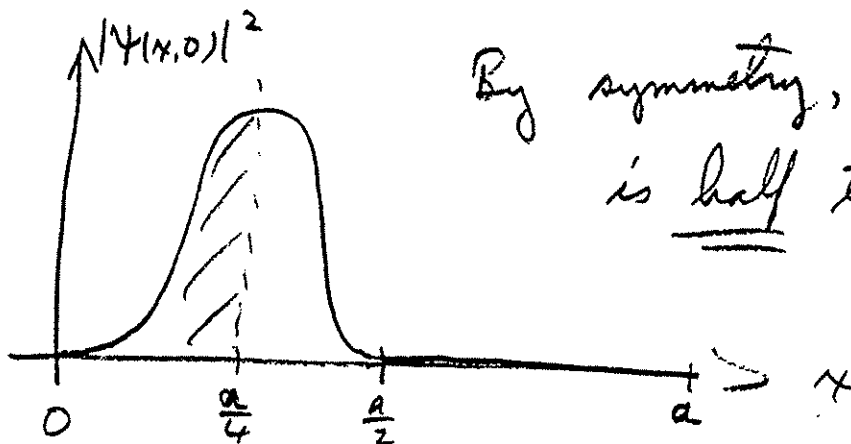
$$= \left(\frac{2}{\sqrt{3a}}\right)^2 \int_0^{a/4} \left(1 - 2\cos\frac{4\pi x}{a} + \cos^2\frac{4\pi x}{a}\right) dx$$

$$= \frac{4}{3a} \left[\frac{a}{4} - 2 \cdot \frac{a}{4\pi} \underbrace{\sin\frac{4\pi x}{a}}_{\text{zero}} \Big|_0^{a/4} + \frac{1}{2} \int_0^{a/4} (1 + \cos\frac{8\pi x}{a}) dx \right]$$

$$= \frac{4}{3a} \left[\underbrace{\frac{a}{4} + \frac{a}{8}}_{\frac{3a}{8}} + \frac{1}{2} \cdot \frac{a}{8\pi} \underbrace{\sin\frac{8\pi x}{a}}_{\text{zero}} \Big|_0^{a/4} \right]$$

$$P = \frac{1}{2}$$

The easy way to obtain this result is to note that $|\psi(x,0)|^2$ is symmetric about $x = \frac{a}{4}$:



By symmetry, the shaded area is half the total area.

QM- (c)

2/2

$$\langle P_x \rangle = \frac{\hbar}{i} \int_{-\infty}^{\infty} \psi^*(x,0) \frac{d}{dx} \psi(x,0) dx$$

where $\psi^* = \psi$.

$$\begin{aligned} \text{Integrate by parts: } \int_{-\infty}^{\infty} \psi \frac{d\psi}{dx} dx &= \int \psi d\psi \\ &= \underbrace{\psi^2 \Big|_{-\infty}^{\infty}}_{\text{zero}} - \int_{-\infty}^{\infty} \psi \frac{d\psi}{dx} dx \end{aligned}$$

$\therefore \int_{-\infty}^{\infty} \psi \frac{d\psi}{dx} dx = 0$ for any real ψ which vanishes at $x = \pm\infty$.

$\therefore \boxed{\langle P_x \rangle = 0}$ (or calculate explicitly using the given ψ)

(d) The only possible measured values of E are the energy eigenvalues.

For the square well, $E_n = \frac{\hbar^2 k_n^2}{2m}$ where $k_n = \frac{n\pi}{a}$.

$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2} ; E_2 = \frac{2\pi^2 \hbar^2}{ma^2} ; \dots$$

The given E lies between E_1 and E_2 .

Therefore the only energy eigenvalue less than E is E_1 . The probability is $P = |\langle \phi_1, \psi(x,0) \rangle|^2$

(continued)

QM-d. (continued) 5/3
 $P = |c_1|^2$, where $c_1 = \langle \phi_1(x), \psi(x,0) \rangle$

$$c_1 = \int_0^{a/2} \frac{2}{\sqrt{3a}} \left(1 - \cos \frac{4\pi x}{a}\right) \cdot \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} dx$$

(Full credit for these results.)

5 extra points for correct integration & finding P:

$$\begin{aligned} c_1 &= \frac{2\sqrt{2}}{\sqrt{3a^2}} \left[\int_0^{a/2} \sin \frac{\pi x}{a} dx - \int_0^{a/2} \sin \frac{\pi x}{a} \cos \frac{4\pi x}{a} dx \right] \\ &= \frac{2\sqrt{2}}{\sqrt{3a^2}} \left[-\frac{a}{\pi} \underbrace{(\cos \frac{\pi}{2} - \cos 0)}_{-1} - \frac{a}{\pi} \int_0^{\pi/2} \sin y \cos 4y dy \right] \\ & \qquad \qquad \qquad -\frac{1}{15} \text{ (given at bottom of page)} \\ &= \frac{2\sqrt{2}}{\sqrt{3a^2}} \cdot \frac{a}{\pi} \left(1 + \frac{1}{15}\right) = \frac{32\sqrt{2}}{15\pi\sqrt{3}} \end{aligned}$$

$$P = |c_1|^2 = \frac{2}{3} \left(\frac{32}{15\pi}\right)^2 = \frac{2048}{675\pi^2}$$

Since $\frac{32}{\pi} \approx 10$, $P \approx \frac{2}{3} \times \left(\frac{10}{15}\right)^2 = \left(\frac{2}{3}\right)^3 = \frac{8}{27} \approx \underline{\underline{0.3}}$

QM - ② Expand $\Psi(x, t) = \sum_n C_n \phi_n(x) e^{-i \frac{E_n}{\hbar} t}$

where $H \phi_n = E_n \phi_n$. The $\{C_n\}$ are given by

$$C_n = \langle \phi_n, \Psi(x, 0) \rangle$$

$$= \int_0^{a/2} \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \cdot \frac{2}{\sqrt{3a}} \left(1 - \cos \frac{4\pi x}{a} \right) dx$$

(f.) Recall that $\frac{d\langle H \rangle}{dt} = 0$ if $\frac{\partial H}{\partial t} = 0$.

Therefore $\langle H \rangle_n$ at time t can be found using $\Psi(x, 0)$, at $t = 0$

$$\langle H \rangle = \langle T \rangle = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \Psi^* \frac{d^2 \Psi}{dx^2} dx$$

$$= +\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \left| \frac{d\Psi}{dx} \right|^2 dx \quad \frac{d\Psi}{dx} = -\frac{2}{\sqrt{3a}} \cdot \frac{4\pi}{a} \sin \frac{4\pi x}{a}$$

$$= \frac{\hbar^2}{2m} \cdot \frac{64\pi^2}{3a^3} \int_0^{\frac{a}{2}} \sin^2 \frac{4\pi x}{a} dx \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$= \frac{32\pi^2 \hbar^2}{3ma^3} \cdot \frac{1}{2} \int_0^{\frac{a}{2}} \left(1 - \cos \frac{4\pi x}{a} \right) dx$$

$$\langle H \rangle = \frac{8\pi^2 \hbar^2}{3ma^2}$$