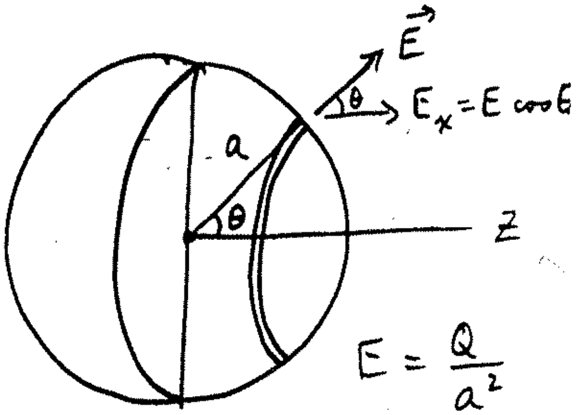
a. A solid conducting sphere of radius a carries a static charge Q. The sphere is cut in half, with the hemispheres remaining in contact. Calculate the force of repulsion between the hemispheres.

b. Three identical isolated uncharged spherical conductors are arranged at the corners of an equilateral triangle. A wire from a battery of unknown voltage is then touched to each sphere in turn. After the first sphere has been touched by the wire, it is found to have a charge Q_1. After the second sphere has been touched by the wire, it is found to have a charge Q_2. In terms of these charges, what is the charge on the third sphere, after it has been touched by the wire?

E & M #1

ⓐ



$$E = \frac{Q}{a^2}$$

$$F_x = \int \tfrac{1}{2}\sigma E_x \, dA \qquad \sigma = \frac{Q}{4\pi a^2}$$

$$= \frac{1}{2}\cdot\frac{Q}{4\pi a^2}\int_{\theta=0}^{\pi/2} E\cos\theta \cdot a^2 \cdot 2\pi\sin\theta \, d\theta$$

$$= \frac{1}{4} Q \cdot \frac{Q}{a^2} \cdot \left.\frac{\cos^2\theta}{2}\right|_{\pi/2}^{0} = \frac{Q^2}{8a^2}$$

ⓑ

$V_i = \sum_j p_{ij} q_j$, where the p_{ij} are the coefficients of potential.

Let V = battery voltage. By symmetry, $\begin{cases} p_{11} = p_{22} = p_{33} \equiv p, \\ p_{21} = p_{31} = p_{32} \equiv p' \end{cases}$

Then
$$V = p_{11}Q_1 \xrightarrow{\qquad} V = pQ_1 \xrightarrow{\qquad} p = V/Q_1$$
$$V = p_{21}Q_1 + p_{22}Q_2 \xrightarrow{\qquad} V = p'Q_1 + pQ_2 \rightarrow p' = \frac{V}{Q_1}\left(1 - \frac{Q_2}{Q_1}\right)$$
$$V = p_{31}Q_1 + p_{32}Q_2 + p_{33}Q_3 \xrightarrow{\qquad} V = p'(Q_1 + Q_2) + pQ_3$$

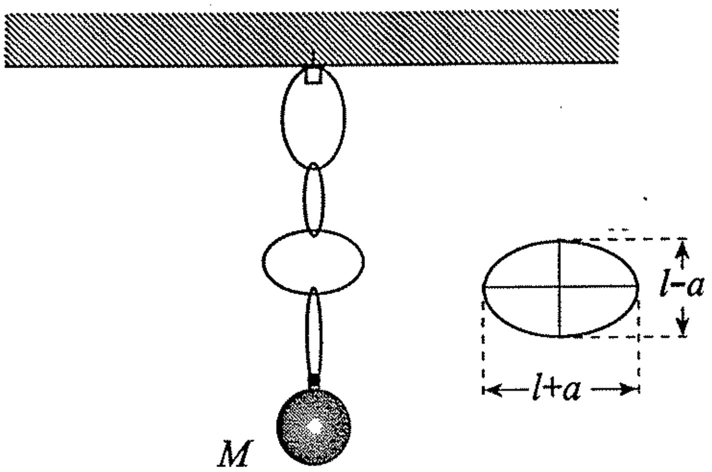
$$V = \frac{V}{Q_1}\left(1 - \frac{Q_2}{Q_1}\right)(Q_1 + Q_2) + \frac{V}{Q_1}Q_3 \implies Q_3 = Q_1 - \left(1 - \frac{Q_2}{Q_1}\right)(Q_1 + Q_2) = \frac{Q_2^2}{Q_1}$$

The upper end of a hanging chain is fixed while the lower end is attached to a mass M. The massless links of the chain are ellipses with major axes $l+a$ and minor axes $l-a$. The links can place themselves only with either the major axis or the minor axis vertical. As an example, the figure below shows a four-link chain in which the major axes of the first and the fourth links and the minor axes of the second and the third links are vertical.

Assume that a chain of this type has N links and is in thermal equilibrium with a heat reservoir at temperature T. Also, assume that the mass M and the center of each link can move only in the vertical direction (i.e., any horizontal motion of the mass or the links — such as, e.g., swinging sideways — is not possible in this system).

(a) Find the partition function for this system.

(b) Find the average length of the chain. What would be the length in the low-temperature limit ($T \to 0$), and in the region of very high temperatures? ($kT \gg Mga$).



(a) Suppose that n links have their <u>major</u> axes vertical, and hence $N-n$ links have their minor axer vertical. In such circumstances the total length of the chain is:

$$L(n) = n(l+a) + (N-n)(l-a)$$

The total energy is:

$$E(n) = -Mg\,L(n) = -Mg\,n(l+a) - Mg(N-n)(l-a)$$

$$= -E_1 n - E_2(N-n)$$

where $E_1 = Mg(l+a)$, $E_2 = Mg(l-a)$

The number of possible states with n links having major axes vertical is:

$$g(n) = \frac{N!}{n!(N-n)!}$$

The partition function is:

$$Z = \sum_{n=0}^{N} g(n)\, e^{-\frac{E(n)}{kT}} = \sum_{n=0}^{N} \frac{N!}{n!(N-n!)}\, e^{\frac{E_1 n}{kT}}\, e^{\frac{E_2(N-n)}{kT}}$$

$$= \left(e^{E_1/kT} + e^{E_2/kT}\right)^N$$

$\left(\text{using } \sum_k \binom{N}{k} a^{N-k} b^k = (a+b)^N\right)$

(b). The average energy is (this is a well-known formula; students can use it without a proof):

$$\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z$$

$$= kT^2 N \frac{\partial}{\partial T} \ln\left(e^{\epsilon_1/kT} + e^{\epsilon_2/kT}\right) =$$

$$= kT^2 N \; \frac{\frac{E_1}{k}\left(-\frac{1}{T^2}\right)e^{\epsilon_1/kT} + \frac{E_2}{k}\left(-\frac{1}{T^2}\right)e^{\epsilon_2/kT}}{e^{\epsilon_1/kT} + e^{\epsilon_2/kT}}$$

$$= -N \; \frac{E_1 e^{\epsilon_1/kT} + E_2 e^{\epsilon_2/kT}}{e^{\epsilon_1/kT} + e^{\epsilon_2/kT}}$$

$$= -NMg \frac{(l+a)e^{Mg(l+a)/kT} + (l-a)e^{Mg(l-a)/kT}}{e^{Mgl/kT} \cdot e^{+Mga/kT} + e^{Mgl/kT} e^{-Mga/kT}}$$

$$= \frac{-NMg}{e^{Mgl/kT}} \times \frac{l e^{Mgl/kT}\left(e^{Mga/kT} + e^{-Mga/kT}\right) + a e^{Mgl/kT}\left(e^{Mga/kT} - e^{-Mga/kT}\right)}{e^{Mga/kT} + e^{-Mga/kT}}$$

$$= -NMg\left[l + a \tanh\left(\frac{Mga}{kT}\right)\right]$$

Since we can also write: $\langle E \rangle = -Mg \langle L \rangle$, we obtain:

$$\langle L \rangle = N\left[l + a \tanh\left(\frac{Mga}{kT}\right)\right]$$

$T \to 0$, tanh → 1, so $\langle L \rangle \to N(l+a)$, i.e., the lowest-$\epsilon$ state;
$kT \gg Mga$, $\langle L \rangle \to N(l + Mga^2/kT) \to Nl$, i.e., ~~so~~ "fifty-fifty" situation.

OSU Physics Comprehensive Exam No. 88 27-28 March 2000 Problem #3

A one-dimensional harmonic oscillator with mass m and natural frequency ω has the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 .$$

At time $t = 0$, the system is in a state for which the probability that a measurement of the energy would yield the ground state energy is 1/2, the probability that a measurement of the energy would yield the first excited state energy is 1/2, and the expectation value of the momentum p_x is $\sqrt{m\hbar\omega/2}$. This information completely specifies the initial state of the system.

a) Determine the initial quantum state of the system ($|\psi(0)\rangle$).

b) Find the expectation value of the position x at a later time t.

c) Find Δx, the root-mean-square deviation of the position x, at time t.

Harmonic Oscillator

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

energies : $E_n = (n+\frac{1}{2})\hbar\omega$ $\quad n = 0, 1, 2, \ldots$

eigenstates : $|n\rangle$ $\quad$ "

w/ $\quad H|n\rangle = E_n|n\rangle$

a) $P(E_0) = \frac{1}{2} = |\langle 0|\psi(0)\rangle|^2$

$P(E_1) = \frac{1}{2} = |\langle 1|\psi(0)\rangle|^2$

expand $|\psi\rangle$ as $\quad |\psi\rangle = \sum_n a_n |E_n\rangle$

$$\Rightarrow \left.\begin{array}{l} |\langle 0|\psi(0)\rangle|^2 = |a_0|^2 = \frac{1}{2} \\ |\langle 1|\psi(0)\rangle|^2 = |a_1|^2 = \frac{1}{2} \end{array}\right\} \begin{array}{l}\text{add to } 1 \\ \Rightarrow \text{rest} = 0\end{array}$$

overall phase not measurable ⇒ let a_0 be real, +

$$\Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} e^{i\phi}|1\rangle$$

$$\langle p\rangle = \langle \psi(0)|p|\psi(0)\rangle = \sqrt{\frac{m\hbar\omega}{2}}$$

from ref sheet: $\langle k|p|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left\{\sqrt{n+1}\,\delta_{k,n+1} - \sqrt{n}\,\delta_{k,n-1}\right\}$

$$\Rightarrow \langle p\rangle = \frac{1}{\sqrt{2}}\left[\langle 0| + e^{-i\phi}\langle 1|\right] p \, \frac{1}{\sqrt{2}}\left[|0\rangle + e^{+i\phi}|1\rangle\right]$$

$$= \frac{1}{2}\left[e^{i\phi}\langle 0|p|1\rangle + e^{-i\phi}\langle 1|p|0\rangle\right]$$; since $\langle n|p|n\rangle = 0$

$$= \frac{1}{2}\left\{e^{i\phi}\, i\sqrt{\frac{m\hbar\omega}{2}}\,(-\sqrt{1}) + e^{-i\phi}\, i\sqrt{\frac{m\hbar\omega}{2}}\,(\sqrt{1})\right]$$

$$= \frac{1}{2}\sqrt{\frac{m\hbar\omega}{2}}\left[-ie^{i\phi} + ie^{-i\phi}\right]$$

$$= \frac{1}{2}\sqrt{\frac{m\hbar\omega}{2}}\left[-i\cdot 2i\sin\phi\right]$$

$$\langle p\rangle = \sqrt{\frac{m\hbar\omega}{2}}\sin\phi = \sqrt{\frac{m\hbar\omega}{2}}$$ from statement of prob

$$\Rightarrow \sin\phi = 1 \Rightarrow \boxed{\phi = \frac{\pi}{2}}$$

$$\Rightarrow |\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{i\frac{\pi}{2}}|1\rangle\right]$$

$$\boxed{|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]}$$

b) at later time

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{E_0}{\hbar}t}|0\rangle + ie^{-i\frac{E_1}{\hbar}t}|1\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{-i\frac{\omega}{2}t}|0\rangle + ie^{-i\frac{3\omega}{2}t}|1\rangle\right]$$

$$\langle x\rangle(t) = \langle \psi(t)|x|\psi(t)\rangle$$

$$\langle k|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}\right)$$ from eqn. sheet

$$\Rightarrow \langle x\rangle(t) = \frac{1}{\sqrt{2}}\left[e^{i\frac{\omega t}{2}}\langle 0| - i e^{i\frac{3\omega}{2}t}\langle 1|\right] x \frac{1}{\sqrt{2}}\left[e^{-i\frac{\omega t}{2}}|0\rangle + i e^{-i\frac{3\omega}{2}t}|1\rangle\right]$$

$$= \frac{1}{2}\left[i e^{-i\omega t}\langle 0|x|1\rangle - i e^{i\omega t}\langle 1|x|0\rangle\right]$$

$$= \frac{1}{2}\, i\sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t} - e^{i\omega t}\right]$$

$$= \frac{1}{2}\, i\sqrt{\frac{\hbar}{2m\omega}}\left(-2i\sin\omega t\right)$$

$$\boxed{\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\sin\omega t}$$

c) $$\Delta x = \sqrt{\langle (x-\langle x\rangle)^2\rangle}$$

$$= \sqrt{\langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\rangle}$$

$$= \sqrt{\langle x^2\rangle - 2\langle x\rangle\langle x\rangle + \langle x\rangle^2}$$

$$= \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

So ~~he~~ we need $\langle x^2\rangle$

$$\langle x^2\rangle(t) = \langle\psi(t)| x^2 |\psi(t)\rangle$$

insert closure

$$= \langle\psi(t)| x\cdot 1\cdot x |\psi(t)\rangle$$

$$= \langle\psi(t)| x \cdot \sum_n |n\rangle\langle n| \cdot x |\psi(t)\rangle$$

$$= \sum_n \langle\psi(t)| x |n\rangle\langle n| x |\psi(t)\rangle$$

$$= \sum_n \left|\langle\psi(t)| x |n\rangle\right|^2$$

$$= \sum_n \left| \frac{1}{\sqrt{2}}\left(\langle 0| e^{i\frac{\omega}{2}t} - i\langle 1| e^{i\frac{\omega^3}{2}t}\right) x |n\rangle\right|^2$$

$$= \sum_n \left| \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{\hbar}{2m\omega}} \left(e^{i\frac{\omega}{2}t}\sqrt{n}\,\delta_{0,n-1} - i e^{i\frac{3\omega}{2}t}\left(\sqrt{n+1}\,\delta_{1,n+1} + \sqrt{n}\,\delta_{1,n}\right.\right.\right.$$

$$= \frac{1}{2}\frac{\hbar}{2m\omega}\cdot\left[1 + 1 + 2\right]$$

$$\langle x^2\rangle(t) = \frac{\hbar}{m\omega}$$

$$\Rightarrow \Delta x(t) = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

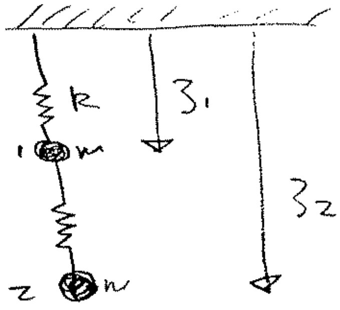
$$= \sqrt{\frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\sin^2\omega t}$$

$$\boxed{\Delta x(t) = \sqrt{\frac{\hbar}{m\omega}}\sqrt{1 - \frac{1}{2}\sin^2\omega t}}$$

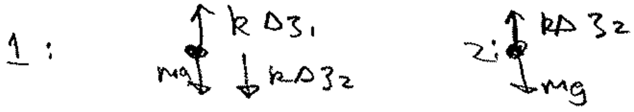
A mass m is hung from a fixed support on a weightless spring of spring constant k and length ℓ. A second equal mass m is hung from the first one by an identical spring. Each spring exerts a force only along the line connecting its ends, but may swing freely in the plane of the paper.

1. Find the equilibrium position of each mass.
2. Find the normal frequencies for small vibrations in the vertical direction and describe the corresponding normal modes.
3. For the special case in which the unstretched length of each spring is $\ell = 2mg/k$, find the normal frequencies for small vibrations in the horizontal direction. (*Hint:* remember that the oscillations are small.)

CM | 500



let l = unstretched length
($= 2mg/k$)



ⓐ Equilibrium: $\sum F_z = 0$

2: $-k(z_2 - z_1 - l) + mg = 0$

1: $k(z_2 - z_1 - l) - k(z_1 - l) + mg = 0$

$$②+① : -k\cancel{z_2} + \cancel{kz_1} + \cancel{kl} + mg + \cancel{kz_2} - \cancel{kz_1} - \cancel{kl} - kz_1 + kl + mg = 0$$

$$2mg + kl = kz_1 \Rightarrow \quad z_1 = l + \frac{2mg}{k} = \frac{4mg}{k}$$

$$② \quad z_2 - z_1 - l = mg/k \Rightarrow z_2 = mg/k + l + z_1 = mg/k + l + l + \frac{2mg}{k} =$$

$$z_2 = 3mg/k + 2l = 7mg/k$$

ⓑ Vertical Oscillation

$$\text{let} \quad \varepsilon_1 = z_1 - \frac{4mg}{k} \quad , \quad \varepsilon_2 = z_2 - 7mg/k$$

$$K = \frac{m}{2}(\dot\varepsilon_1^2 + \dot\varepsilon_2^2) \quad , \quad V = \tfrac{1}{2}k\varepsilon_1^2 + \tfrac{1}{2}k(\varepsilon_2 - \varepsilon_1)^2$$

$$L = T - V = \frac{m}{2}(\dot\varepsilon_1^2 + \dot\varepsilon_2^2) - \tfrac{1}{2}k\varepsilon_1^2 - \tfrac{1}{2}k(\varepsilon_2 - \varepsilon_1)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{dL}{dq} = 0 \ : \quad m\ddot\varepsilon_1 + k\varepsilon_1 - k(\varepsilon_2 - \varepsilon_1) = 0 \qquad -m\omega^2 A + \cancel{kA} - k(B-A) = 0$$

$$m\ddot\varepsilon_2 + k(\varepsilon_2 - \varepsilon_1) = 0 \qquad -m\omega^2 B + k(B-A)$$

$$\text{let} \quad \varepsilon_1 = Ae^{i\omega t} \qquad \varepsilon_2 = Be^{i\omega t} \longrightarrow \begin{pmatrix} -m\omega^2 + 2k & -k \\ -k & k - m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\det(\,) = 0 : -3m\omega^2 k + m^2\omega^4 + k^2 : \quad \omega^2 = \frac{3mk \pm \sqrt{9m^2k^2 - 4k^2m^2}}{2m^2} = \frac{k}{2m}(3 \pm \sqrt{5})$$

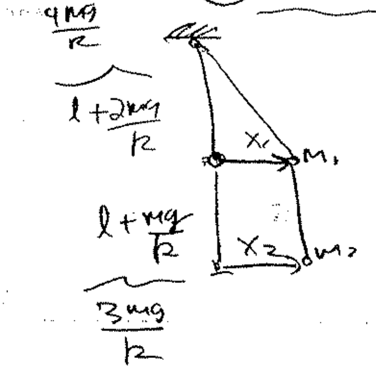
$$\begin{pmatrix} 2k - \frac{k}{2}(3 \pm \sqrt{5}) & -k \\ -k & k - \frac{k}{2}(3 \pm \sqrt{5}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \qquad B =$$

$\left(2-\frac{3\pm\sqrt{5}}{2}\right)A \quad = B$

$+ \Rightarrow$ m_1 and m_2 in same direct

$- \Rightarrow$ m_1 & m_2 opposite

ⓒ Horizontal Oscillation



$$K = \tfrac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2)$$

$$V = \tfrac{1}{2}k\left[\sqrt{\frac{16m^2g^2}{k^2}+x_1^2}-\frac{2mg}{k}\right]^2+\tfrac{1}{2}k\left(\sqrt{\frac{9m^2g^2}{k^2}+(x_2-x_1)^2}-\frac{3mg}{k}\right)^2$$

$$=\tfrac{1}{2}k\left[\frac{16m^2g^2}{k^2}+x_1^2+\frac{4m^2g^2}{k^2}-\frac{4mg}{k}\left(\frac{4mg}{k}\right)\left(1+\frac{x_1^2k^2}{2\cdot16m^2g^2}\right)\right]$$

$$+\tfrac{1}{2}k\left[\frac{9m^2g^2}{k^2}+(x_2-x_1)^2-\frac{6mg}{k}\left(\frac{3mg}{k}\right)\left(1+\frac{1}{2}\frac{(x_2-x_1)^2k^2}{9m^2g^2}\right)\right]$$

only need x^2 terms:

$$V = \text{const} + \tfrac{1}{2}k\left[x_1^2-\frac{16m^2g^2}{k^2}\,\frac{x_1^2k^2}{32m^2g^2}\right]$$

$$+\tfrac{1}{2}k\left[(x_2-x_1)^2-\frac{18m^2g^2}{k^2}\left(\frac{(x_2-x_1)^2k^2}{18m^2g^2}\right)\right]$$

$$=\text{const}+\frac{1}{4}kx_1^2+\frac{k}{2}\left[1-\frac{2}{3}\right](x_1-x_2)^2$$

$$L = K-V = \tfrac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2)-\tfrac{1}{4}kx_1^2-\frac{k}{6}(x_1-x_2)^2$$

x_1: $\quad m\ddot{x}_1+\frac{k}{2}x_1+\frac{k}{3}(x_1-x_2)=0 \quad$ or $\quad m\ddot{x}_1+x_1\overbrace{\left(\frac{k}{2}+\frac{k}{3}\right)}^{\frac{5k}{6}}-\frac{k}{3}x_2=0$

x_2: $\quad m\ddot{x}_2-\frac{k}{3}(x_1-x_2)=0 \quad\quad m\ddot{x}_2-\frac{k}{3}x_1+\frac{k}{3}x_2=0$

let $x_1=Ae^{i\omega t}$, $x_2=Be^{i\omega t}$:

$$-\omega^2A+\frac{5}{6}\frac{k}{m}A-\frac{k}{3m}B=0$$

$$-\omega^2B-\frac{k}{3m}A+\frac{k}{3m}B=0$$

$$\begin{pmatrix}\frac{5}{6}\frac{k}{m}-\omega^2 & -\frac{k}{3m}\\ -\frac{k}{3m} & \frac{k}{3m}-\omega^2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0$$

$$\det(\;)=0 \Rightarrow \left(\frac{5}{6}\frac{k}{m}-\omega^2\right)\left(\frac{k}{3m}-\omega^2\right)-\frac{k^2}{9m^2}=0=\frac{5}{18}\frac{k^2}{m^2}-\omega^2\left(\frac{5}{6}\frac{k}{m}-\frac{k}{3m}\right)+\omega^4-$$

$\frac{5}{6m}$

$$\frac{1}{6}\frac{k^2}{m^2}-\frac{\omega^2k}{2m}+\omega^4=0 \Rightarrow \omega^2=\frac{\frac{k}{2m}\pm\sqrt{\frac{k^2}{4m^2}-\frac{2}{3}\frac{k^2}{m^2}}}{2}=\frac{k}{2m}\left[\frac{1}{2}\pm\sqrt{}\right.$$

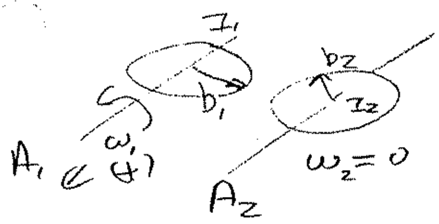
k/m, $k/6m$

Two gears wheels of radii b_1 and b_2 and axial moments of inertia (rotational inertia) I_1 and I_2, respectively, can rotate freely about fixed parallel axes. Initially the wheel of radius b_1 is rotating with an angular velocity ω_1, while the other wheel is at rest. The axes are such that an infinitesmal displacement would engage the wheels.

1. Find the total angular momentum of the system relative to the axis of wheel 2.
2. The wheels are suddenly engaged with the axes remaining fixed (except for the infinitesmal displacement to place them in contact). Find the angular velocity of each wheel afterward.
3. Find the total angular momentum of the system about the axis of wheel 1 after the gears have engaged.
4. Explain whether angular momentum is or is not conserved here.
5. Explain whether this is an "elastic collision".

(UG)

5001CM Gears



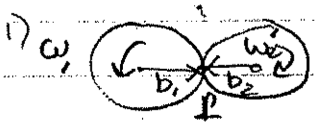
a) L relative to A2

The angular momentum is the same for all axis in the CM frame

$$L_{A2} = L_{A1} = \omega_1 I_1$$

b) Constraints

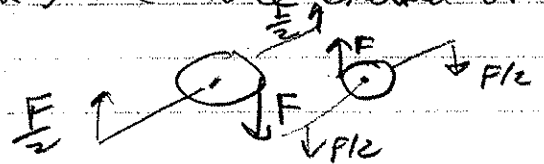
1)



The contact point has same velocity (at P)

$$\omega_1' b_1 = -\omega_2' b_2$$

2) The Force exerted at P is = and opposite for I_1 & I_2 (Newton III)



So torques differ

$$N_1 = -b_1 F \quad , \quad N_2 = -b_2 F$$

The torque is not constant, yet $N = \frac{\Delta L}{\Delta t} \Rightarrow \Delta L = \int N dt$

$$\Rightarrow \Delta L_1 = \int N_1 dt = -\int b_1 F dt = I_1 \Delta\omega_1$$

$$\Delta L_2 = \int N_2 dt = -\int b_2 F dt = I_2 \Delta\omega_2$$

$$\Rightarrow \frac{I_1 \Delta\omega_1}{-b_1} = \frac{I_2 \Delta\omega_2}{-b_2} \Rightarrow \Delta\omega_1 = +\frac{I_2}{I_1}\frac{b_1}{b_2}\Delta\omega_2$$

$$\omega_1' - \omega_1 = +\frac{I_2}{I_1}\frac{b_1}{b_2}(\omega_2' - 0)$$

2 eqtns to solve

1) $\omega_1' b_1 + \omega_2' b_2 = 0$

2) $\omega_1' - \left(\frac{I_2}{I_1}\frac{b_1}{b_2}\right)\omega_2' = \omega_1$

① − $b_1$②: $\omega_1' \cancel{b_1} + \omega_2' b_2 - \cancel{b_1}\omega_1' + \frac{I_2 b_1^2}{I_1 b_2}\omega_2' = \omega_1$

$$\omega_2'\left(\frac{I_1 b_2^2 + I_2 b_1^2}{I_1 b_2}\right) = \omega_1 b_1 \Rightarrow \omega_2' = \frac{I_1 b_1 b_2}{b_2^2 I_1 + b_1^2 I_2}\omega_1$$

$$\omega_1' = -\frac{b_2}{b_1}\omega_2' = \frac{-I_1 b_2^2}{b^2 I_1 + b_1^2 I_2}\omega_1$$

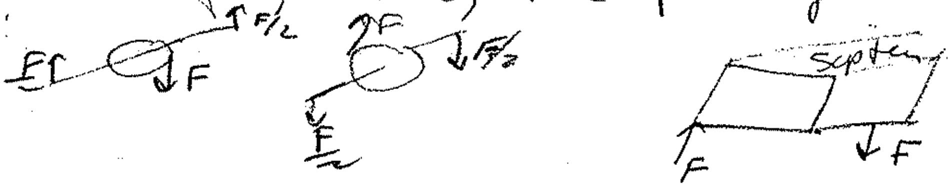
c) Just add the 2 L's in COM

$$L_1 = \omega_1' I_1 = \frac{b_2^2 I_1^2}{b_1^2 I_2 + I_1 b_2^2} \quad , L_2 = +\omega_2' I_2 = \frac{-b_1 b_2 I_1 \omega_1 I_2}{b_1^2 I_2 + b_2^2 I_1}$$

$$L = L_1 + L_2 = \frac{b_2^2 I_1^2 - b_1 b_2 I_1 I_2}{b_1^2 I_2 + b_2^2 I_1} \omega_1$$

d) $$\frac{d\vec{L}_{TOT}}{dt} = \vec{N}_{\substack{EXT \\ net}}$$

There is external torque, namely the pair of F/2 forces on the mount:



Couple exerts Torque F(b₁+b₂)

e) KE conser. req $I_1\omega_1^2 = I_1\omega_1'^2 + I_2\omega_2'^2$
not the case unless the values of $b_1 + b_2$ are picked (machined) – So can be, but not general.

Part (a). The "Gibbs Theorem" states that *the entropy of a mixture of ideal gases occupying a volume V at temperature T is the sum of the entropies that each gas would have if it alone were to occupy the volume V at temperature T.* Using this theorem, show that the pressure of a multicomponent simple ideal gas can be written as the sum of "partial pressures" P_j:

$$P_j \equiv N_j RT/V,$$

where N_j is the number of moles of the jth component in the mixture, and R is the universal gas constant (the above rule is often referred to as the "Dalton's Law for gas mixtures"). What is the physical interpretation of P_j?

Part (b). Show that μ_j, the electrochemical potential of the jth component in a multicomponent simple ideal gas, satisfies

$$\frac{\mu_j}{T} = R \ln\left(\frac{N_j v_0}{V}\right) + (\text{function of } T)$$

where v_0 is the molar volume of a simple ideal gas in a fixed reference state. Find the explicit form of the "function of T."

Single-component ideal gas equations:

$pV = NRT$ (N – number of moles).

Energy: $U = cNRT$ where $c = \frac{3}{2}$ for monoatomic gases,
$\frac{5}{2}$ " diatomic "
$\frac{6}{2}$ " $\geqslant 3$ atoms in the molecule.

Entropy:
$$S = S_0 + RN \ln\left[\left(\frac{U}{U_0}\right)^{c}\left(\frac{V}{V_0}\right)\left(\frac{N}{N_0}\right)^{-c-1}\right]$$

Part (a): Mixture of ideal gases; N_i – number of moles of the i^{th} component.

Total energy: $U_{TOT} = \sum_i U_i = RT \sum N_i c_i$ (1)

"Gibbs Law" for the total entropy of the mixture:

$$S_{TOT} = \sum_i S_i = \sum_i \left\{ S_{0i} + N_i R \ln\left[\left(\frac{U_i}{U_{0i}}\right)^{c_i}\left(\frac{V_i}{V_{0i}}\right)\left(\frac{N_i}{N_{0i}}\right)^{-c_i-1}\right]\right\}$$

Since all gases occupy the same volume V, each $V_i = V$.

Using $U_i = c_i R N_i T$, one can write:

~~$S_{TOT} = \sum$~~ $$\ln\left(\frac{U_i}{U_{0i}}\right)^{c_i} = \ln\left(\frac{c_i R N_i T}{c_i R N_{0i} T_0}\right)^{c_i} = \ln\left[\left(\frac{T}{T_0}\right)^{c_i}\left(\frac{N_i}{N_{0i}}\right)^{c_i}\right]$$

Hence:
$$S_{TOT} = \sum_i \left\{ S_{0i} + N_i R \ln\left[\left(\frac{T}{T_0}\right)^{c_i}\left(\frac{N_i}{N_{0i}}\right)^{c_i}\left(\frac{V}{V_{0i}}\right)\left(\frac{N_i}{N_{0i}}\right)^{-c_i-1}\right]\right\}$$

$$= \sum_i \left[S_{0i} + c_i N_i R \ln\left(\frac{T}{T_0}\right) + N_i R \ln\left(\frac{V}{V_{0i}}\right) - N_i R \ln\left(\frac{N_i}{N_{0i}}\right)\right] \quad (2)$$

Pressure: let's calculate the entropic intensive parameter $\frac{P}{T}$:

$$\frac{P}{T} = \left[\frac{\partial}{\partial V} S(U, V, N_1, N_2 \dots N_K)\right]_{U, N_1, N_2 \dots N_K}$$

But Eq. (1) implies that if $U = const$, then $T = const$.

So, one can also write:

$$\frac{P}{T} = \left[\frac{\partial}{\partial V} S(T, V, N_1, N_2, \dots N_K)\right]_{T, N_1, N_2, \dots N_K} \tag{3}$$

Let's insert Eq. (2) into Eq. (3). Since $T, N_1, N_2 \dots N_K = const$, the derivatives of all terms, exept $N_i R \ln\left(\frac{V}{V_{0i}}\right)$, are zeros.

Hence:

$$\frac{P}{T} = \frac{\partial}{\partial V} \sum_i N_i R \ln\left(\frac{V}{V_{0i}}\right) = \sum_i \frac{N_i R}{V}$$

and:

$$P = \sum_i \frac{N_i R T}{V} = \sum_i P_i$$

$P_i = \frac{N_i R T}{V}$, taking into account the single-component gas equation $pV = RNT$, is the pressure the i^{th} component would have if it occupied the volume V alone at temperature T.

Part (a) – another possible way is to use the Euler Equations in entropic representation:

Single-component gas: $S = \frac{U}{T} + \frac{P}{T}V - \frac{\mu}{T}N$

$$= cRN + \frac{P}{T}V - \frac{\mu}{T}N = cRN + RN - \frac{\mu}{T}N$$

Multi-component gas:

$$S_{TOT} = \frac{U_{TOT}}{T} + \frac{P}{T}V - \sum_i \frac{\mu_i}{T}N_i$$

$$= \sum_i c_i R N_i + \frac{P}{T}V - \sum_i \frac{\mu_i}{T}N_i \qquad (E1)$$

But from the Gibbs Law:

$$S_{TOT} = \sum_i S_i = \sum_i \left(\cancel{\frac{U_i}{T} +} c_i R N_i + R N_i - \frac{\mu_i}{T} N_i \right)$$

$$= \sum_i c_i R N_i + \sum_i R N_i - \sum_i \frac{\mu_i}{T} N_i \qquad (E2)$$

Comparing (E1) and (E2):

$$\frac{P}{T}V = \sum_i R N_i$$

Hence:

$$P = \sum_i \frac{R N_i T}{V} = \sum_i P_i$$

Part (b): Here we can use the entropic intensive parameter:

$$\frac{\mu_i}{T} = -\left[\frac{\partial}{\partial N_i} S(U,V,N_1\ldots N_k)\right]_{U,V,N_{j\neq i}} = -\left[\frac{\partial}{\partial N_i} S(T,V,N_1,\ldots N_k)\right]_{T,V,N_{j\neq}}$$

Again, we use Eq. (3) – however, ~~the "zero entropy"~~ we should keep in mind now that the "zero-entropy" S_{0i} of the i^{th} component is a function of N_i : $S_{0i} = N_i s_{0i}$ where s_{0i} is the molar "zero-entropy".

$$\frac{\mu_i}{T} = -\frac{\partial}{\partial N_i} \sum_i \left[N_i s_{0i} + c_i N_i R \ln\left(\frac{T}{T_0}\right) + N_i R \ln\left(\frac{V}{V_{0i}}\right) - N_i R \ln\left(\frac{N_i}{N_{0i}}\right)\right]$$

$$= -s_{0i} - c_i R \ln\left(\frac{T}{T_0}\right) - R \ln\left(\frac{V}{V_{0i}}\right) + R \ln\left(\frac{N_i}{N_{0i}}\right) + R$$

$$= (R - s_{0i}) - c_i R \ln\left(\frac{T}{T_0}\right) + R \ln\left(\frac{N_i}{V}\frac{V_{0i}}{N_{0i}}\right)$$

$$= (R - s_{0i}) - c_i R \ln\left(\frac{T}{T_0}\right) + R \ln\left(\frac{N_i v_{0i}}{V}\right)$$

where $v_{0i} = V_{0i}/N_{0i}$ is the standardized molar volume. (assuming that the "reference state" parameters are the same for all components, and can drop i in v_{0i}).

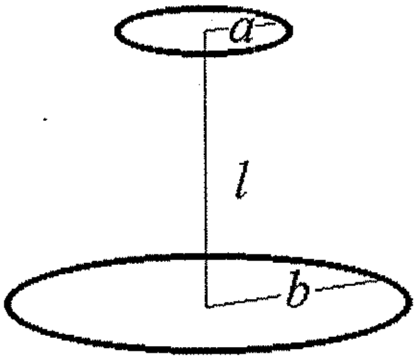
Thus:

$$\mu_i = RT \ln\left(\frac{N_i v_0}{V}\right) + (R - s_{0i})T - c_i RT \ln\left(\frac{T}{T_0}\right)$$

Thus the "function of T" in question is:

$$f(T) = (R - s_{0i})T - c_i RT \ln\left(\frac{T}{T_0}\right).$$

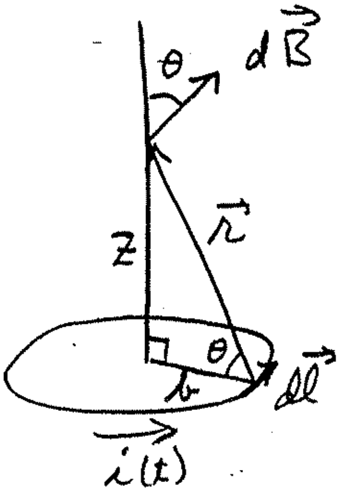
Two circular loops, formed from single turns of copper wire, have a common axis. Their planes are parallel and separated by a distance l. A known current $i(t)$, which is increasing with time, flows through the lower wire loop, which has radius b. Calculate the magnitude and direction of the magnetic force **F** on the upper loop, which has resistance R and radius a. You may assume $a << b$ and $a << l$.



E&M #2:

This question involves several parts:

1. Biot-Savart: The current in the lower ring generates a time-dependent magnetic field.
2. Faraday: The magnetic flux induces an emf in the upper ring.
3. Ohm: The emf causes a current to flow in the upper ring.
4. Lorentz: A magnetic force acts on this induced current.



1. Magnetic field due to lower loop:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{l}\times\hat{r}}{r^2}$$

$$B_z = \int dB\cos\theta = \frac{\mu_0 i}{4\pi}\frac{2\pi\ {}^2}{r^3}$$

$$= \frac{\mu_0 i b^2}{2(z^2+b^2)^{3/2}}$$

2. Flux through upper loop: $\Phi = \pi a^2 B_z\Big|_{\text{on axis}}$ since $\underline{a}$ is small.

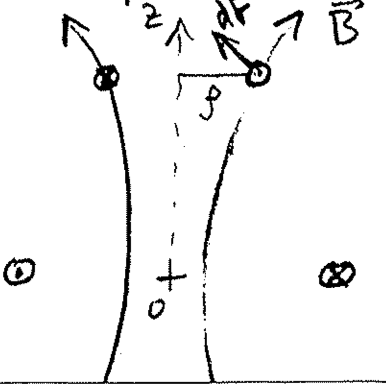
Emf in upper loop: $\mathcal{E} = \frac{d\Phi}{dt}$.

3. Then the current in the upper loop is $I = \frac{\mathcal{E}}{R}$.

$$I = \frac{\pi a^2\mu_0 b^2}{2R(l^2+b^2)^{3/2}}\frac{di}{dt}$$

Direction: Clockwise in the figure. (Lenz)

4. The force on the upper loop is $\vec{F} = I\int d\vec{l}'\times\vec{B}$.



The horizontal components cancel. We must therefore determine the radial component of $\vec{B}$ due to the lower loop. Cylindrical coordinates ρ, ϕ, z.

If the radial field component is B_ρ, then the net force acts in the z-direction (upward) and is $F_z = I \cdot 2\pi a B_\rho$. "JUMPING RING"

B_ρ (for small ρ) can be found from $\vec{\nabla} \cdot \vec{B} = 0$, which can be written as $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0$ since $B_\phi = 0$.

$$\frac{\partial B_z}{\partial z} = \frac{\mu_0 i b^2}{2} \frac{\partial}{\partial z}(z^2+b^2)^{-3/2} = \frac{\mu_0 i b^2}{\cancel{2}}\left(-\frac{3}{2}\right) \cdot \frac{\cancel{2} z}{(z^2+b^2)^{5/2}}$$

Since this does not depend on ρ,

$$\frac{\partial}{\partial \rho}(\rho B_\rho) = -\left(\frac{\partial B_z}{\partial z}\right)\rho \longrightarrow \rho B_\rho = -\left(\frac{\partial B_z}{\partial z}\right)\frac{\rho^2}{2}$$

or $$B_\rho = -\frac{\rho}{2}\frac{\partial B_z}{\partial z} = +\frac{3\mu_0 i b^2 z \rho}{4(z^2+b^2)^{5/2}}$$ with $z = \ell$ and $\rho = a$.

Therefore,

$$F_z = I \cdot 2\pi a B_\rho = \frac{\pi a^2 \mu_0 b^2}{\cancel{2} R (\ell^2+b^2)^{3/2}} \frac{di}{dt} \cdot \cancel{2}\pi a \cdot \frac{3\mu_0 i b^2 \ell a}{4(\ell^2+b^2)^{5/2}}$$

$$\boxed{F_z = \frac{3\pi^2 \mu_0^2 a^4 b^4 \ell}{4R(\ell^2+b^2)^4} \, i(t) \frac{di}{dt}}$$

An electron and a positron are situated in a uniform external magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{z}}$. Let $\vec{\mathbf{S}}_1$ be the spin of the electron, $\vec{\mathbf{S}}_2$ be the spin of the positron, and $\vec{\mathbf{S}} = \vec{\mathbf{S}}_1 + \vec{\mathbf{S}}_2$ be the total spin of the two-particle system. Each particle has a magnetic moment, which can be written as

$$\vec{\mu}_i = \frac{q_i}{m_i c}\,\vec{\mathbf{S}}_i\ ,$$

where $i = 1, 2$ for the electron and positron respectively. For this problem, ignore any interactions between the electron and positron (*i.e.*, the Coulomb interaction, dipole-dipole coupling, etc.).

a) Find the Hamiltonian of the system and its eigenvalues and eigenstates.

At $t = 0$, the system is in a singlet state with total spin equal to zero.

b) Find the probability that the system is in a total spin 1 state (triplet) at a later time t. Do this separately for the three possible values (1,0,-1) of the projection of the total spin along the z-axis.

c) Find the probability that the spin projections of both particles along the x-axis are $+\hbar/2$ at time t. (*i.e.*, $S_{1x} = +\hbar/2$ and $S_{2x} = +\hbar/2$)

$e^+ e^-$

$\vec{S} = \vec{S}_1 + \vec{S}_2$ $\quad$ $1 = e^-$, $2 = e^+$

a) $H = -(\vec{\mu}_1 + \vec{\mu}_2) \cdot \vec{B}$

$$= -\left[\frac{-e}{mc}\vec{S}_1 + \frac{+e}{mc}\vec{S}_2\right] \cdot B_0 \hat{z}$$

$$= \frac{eB_0}{mc} S_{1z} - \frac{eB_0}{mc} S_{2z}$$

$$\boxed{H = \omega_0 \left[S_{1z} - S_{2z}\right]} \quad \text{w/ } \omega_0 \equiv \frac{eB_0}{mc}$$

⇒ eigenstates of H are eigenstates of S_{1z}, S_{2z}

let $|++\rangle$ be state $|+\rangle_1 |+\rangle_2$ etc

⇒ 4 states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$

w/ $S_{1z}|++\rangle = +\frac{\hbar}{2}|++\rangle$ etc

$$\Rightarrow H|++\rangle = \omega_0\left[+\tfrac{\hbar}{2} - \tfrac{\hbar}{2}\right]|++\rangle = 0$$

$$H|+-\rangle = \omega_0\left[+\tfrac{\hbar}{2} - (-\tfrac{\hbar}{2})\right]|+-\rangle = \hbar\omega_0|+-\rangle$$

$$H|-+\rangle = \omega_0\left[-\tfrac{\hbar}{2} - \tfrac{\hbar}{2}\right]|-+\rangle = -\hbar\omega_0|-+\rangle$$

$$H|--\rangle = \omega_0\left[-\tfrac{\hbar}{2} - (-\tfrac{\hbar}{2})\right]|--\rangle = 0$$

$\Rightarrow$ if we label states

$|1\rangle = |++\rangle$

$|2\rangle = |+-\rangle$

$|3\rangle = |-+\rangle$

$|4\rangle = |--\rangle$

then $\boxed{\begin{aligned} E_1 &= E_4 = 0 \\ E_2 &= +\hbar\omega_0 \\ E_3 &= -\hbar\omega_0 \end{aligned}}$ are eigenvalues of H.

b) $\vec{S}^2$, S_z eigenstates are

$|s,m\rangle$ w/ $\vec{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$

$S_z|s,m\rangle = m\hbar|s,m\rangle$

$$|s,m\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$|1,1\rangle = |++\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$$

$$|1,-1\rangle = |--\rangle$$

$$\Rightarrow |\psi(0)\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$\Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega_0 t}|+-\rangle - e^{+i\omega_0 t}|-+\rangle\right]$$

$$P(S=1, m=1) = |\langle 1,1|\psi(t)\rangle|^2$$

$$= |\langle ++|\psi(t)\rangle|^2 = \underline{0}$$

$$P(1,-1) = |\langle --|\psi(t)\rangle|^2 = \underline{0}$$

$$P(1,0) = |\langle 1,0|\psi(t)\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}\{\langle +-| + \langle -+|\} \cdot \frac{1}{\sqrt{2}}\{e^{-i\omega_0 t}|+-\rangle - e^{+i\omega_0 t}|-+\rangle\}\right|^2$$

$$= \left|\frac{1}{2}(e^{-i\omega_0 t} - e^{+i\omega_0 t})\right|^2$$

$$= \left|\frac{1}{2}(-2i\sin\omega_0 t)\right|^2$$

$$\boxed{P(1,0) = \sin^2\omega_0 t}$$

c) $$P\left(S_{1x} = +\frac{\hbar}{2}, S_{2x} = +\frac{\hbar}{2}\right) = |{}_{1x}\langle +|{}_{2x}\langle +| \cdot \psi(t)\rangle|^2$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]$$

$$\Rightarrow |+\rangle_{1x}|+\rangle_{2x} = \frac{1}{\sqrt{2}}[|+\rangle_1 + |-\rangle_1]\frac{1}{\sqrt{2}}[|+\rangle_2 + |-\rangle_2]$$

$$= \frac{1}{2}[|++\rangle + |+-\rangle + |-+\rangle + |--\rangle]$$

$$P(+_{1x}, +_{2x}) = \left| \frac{1}{2}\left(\langle ++| + \langle -+| + \langle +-| + \langle --|\right) \cdot \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|+-\rangle - e^{+i\omega_0 t}|-+\rangle\right)\right|^2$$

$$= \left| \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t} - e^{+i\omega_0 t}\right)\right|^2$$

$$\boxed{P(+_{1x}, +_{2x}) = \frac{1}{2}\sin^2\omega_0 t}$$