

OSU Physics Comprehensive Exam No. 87 27-28 September 1999

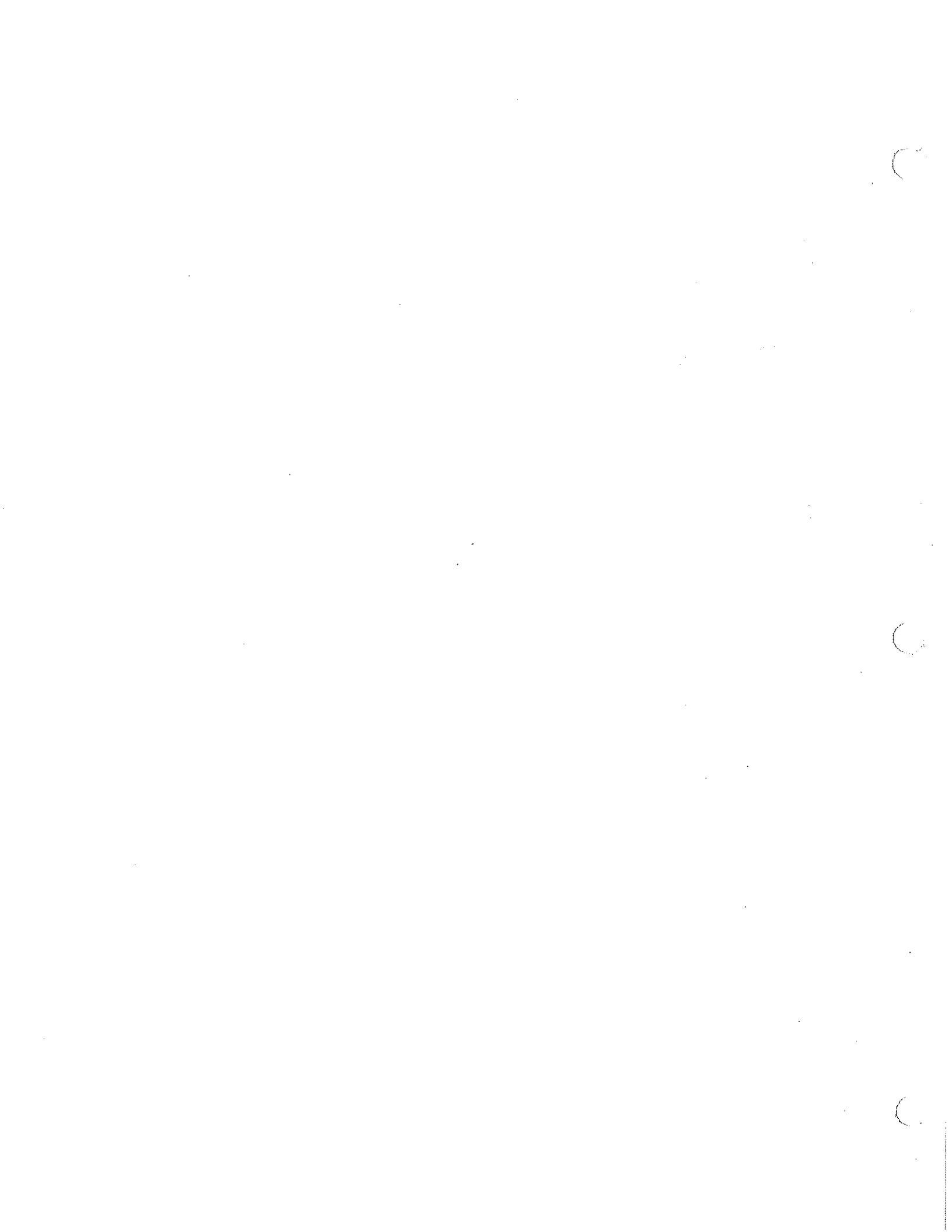
Problem No. 1

A particle of charge q and mass m is confined to move on a circle of fixed radius r in the x - y plane.

- a) What is the quantum mechanical Hamiltonian for this system?
- b) Find the quantum eigenstates and eigenenergies of this system.

An electric field $\vec{E} = F\hat{x}$ is applied to the system.

- c) Find the corrections to the energies to first order in the electric field F .
- d) Find the first-order correction to the ground state wavefunction.



a) rotor confined in x-y plane
so L_z is only ang mom

$$\Rightarrow H = \frac{L_z^2}{2mr^2} = \frac{L_z^2}{2I}$$

$$\text{w/ } I = 2mr^2$$

$$\Rightarrow \boxed{H = \frac{L_z^2}{2I}}$$

b) $H\psi = E\psi$

$$\frac{L_z^2}{2I}\psi = E\psi$$

$$\text{or } -\frac{\hbar^2}{2I} \frac{\partial^2}{\partial \phi^2} \psi = E\psi$$

\Rightarrow solns are eigenfunctions of L_z

$$\Rightarrow \psi = A e^{im\phi}$$

$$m = 0, \pm 1, \pm 2, \dots$$

$$\boxed{\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$$

$$L_z \psi = m\hbar \psi$$

$$\Rightarrow \frac{L_z^2}{2I} \psi = \frac{m^2 \hbar^2}{2I} \psi = E \psi$$

$$\Rightarrow \boxed{E_m = \frac{m^2 \hbar^2}{2I}} \quad m = 0, \pm 1, \pm 2, \dots$$

c) $\vec{E} = F \hat{x}$

potential $V = -q \int \vec{E} \cdot d\vec{l} = -qFx = -qFr \cos \phi$

$$\Delta E_m^{(1)} = \langle m|V|m \rangle$$

$$\begin{aligned} \langle m|V|m \rangle &= \int_0^{2\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{-im\phi} (-qFr \cos \phi) \frac{1}{\sqrt{2\pi}} e^{im\phi} \\ &= -\frac{qFr}{2\pi} \int_0^{2\pi} \cos \phi d\phi \end{aligned}$$

$$\langle m|V|m \rangle = 0 \quad \text{for all } m \Rightarrow \boxed{\text{no 1st order shift}}$$

d) Ground state: $m=0$

$$\psi_0^{(0)}(\phi) = \frac{1}{\sqrt{2\pi}}$$

$$\psi_0^{(1)}(\phi) = \sum_{m \neq 0} \frac{\langle m|V|0 \rangle}{E_0^{(0)} - E_m^{(0)}} \psi_m(\phi)$$

$$\langle m|V|0\rangle = \int_0^{2\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{-im\phi} (-gFr \cos\phi) \frac{1}{\sqrt{2\pi}}$$

$$= \frac{-gFr}{2\pi} \int_0^{2\pi} e^{-im\phi} \cos\phi d\phi$$

$$= \frac{-gFr}{2\pi} \int_0^{2\pi} e^{-im\phi} \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) d\phi$$

$$= \frac{-gFr}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(e^{-i(m-1)\phi} + e^{-i(m+1)\phi} \right) d\phi$$

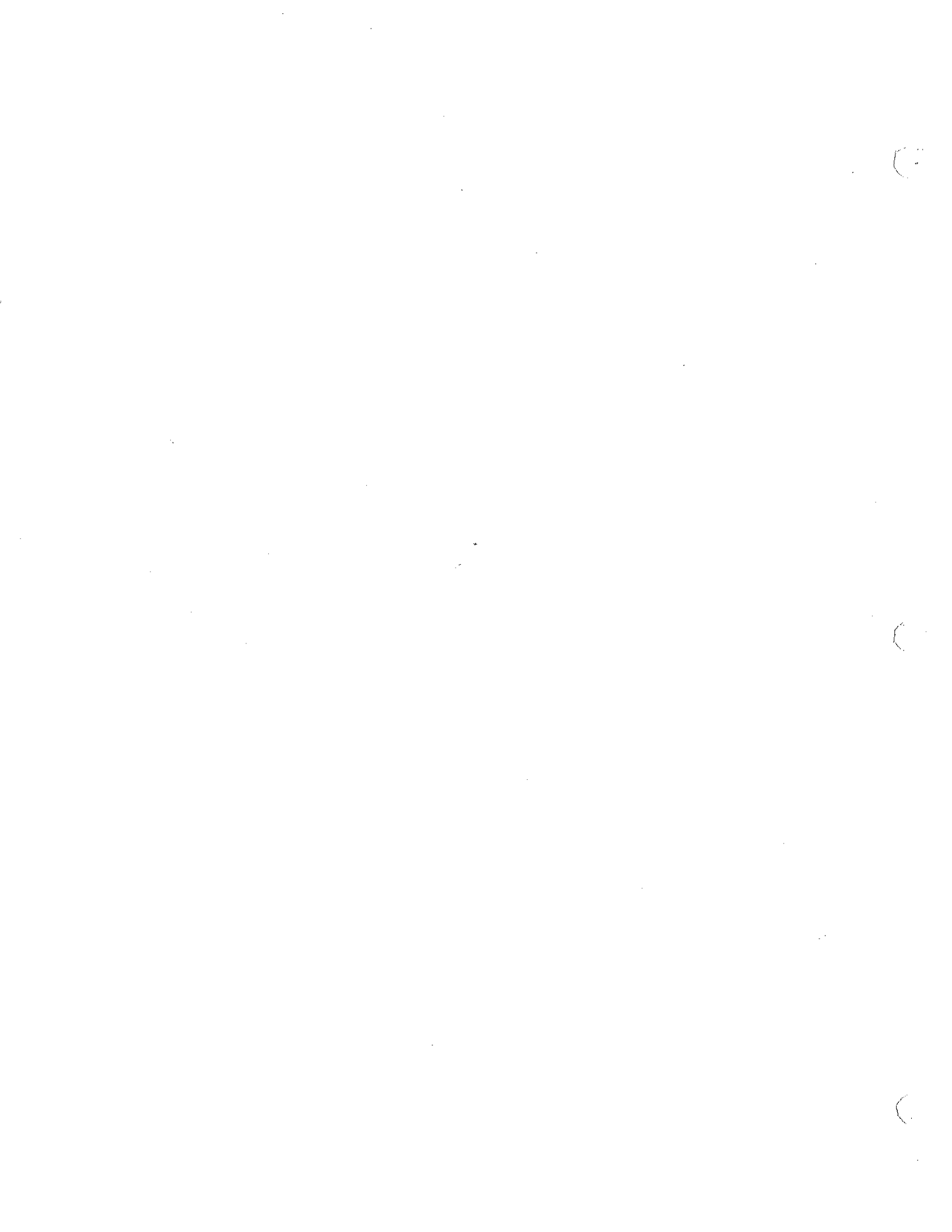
$$= \frac{-gFr}{2\pi} \frac{1}{2} \cdot 2\pi (\delta_{m,1} + \delta_{m,-1})$$

$$= \frac{-gFr}{2} (\delta_{m,1} + \delta_{m,-1})$$

$$\Rightarrow \psi_0^{(1)}(\phi) = \frac{-gFr}{2} \frac{1}{\left(0 - \frac{\hbar^2}{2I}\right)} \frac{1}{\sqrt{2\pi}} (e^{i\phi} + e^{-i\phi})$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2gFr I}{\hbar^2} \cos\phi$$

$$\Rightarrow \psi_0(\phi) = \frac{1}{\sqrt{2\pi}} \left[1 + \frac{2gFr I}{\hbar^2} \cos\phi \right]$$



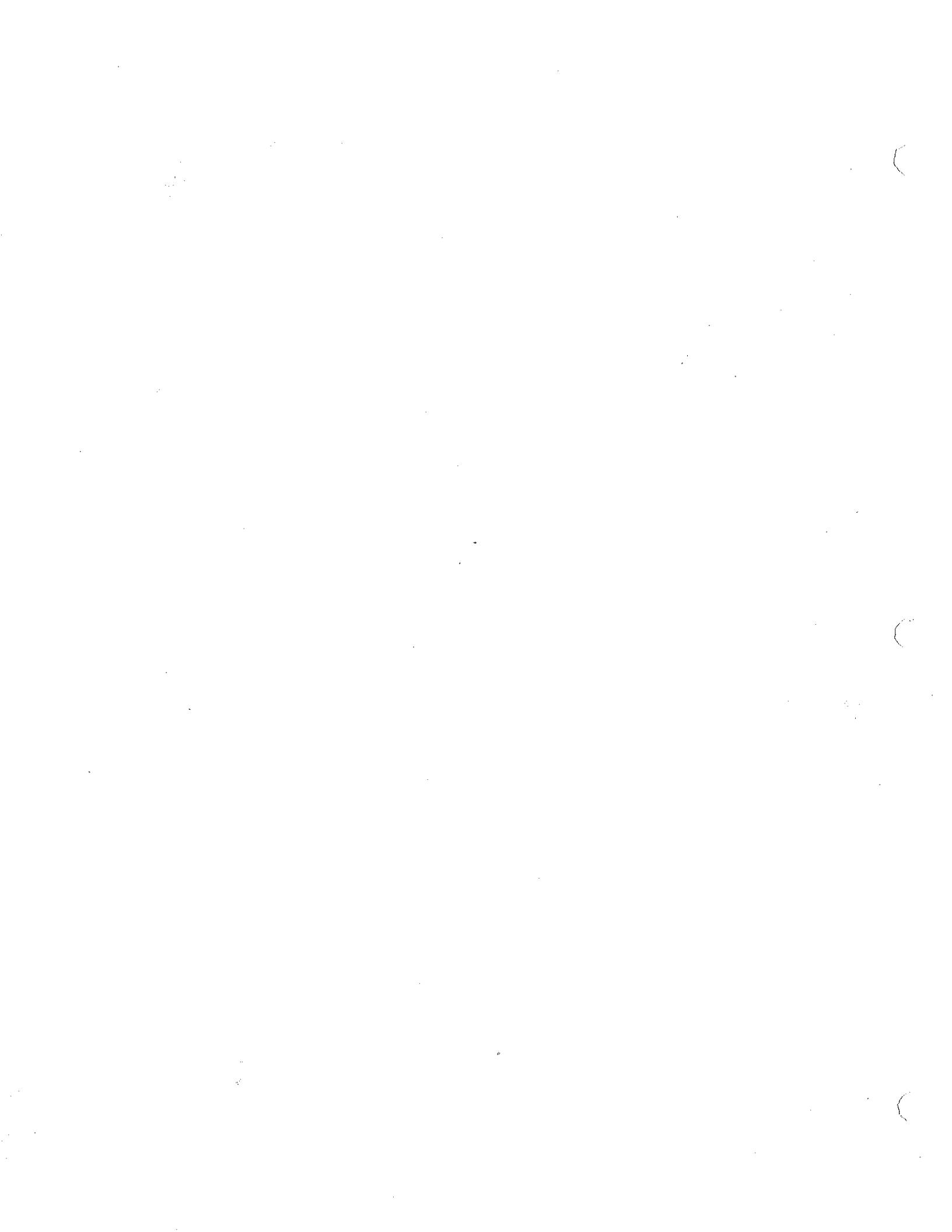
Problem No. 2

One mole of an ideal van der Waals gas is contained in an adiabatic vessel of volume V_i at temperature T_i . The gas satisfies the equation of state:

$$P = \frac{RT}{v - b} - \frac{a}{v^2},$$

where a and b are positive constants, P is the pressure, T is the temperature, and v is the volume per one mole. A valve is opened, permitting the gas to expand into an initially evacuated adiabatic vessel, so that the final volume is V_f .

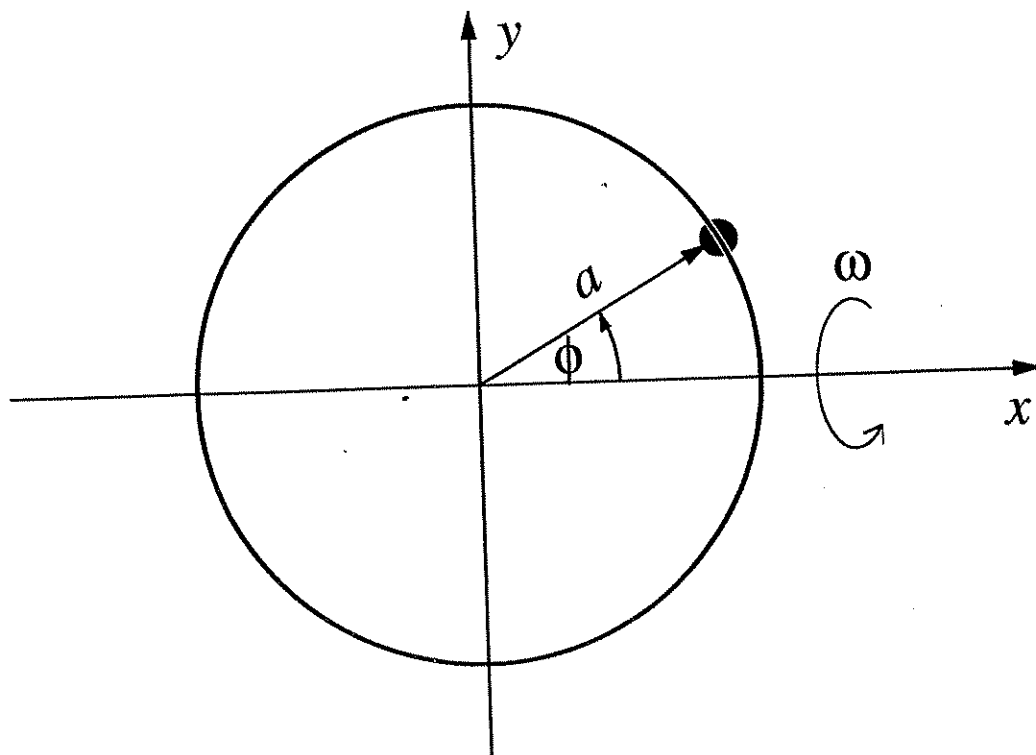
- (a) Find the final temperature T_f .
- (b) What is the sign of the temperature change, $\Delta T = T_f - T_i$? Does the expansion process lead to a heating, or to a cooling of the gas? Present a brief physical explanation of your result, based on the assumptions underlying the van der Waals model.
- (c) What would the temperature change be in the given process in the case of a simple ideal gas? (i.e., a gas strictly obeying the $Pv = RT$ and $u = cRT$ equations, where u is the internal energy per one mole).



Problem No. 3

A bead slides on a smooth circular wire of radius a that is rotating with constant angular velocity ω about a fixed horizontal diameter. At $t = 0$ the wire is vertical. Let ϕ be the angle that the radius drawn to the bead makes with the axis of rotation.

- (a) Determine the kinetic and potential energies of the bead in terms of generalized coordinate(s).
- (b) What is (are) the generalized applied force(s) Q_i ?
- (c) Explain whether $Q_i = -\partial V/\partial\phi$, even though there is a time-dependent constraint?
- (d) Determine the equation(s) of motion for this system.
- (e) If the wire were not rotating, what would be the equation of motion for ϕ ? Discuss the character of the motion.
- (f) If the experiment were carried out in a laboratory orbiting the earth, what would be the equation of motion for ϕ ? Discuss the character of the motion. (Do not assume $\omega = 0$).





Solution. In adiabatic free expansion the total internal energy of the gas remains constant during the process because neither heat nor work is transferred to the system by any external agency.

One can say that classical thermodynamics is “highly redundant” because most functions and coefficients used in this theory are strongly “interrelated”. So, it is not unusual that there are many possible “solution schemes” for a given problem, and it seems to be the case in the present one. Here we present a relatively simple method, based on the treatment of the van der Waals gas in *Thermodynamics and an introduction to thermostatistics, Second Edition* by Herbert Callen (this material should be known to the students because the Callen’s book was used in the Ph641 course for the last four years).

Note that all one needs to know for solving the present problem is a state equation expressing the relation between the temperature, and the molar energy and volume of the expanding gas — in general form, this equation can be written as $1/T = f(u, v)$. Such an equation of state is not particularly important in the standard presentation of the van der Waals model, and therefore it is seldom discussed in the textbooks. However, it can be obtained as a “byproduct” in the process of deriving the fundamental relation $s = s(u, v)$ for the van der Waals gas used in the Callen’s book. In this method, one first considers the mixed second-order derivative of the entropy with respect to the internal energy and volume. It is required, of course, that:

$$\frac{\partial^2 s}{\partial v \partial u} = \frac{\partial^2 s}{\partial u \partial v}, \quad (1)$$

which can also be written as:

$$\frac{\partial}{\partial v} \left[\left(\frac{\partial s}{\partial u} \right)_v \right]_u = \frac{\partial}{\partial u} \left[\left(\frac{\partial s}{\partial v} \right)_u \right]_v. \quad (2)$$

But the two derivatives inside the square brackets are entropic intensive parameters, related to the temperature and pressure as:

$$\left(\frac{\partial s}{\partial u} \right)_v = \frac{1}{T} \quad \text{and} \quad \left(\frac{\partial s}{\partial v} \right)_u = \frac{P}{T}. \quad (3)$$

Hence, Eq. (2) can be rewritten as:

$$\frac{\partial}{\partial v} \left(\frac{1}{T} \right)_u = \frac{\partial}{\partial u} \left(\frac{P}{T} \right)_v \quad (4)$$

Now, one can rewrite the van der Waals formula as:

$$\frac{P}{T} = \frac{R}{v-b} - \frac{a}{Tv^2}, \quad (5)$$

and plug this into the right-side derivative in Eq. 4:

$$\frac{\partial}{\partial u} \left(\frac{P}{T} \right)_v = \frac{\partial}{\partial u} \left(\frac{R}{v-b} - \frac{a}{Tv^2} \right)_v = -\frac{a}{v^2} \frac{\partial}{\partial u} \left(\frac{1}{T} \right)_v \quad (6)$$

So, Eq. 4 becomes:

$$\frac{\partial}{\partial v} \left(\frac{1}{T} \right)_u = -\frac{a}{v^2} \frac{\partial}{\partial u} \left(\frac{1}{T} \right)_v \quad (7)$$

Furthermore, the left-side partial derivative with respect to v can be changed to a one with respect to $1/v$ by using the elementary operation:

$$\frac{\partial}{\partial v} \left(\frac{1}{T} \right)_u = \frac{d(1/v)}{dv} \frac{\partial}{\partial(1/v)} \left(\frac{1}{T} \right)_u = -\frac{1}{v^2} \frac{\partial}{\partial(1/v)} \left(\frac{1}{T} \right)_u.$$

After all the above manipulations, the initial condition expressed by Eq. 1 takes the compact form:

$$\frac{\partial}{\partial(1/v)} \left(\frac{1}{T} \right)_u = \frac{\partial}{\partial(u/a)} \left(\frac{1}{T} \right)_v \quad (8)$$

A conclusion from this result is that the $1/T$ function should depend on the $1/v$ and u/a variables in such a way that the derivatives with respect to these two variables are equal. The simplest solution satisfying this requirement is a function depending only on the $(1/v + u/a)$ sum — or, which is equivalent, on $a/v + u$. Furthermore, in the $a \rightarrow 0$ we want our equation to be consistent with the corresponding molar equation for a simple ideal gas, $1/T = cR/u$ (where R is the gas constant, and the coefficient c is $\frac{3}{2}$ for a monoatomic gas, $\frac{5}{2}$ for a gas consisting of diatomic molecules, etc.). As we can conclude from the above, the equation of state we are looking for has the following form:

$$\frac{1}{T} = \frac{cR}{u + a/v} \quad (9)$$

Having derived this equation, one can immediately find the temperature change in the free expansion process. From the equation, we obtain the following two relations for the initial and the final state:

$$T_i = \frac{u}{cR} + \frac{a}{cRV_i} \quad \text{and} \quad T_f = \frac{u}{cR} + \frac{a}{cRV_f}, \quad (10)$$

and by subtracting the first from the second, we get:

$$\Delta T = T_f - T_i = \frac{a}{cR} \left(\frac{1}{V_f} - \frac{1}{V_i} \right). \quad (11)$$

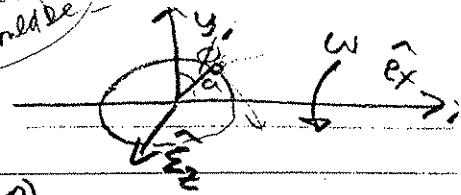
Since a is positive, and $V_f > V_i$, we conclude that the sign of ΔT is negative — in other words, the gas temperature *decreases* in the free expansion process.

This result can be easily explained in physical terms. The a constant in the van der Waals model is associated with the *attractive* forces between the gas molecules. These are the famous *van der Waals forces*, corresponding to the “rising tail” in the molecule-molecule interaction potential. When the gas volume and the average distance between the molecules increase, the total energy of intermolecular interactions also increases (i.e., it becomes “less negative”). But the total *system energy* cannot change (let’s keep in mind that the process is adiabatic), so that the average *kinetic energy* must decrease. It explains the cooling effect, as the gas temperature is proportional to the average kinetic energy of the molecules.

In the case of a truly ideal gas, $a = 0$, and from the formula we derived for the van der Waals gas it follows that $\Delta T = 0$. This is, of course, the expected result because in ideal gas, by definition, there are no interactions of any kind between the particles. Hence, there is no physical mechanism capable of changing their kinetic energy during the expansion process.

#3 Bead

Turn ϕ with ω , should be \times



In rotating frame, bead is at

$$\vec{r} = a(\hat{e}_x \sin \phi + \hat{e}_y \cos \phi)$$

$$\text{Velocity} = \frac{d\vec{r}}{dt} = \frac{d}{dt} \Big|_{\text{notate}} \vec{r} + \vec{\omega} \times \vec{r}$$

where $\vec{\omega} = \omega \hat{e}_x$

$$\Rightarrow \frac{d\vec{r}}{dt} = a[\hat{e}_x \cos \phi \dot{\phi} + \hat{e}_y (-\sin \phi) \dot{\phi}] + \omega a \cos \phi \hat{e}_z$$

a) $KE = \frac{1}{2} m \left| \frac{d\vec{r}}{dt} \right|^2 = \frac{1}{2} m (\dot{\phi}^2 a^2 + \omega^2 a^2 \cos^2 \phi)$

$$V = mgh = mg a \cos \phi \cos \omega t$$

b) $\delta W = Q_\phi \delta \phi$ (basic def. of generalized force)

height above \vec{x}_{00} = $a \cos \phi \cos \omega t$, $W = mgh$

$$\Rightarrow \delta W = -a \cos \omega t \sin \phi m g \delta \phi \Rightarrow Q_\phi = -mg a \cos \omega t \sin \phi$$

c) $V = mg a \cos \phi \cos \omega t$, $\frac{\partial V}{\partial \phi} = -mg a \sin \phi$ yes generally true for conservative system, $\therefore V$ depends only on position, independent even though t

d) $L = T - V = \frac{1}{2} m a^2 (\dot{\phi}^2 + \omega^2 \cos^2 \phi) - mg a \cos \phi \cos \omega t$

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = 0$$

$$\frac{\partial L}{\partial \phi} = -\frac{1}{2} m a^2 \omega^2 2 \cos \phi \sin \phi \dot{\phi} + mg a \sin \phi \cos \omega t \dot{\phi}$$

$$\frac{\partial L}{\partial \dot{\phi}} = m a \dot{\phi}; \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = m a \ddot{\phi}$$

$$\Rightarrow \ddot{\phi} + \omega^2 \cos \phi \sin \phi - g/a \sin \phi \cos \omega t = 0$$

e) $\omega = 0 \Rightarrow \ddot{\phi} - g/a \sin \phi = 0$

Bead falls down in upper quadrants
oscillates in lower quadrants (restoring force)

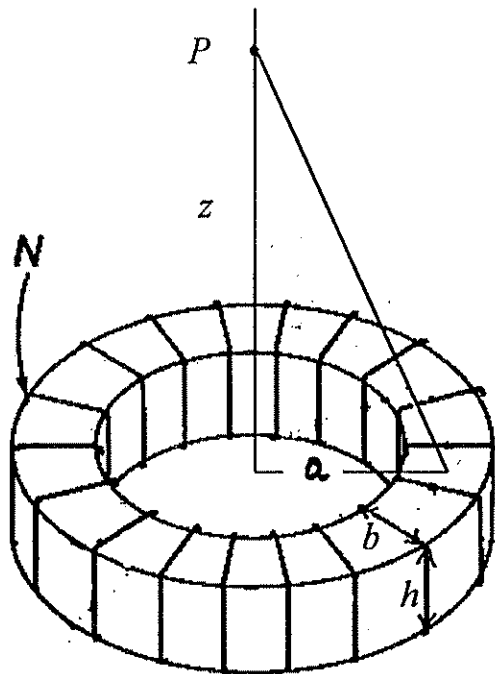
$\Rightarrow g = 0$
 $\ddot{\phi} = -\omega^2 \cos \phi \sin \phi \approx -\omega^2 \phi$ } SHM about small ϕ



Problem No. 4

A toroidal solenoid consists of many turns of fine insulated wire, wound uniformly around a doughnut-shaped form. Consider the case in which the number of turns is N , the distance from the symmetry axis to the center of the winding is a , and the dimensions of the rectangular coil cross section are b and h , as shown in the figure. No magnetic materials are present. Assume $b \ll a$, $h \ll a$, and $N \gg 1$.

- a) (6 points) Suppose that this coil is connected as an element in an electronic circuit. Calculate its self-inductance L .
- b) (14 points) If the current through the solenoid is $i(t)$, with a known time dependence, find (at time t) the magnitude of the electric field E at a point P on the symmetry axis, at a distance z from the center of the toroid. Ignore retardation effects.





Toroidal solenoid:

(a) $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 N i$ for path C = circle within the windings, at radius a.
 Ampere's law. i = current through coil

$B \cdot 2\pi a = \mu_0 N i$

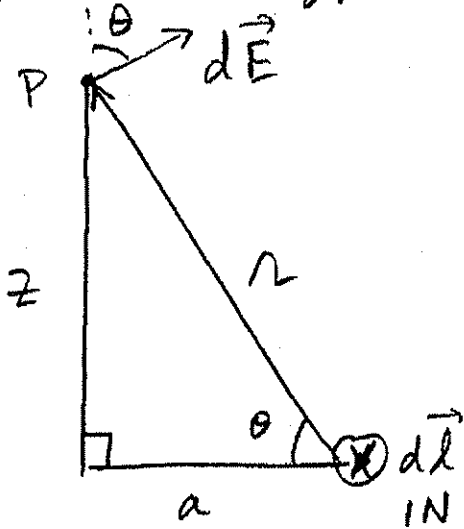
Enclosed flux: $\Phi = B \cdot b \cdot h = \frac{\mu_0 N i b h}{2\pi a}$

Self-inductance: $L = \frac{N\Phi}{i} \Rightarrow$

$L = \frac{\mu_0 N^2 b h}{2\pi a}$

(b) $\oint \vec{B} \cdot d\vec{l} = \mu_0 i \rightarrow d\vec{B} = \frac{\mu_0}{4\pi} i \frac{d\vec{l} \times \hat{r}}{r^2}$

Therefore $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\dot{\Phi} \rightarrow d\vec{E} = -\frac{1}{4\pi} \frac{\dot{\Phi} d\vec{l} \times \hat{r}}{r^2}$



$E_z = \int dE \cos\theta$ axial component

$= \frac{1}{4\pi} \frac{a}{r} \dot{\Phi} \int \frac{dl}{r^2}$

$= \frac{1}{4\pi} \frac{a \cdot 2\pi a}{r^3} \dot{\Phi}$

$E_z = \frac{a^2}{2r^3} \cdot \frac{\mu_0 N \frac{di}{dt} b h}{2\pi a}$

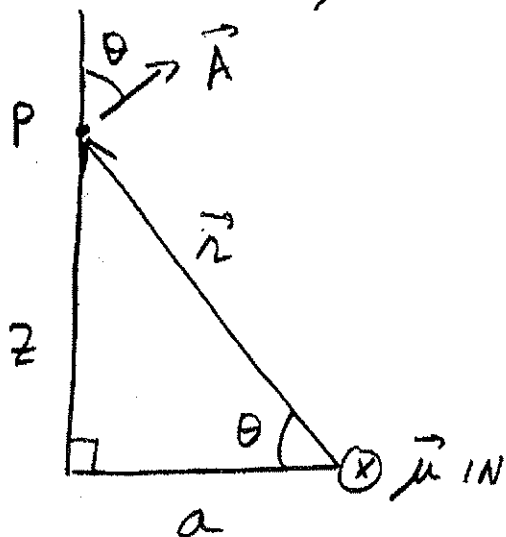
$E_z = \frac{\mu_0 N a b h}{4\pi (z^2 + a^2)^{3/2}} \frac{di}{dt}$

Alternative solution to (b):

Treat each wire loop as a point-dipole: $\mu = i b h$
dipole moment.

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \hat{r}}{r^2}$$

vector potential at P.
due to one turn.



For N turns,

$$A_z = N A \cos \theta$$

$$= N \frac{\mu_0}{4\pi} \frac{\mu}{r^2} \cdot \frac{a}{r}$$

$$= \frac{\mu_0}{4\pi} \frac{N a}{r^3} b h i$$

Field magnitude at P:

$$\vec{E} = - \frac{\partial \vec{A}}{\partial t} \quad \Rightarrow \quad E = \left| \frac{\partial A_z}{\partial t} \right|$$

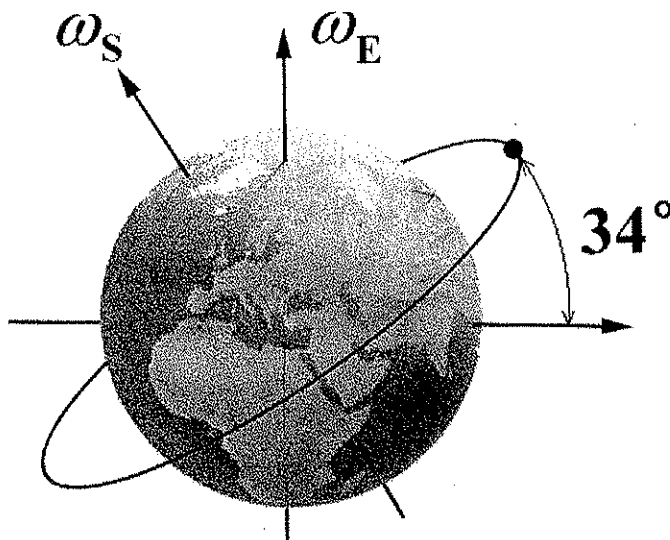
$$E = \frac{\mu_0 N a b h}{4\pi (z^2 + a^2)^{3/2}} \frac{di}{dt}$$

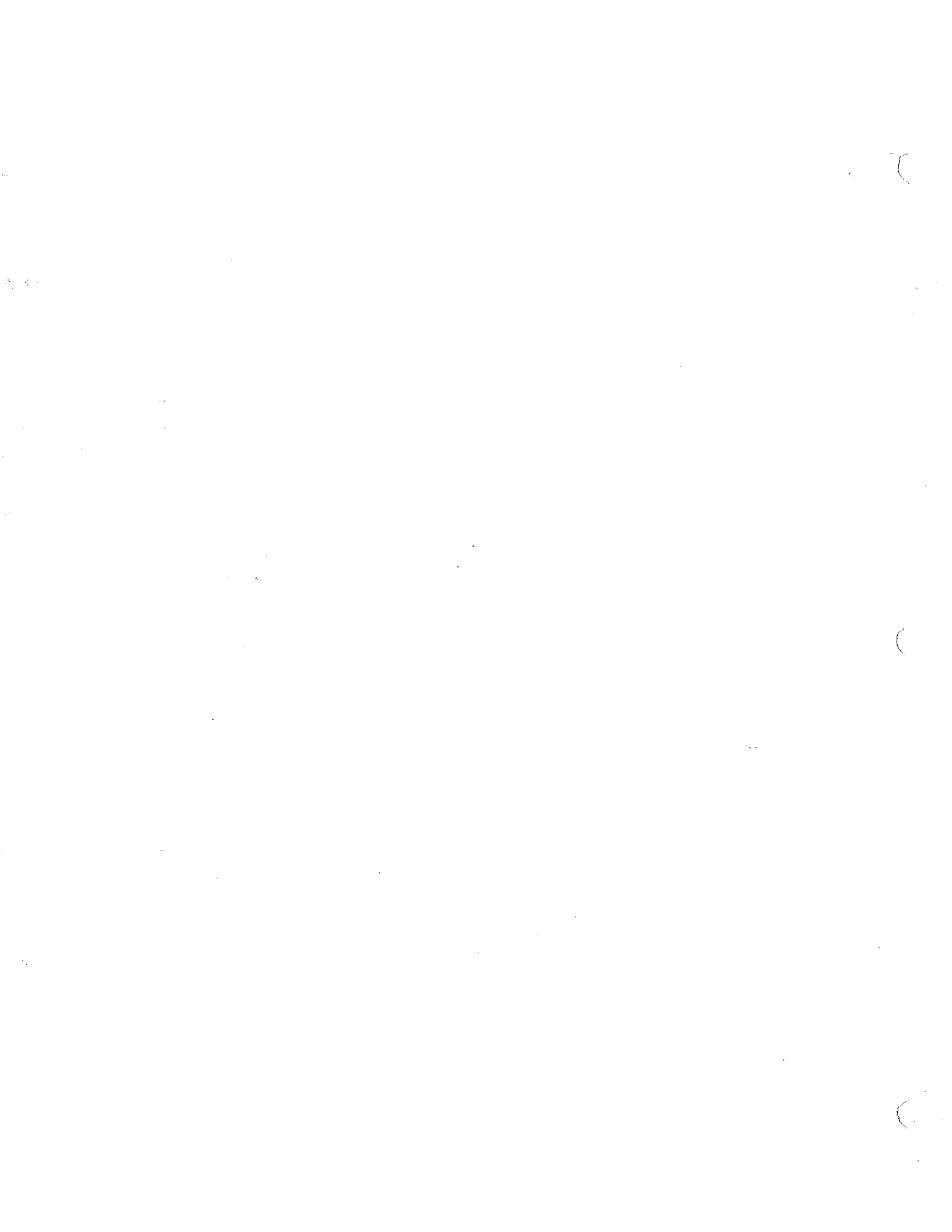
in agreement with first solution.

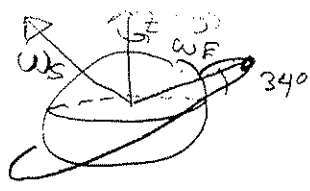
Problem No. 5

Consider a reference frame O with its origin at the center of mass of the spherical earth, but not rotating relative to the fixed stars. In this frame, a satellite is moving in a circular orbit inclined 34° to the earth's equatorial plane, with an angular frequency of ω_S . (Ignore motion of the earth's center of mass.)

- Discuss the applicability of each of Kepler's laws in a reference frame O' attached to the earth's crust.
- Discuss the effects of the fictitious (noninertial-frame) forces on the kinetic energy and angular momentum of the satellite as observed in O' .
- Determine an expression for the position of the satellite as seen in O' .
- Describe the motion of the axis of rotation of the satellite as seen in O' .







5

ⓐ) Kepler's laws: these are based on central forces
 We now have centrifugal and Coriolis forces - which are not central.

KI: Since orbit isn't precesses, there are no closed ellipses or conic sections

KII: (areal velocity): this follows from angular momentum conservation, which is not true $\Rightarrow 0$ for non central forces.

KIII: $R^3 \propto T^2$: only for central $F \propto \frac{1}{r^2}$

ⓑ) New forces, in addition to central forces, are from

$$\vec{F}_{\text{net}} = m\vec{a} = \underbrace{-2m\vec{\omega} \times \vec{v}'}_{\text{Coriolis}} - \underbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r}')}_{\text{centrifugal}}$$

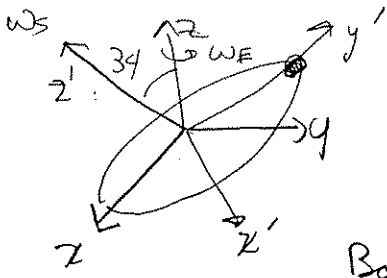
since $\vec{\omega} \times \vec{v}'$ is perpendicular to \vec{v}' , it does no work \Rightarrow no effect on KE

yet $\vec{\omega} \times (\vec{\omega} \times \vec{r}')$ does work

we see this by evaluating $\frac{dV}{dt} \propto \frac{d}{dt} \frac{1}{2} v'^2$ which has periodic ($\sin^2 \omega t$) terms

$$\frac{d\vec{L}'}{dt} = \vec{r}' \times \vec{F}'$$

both forces create torques (yet centrifugal averages to zero since vector parallel to \vec{r}')
 \therefore affect angular momentum



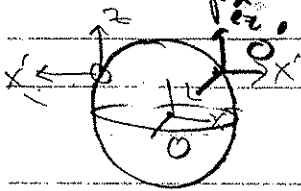
ω_E = earth frequency
 ω_s = satellite frequency

projection into xy plane = $a \cos 34^\circ$
 yz " = $a \sin 34^\circ$

Both planes have circular motion, start x axis $t=0$

Fixed at com. in O : $\vec{r} = a \cos 34^\circ \cos \omega_s t \hat{e}_x + a \sin \omega_s t \cos 34^\circ \hat{e}_y + a \sin \omega_s t \sin 34^\circ \hat{e}_z$

We rotate this with ω_E about \hat{e}_z to obtain position of satellite in frame attached to the earth's crust, aligned at $t=0$



$$\begin{aligned} \hat{e}_x &\rightarrow \cos \omega_E t \hat{e}'_x - \sin \omega_E t \hat{e}'_y & (2) \\ \hat{e}_y &\rightarrow \sin \omega_E t \hat{e}'_x + \cos \omega_E t \hat{e}'_y & (3) \end{aligned}$$

$$\Rightarrow \vec{r}' = a \cos^2 \omega_s t (\cos \omega_E t \hat{e}'_x - \sin \omega_E t \hat{e}'_y)$$

$$+ a \sin \omega_s t \cos 34^\circ (\sin \omega_s t \hat{e}'_x + \cos \omega_E t \hat{e}'_y) + a \sin \omega_s t \sin 34^\circ \hat{e}'_z$$

$$\vec{r}' = (a \cos^2 34^\circ \cos \omega_s t \cos \omega_E t + a \sin \omega_s t \sin \omega_E t \cos 34^\circ + a \sin \omega_s t \cos \omega_E t \cos 34^\circ - a \cos^2 \omega_s t \sin \omega_E t \cos 34^\circ) \hat{e}'_x + a \sin \omega_s t \sin 34^\circ \hat{e}'_z$$

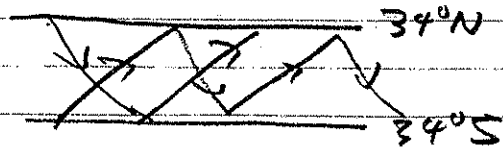
(d) The axis of rotation of the satellite is

$$O: \vec{A}' = -\sin 34^\circ \hat{e}'_y + \cos 34^\circ \hat{e}'_z \quad \text{as seen in } O$$

now substitute (2) + (3)

$$O': \vec{A}' = -\sin 34^\circ \sin \omega_E t \hat{e}'_x - \sin 34^\circ \cos \omega_E t \hat{e}'_y + \cos 34^\circ \hat{e}'_z$$

ie. \vec{A}' precesses (rotates) about the z axis with angular velocity ω_E :



Problem No. 6

At high temperatures electrons “evaporate” from metals. Consider an evacuated glass tube of volume V containing a thin metal wire heated to temperature T . Assume that:

- The electrons outside the metal constitute an ideal monoatomic gas, with a Maxwell distribution of particle velocities;
 - The chemical potential μ_s of the electrons in metal is constant;
 - The electron gases inside and outside the metal are in thermal equilibrium.
- (a) Find the pressure of the electron gas in the tube if the energy needed to take an electron from the Fermi level inside the metal into vacuum (the “work function”) is Φ ;
- (b) Next, find the emission current J per unit area of the metal surface, assuming that each electron striking the wire from outside returns to the metal.

Hints: You may want to recall the formula for “quantum concentration”:

$$n_Q = \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2}$$

The Maxwell distribution function of particle velocities is:

$$\frac{dN(v)}{N} = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT} dv$$

If a gas contains n particles per unit volume, and the average particle velocity is $\langle v \rangle$, the number ρ of particles passing through a surface element dS per unit time is:

$$\rho = \frac{1}{4} n \langle v \rangle dS$$

You may use the above formulae without deriving them.



Solution: The partition function for an ideal gas of volume V consisting of N particles is:

$$Z = \frac{1}{N!} (Vn_Q)^N. \quad (1)$$

From Z one obtains the Helmholtz free energy: $F = -kT \ln Z$. Using the Stirling approximation, $\ln N! \cong N \ln N - N$, we get:

$$F = kT \ln N! - NkT \ln(Vn_Q) \cong NkT \ln N - NkT \ln(Vn_Q) - NkT = NkT \left(\ln \frac{N}{Vn_Q} - 1 \right). \quad (2)$$

Next, we calculate the chemical potential of the electron gas μ_g :

$$\mu_g = \left(\frac{\partial F}{\partial N} \right)_{T,V} = kT [\ln N + kT - kT \ln(Vn_Q) - kT] = kT \ln \frac{N}{Vn_Q}. \quad (3)$$

In this particular problem, we are interested in the gas pressure P . Combining Eq. (3) and the ideal gas equation, $PV = NkT$, we can express μ_g in terms of P :

$$\mu_g = kT \ln \frac{P}{kTn_Q} = kT \ln \left[\frac{P}{(kT)^{5/2}} \left(\frac{2\pi\hbar^2}{m} \right)^{3/2} \right]. \quad (4)$$

The electron gas pressure: the state of equilibrium requires that the gas chemical potential μ_g be equal to the potential inside the metal. Taking the vacuum potential as zero, the chemical potential inside the metal is $-\Phi$; thus, the gas potential should be:

$$\mu_g = -\Phi. \quad (5)$$

By solving Eq. (4) for P , we obtain the answer to the first question in the problem:

$$P = \left(\frac{m}{2\pi\hbar^2} \right)^{3/2} (kT)^{5/2} e^{-\Phi/kT}. \quad (6)$$

The emission current: The Maxwell distribution function of the particle velocities can be derived in a straightforward manner using the classical formalism. The probability of finding a gas particle with the momentum vector from the volume element $dp_x dp_y dp_z$ of the momentum space is:

$$\frac{dN(p_x, p_y, p_z)}{N} = \frac{e^{-(p_x^2 + p_y^2 + p_z^2)/2mkT} dp_x dp_y dp_z}{\int \int \int e^{-(p_x^2 + p_y^2 + p_z^2)/2mkT} dp_x dp_y dp_z}. \quad (7)$$

The denominator can be simply rewritten as a product of three integrals of Gaussian-like functions; using the familiar result, $\int_{-\infty}^{\infty} \exp(-ax^2) = \sqrt{\pi/a}$, we obtain:

$$\int \int \int e^{-(p_x^2 + p_y^2 + p_z^2)/2mkT} dp_x dp_y dp_z = \int e^{-p_x^2/2mkT} dp_x \int e^{-p_y^2/2mkT} dp_y \int e^{-p_z^2/2mkT} dp_z = \sqrt{2\pi mkT}^3. \quad (8)$$

By carrying out a simple variable transformation, we obtain the fraction of particles with the velocity vectors in the $dv_x dv_y dv_z$ element:

$$\frac{dN(v_x, v_y, v_z)}{N} = \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-(v_x^2 + v_y^2 + v_z^2)/2mkT} dv_x dv_y dv_z. \quad (9)$$

very interesting results given in the HINTS

Switching from Cartesian coordinates to spherical polar coordinates, $dv_x dv_y dv_z \rightarrow v^2 \sin \theta d\theta d\phi dv$ (where $v^2 = v_x^2 + v_y^2 + v_z^2$), and integrating Eq. (9) over the angular variables, we finally obtain the known Maxwell distribution function of particle velocities:

$$\frac{dN(v)}{N} = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT} dv \quad (10)$$

From this function, recalling another familiar integral: $\int_0^\infty x^2 \exp(-a^2 x^2) = \sqrt{\pi}/4a^3$, we readily get the average particle velocity:

$$\langle v \rangle = \int_0^\infty v \frac{dN(v)}{N} = \left(\frac{8kT}{\pi m} \right)^{1/2}. \quad (11)$$

In the state of equilibrium, the emission current must be equal to the current of electrons "returning" from the vacuum to the wire. Considering the formula given in the hint, this latter current (per unit surface area) is:

$$I = \frac{q_e n}{4} \langle v \rangle = \frac{q_e n}{4} \left(\frac{8kT}{\pi m} \right)^{1/2}, \quad (12)$$

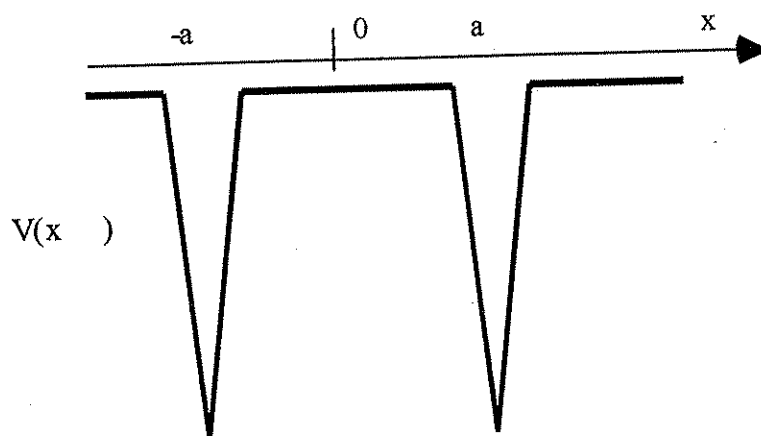
where q_e is the electronic charge. Again using the ideal gas equation, $PV = NkT$, we can express the gas concentration as $n = N/V = P/kT$. Inserting this result into Eq. (12), and then substituting P with Eq. (6), we obtain the final result for the emission current per unit area:

$$I = \frac{q_e P}{4kT} \left(\frac{8kT}{\pi m} \right)^{1/2} = \frac{q_e}{4kT} \left(\frac{8kT}{\pi m} \right)^{1/2} \left(\frac{m}{2\pi \hbar^2} \right)^{3/2} (kT)^{5/2} e^{-\Phi/kT} = \frac{mq_e}{4\pi^2 \hbar^3} (kT)^2 e^{-\Phi/kT}. \quad (13)$$

It should be noted that this result differs by a factor of 2 from the formula one can find in the textbooks (the so-called "Richardson equation"). The reason is that in the present solution we have used a partition function for the simplest quantum model of an ideal gas, in which the electron spin is not taken into account.

Problem No. 7

Consider a particle of mass m placed in a one-dimensional double delta function potential as shown below

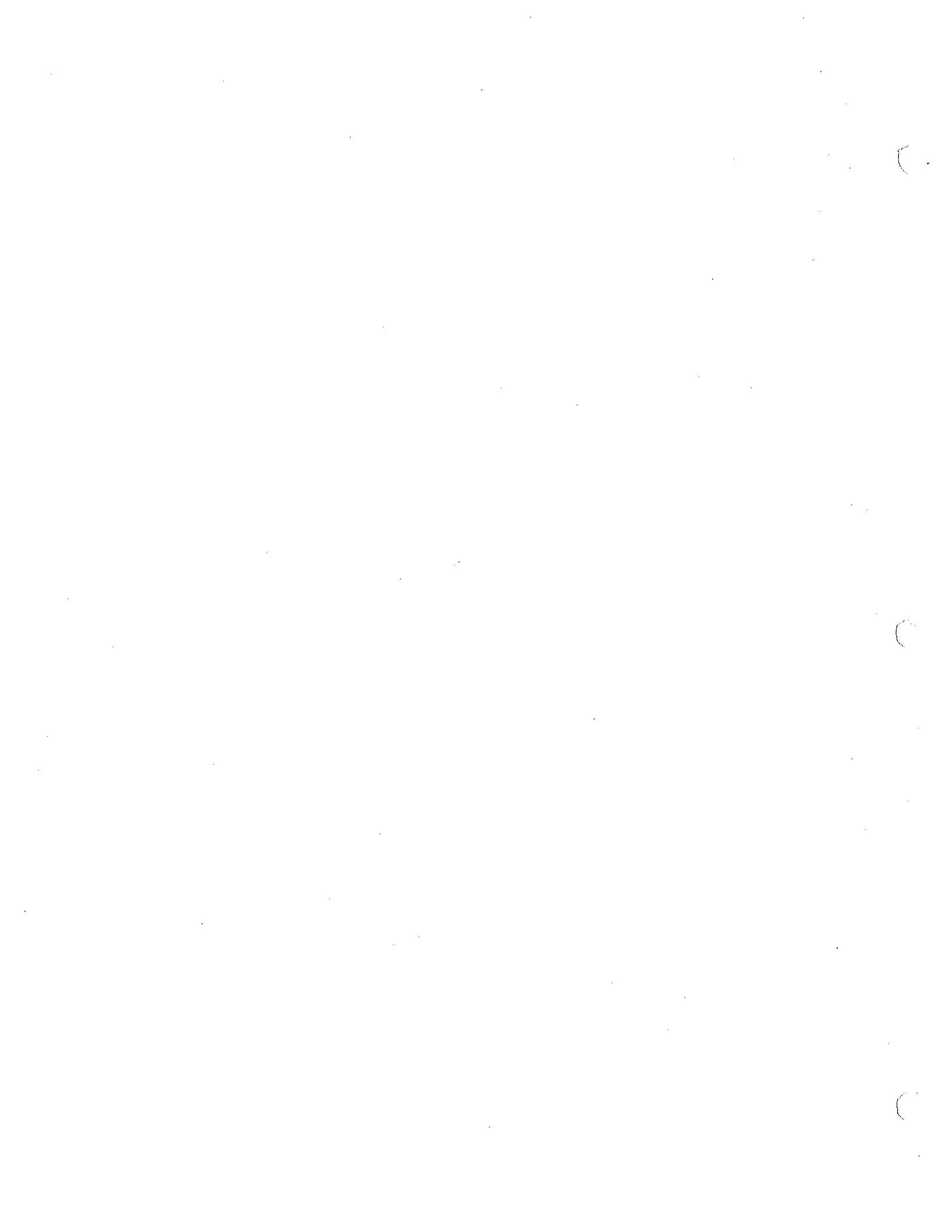


This attractive potential can be written as

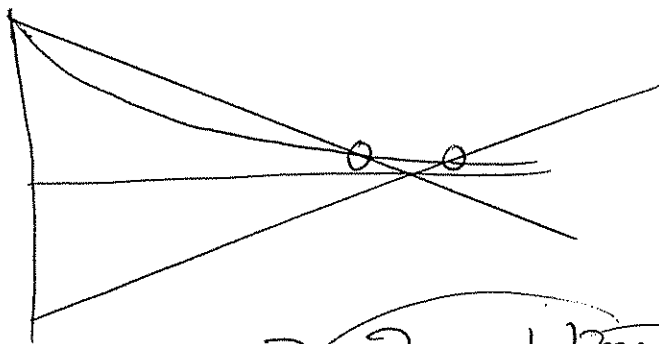
$$V(x) = -\lambda \frac{\hbar^2}{2ma} [\delta(x-a) + \delta(x+a)],$$

where λ is a dimensionless parameter that characterizes the strength of the wells and a is the same length as in the diagram.

- Find the equations that determine the bound state energy level spectrum of this quantum mechanical system. You can expect to encounter one or more transcendental equations (as in the finite square well problem). Show how the eigenvalues can be found graphically from these equations.
- How many bound states does this system have when $\lambda \gg 1$ and when $\lambda \ll 1$?
- Calculate the energy separation of the two lowest bound states in this system for the case of very large λ (strong wells). Sketch the wave functions for these states.

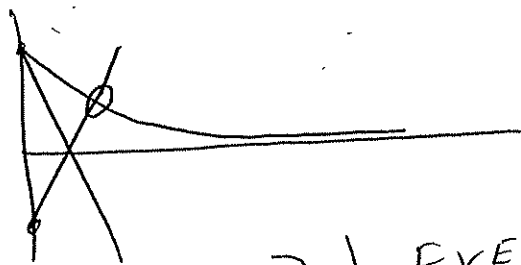


b) for $\lambda \gg 1$, slopes ($\frac{1}{\lambda}$) are shallow:



\Rightarrow 2 solutions
1 even, 1 odd

for $\lambda \ll 1$, slopes are steep



\Rightarrow 1 EVEN soln only

c) for $\lambda \gg 1$, $x \approx \lambda$ for both solutions

$$\text{let } \begin{cases} x_1 = \lambda + \epsilon & \text{for even soln} \\ x_2 = \lambda - \eta & \text{for odd soln} \end{cases} \quad \left. \begin{matrix} \epsilon, \eta \ll 1 \end{matrix} \right\}$$

$$\Rightarrow e^{-x_1} = \frac{x_1}{\lambda} - 1 \rightarrow e^{-\lambda - \epsilon} = \frac{\lambda + \epsilon}{\lambda} - 1 = \frac{\epsilon}{\lambda}$$

$$e^{-\lambda} (1 - \epsilon) \approx \frac{\epsilon}{\lambda}$$

$$e^{-\lambda} \approx \epsilon \left(\frac{1}{\lambda} + e^{-\lambda} \right)$$

\uparrow much smaller \Rightarrow neglect

$$\Rightarrow \lambda e^{-\lambda} \simeq \epsilon = x_1 - \lambda = 2k_1 a - \lambda$$

$$\Rightarrow 2k_1 a = \lambda e^{-\lambda} + \lambda$$

$$E_1 = -\frac{\hbar^2 k_1^2}{2m} = -\frac{\hbar^2}{2m} \frac{1}{4a^2} [\lambda + \lambda e^{-\lambda}]^2$$

$$E_1 = -\frac{\hbar^2}{8ma^2} \lambda^2 (1 + 2e^{-\lambda})$$

$$x_2: e^{-x_2} = 1 - \frac{x_2}{\lambda} \rightarrow e^{-\lambda} e^n = 1 - \frac{\lambda - n}{\lambda} = \frac{n}{\lambda}$$

$$e^{-\lambda} (1 + n) \simeq \frac{n}{\lambda}$$

$$e^{-\lambda} \simeq n \left(\frac{1}{\lambda} - e^{-\lambda} \right)$$

↑ neglect

$$\lambda e^{-\lambda} \simeq n = \lambda - x_2 = \lambda - 2k_2 a$$

$$2k_2 a = \lambda - \lambda e^{-\lambda}$$

$$E_2 = -\frac{\hbar^2}{8ma^2} \lambda^2 (1 - 2e^{-\lambda})$$

$$\Rightarrow \Delta E = E_2 - E_1 = -\frac{\hbar^2}{8ma^2} \lambda^2 [1 - 2e^{-\lambda} - (1 + 2e^{-\lambda})]$$

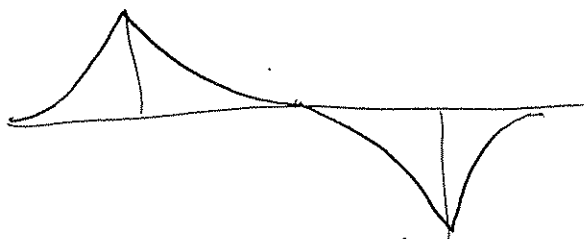
$$\Delta E = -\frac{\hbar^2}{2ma^2} \lambda^2 e^{-\lambda}$$

Wave functions:

even



odd



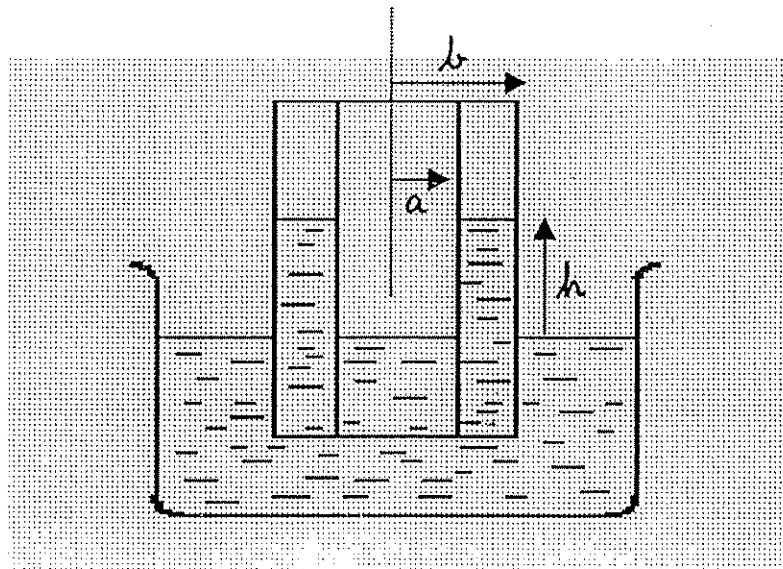
more curvature
 \Rightarrow
more energy

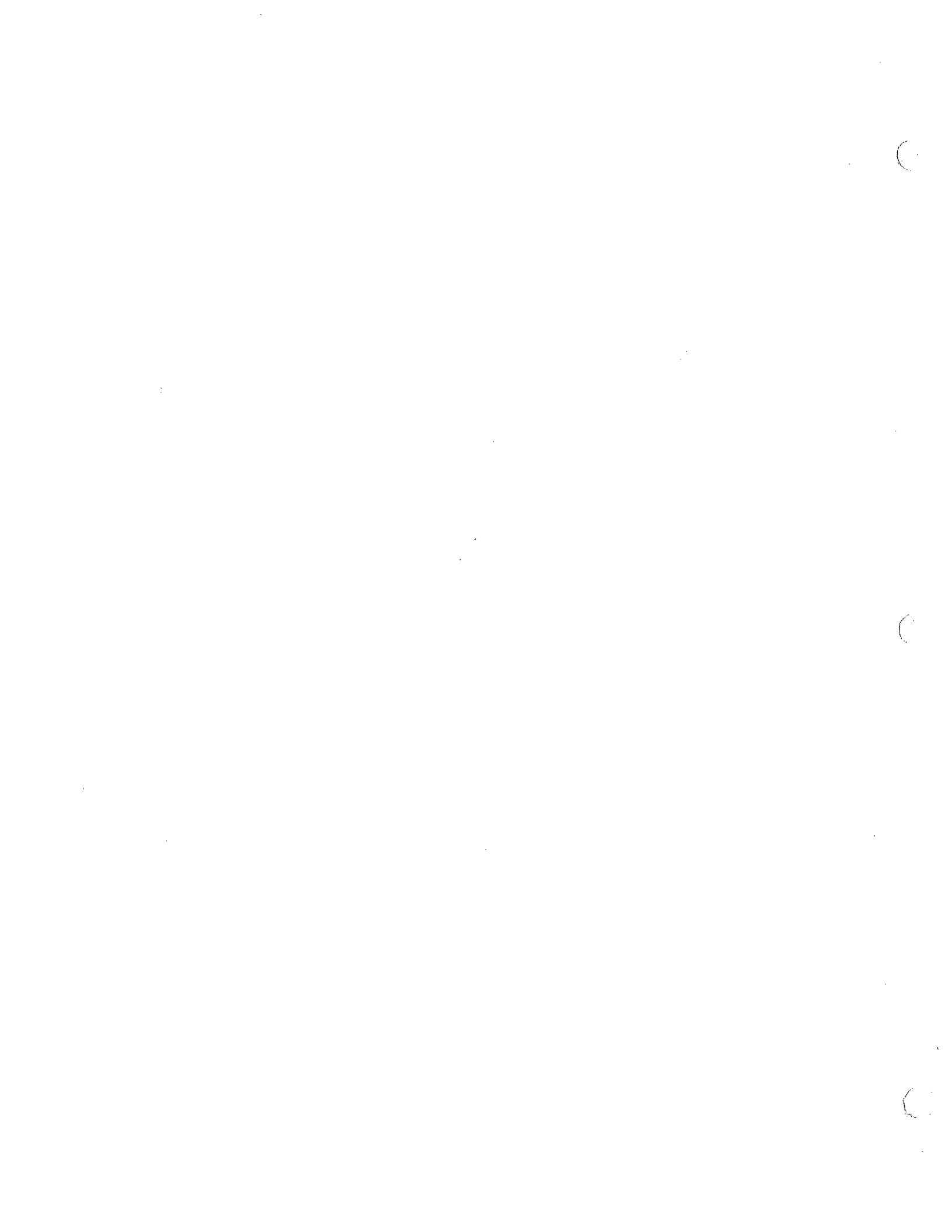


Problem No. 8

- a) (6 points) A capacitor is made from two thin coaxial cylindrical conducting pipes, fixed in position by an insulating disk at the upper end. The conductors have length ℓ and radii a and b , with $a < b$. What is the capacitance C , if no dielectric is present?
- b) (14 points) The capacitor in part (a) is lowered, with its axis vertical, into a jar of oil with mass density ρ and dielectric constant κ . When a battery, with a known potential difference V , is connected between the two conducting pipes, the oil level rises to a height h above the level outside the capacitor. Calculate h in terms of the given quantities and the gravitational acceleration g . (For simplicity, you may assume $b - a \ll a$.)

Cross-sectional view:



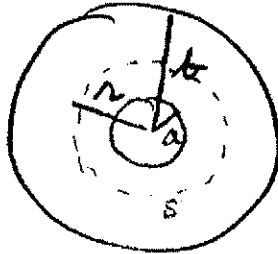


Coaxial Capacitor :

- (a) If we put a charge Q on the inner pipe and $-Q$ on the outer pipe, then the internal field is given by

Gauss's law :

$$\oint_S \vec{E} \cdot \hat{n} dA = \frac{Q}{\epsilon_0}$$



$$E \cdot 2\pi r l = \frac{Q}{\epsilon_0} \quad \rightarrow \quad E = \frac{Q}{2\pi \epsilon_0 l} \cdot \frac{1}{r} \quad (\text{radial})$$

$$\text{Potential difference: } V = \int_a^b E dr = \frac{Q}{2\pi \epsilon_0 l} \ln \frac{b}{a}$$

$$\text{Capacitance: } Q = CV \quad \Rightarrow \quad C = \frac{2\pi \epsilon_0 l}{\ln b/a}$$

- (b) The force F required to pull oil to a height h above the level outside the capacitor is given by

$$F = \rho g h \cdot \pi (b^2 - a^2).$$

If the oil level rises by δh , then work $F \delta h$ is done on the oil, the energy $U = \frac{1}{2} CV^2$ stored in the capacitor changes by δU , and the battery loses energy $V \delta Q$.

$$\text{Energy conservation} \Rightarrow F \delta h + \delta U = V \delta Q.$$

Since $Q = CV$,

$$V \delta Q = V^2 \delta C = 2 \delta \left(\frac{1}{2} CV^2 \right) = 2 \delta U.$$

$$\therefore F \delta h = + \delta U$$

$$\therefore F = \left. \frac{\partial U}{\partial h} \right|_V = \frac{V^2}{2} \frac{\partial C}{\partial h}$$

The partially filled capacitor can be regarded as a parallel combination of a dielectric-filled capacitor of height h and a vacuum-filled one with height $(l-h)$.

$$C = \frac{2\pi\epsilon_0}{\ln \frac{b}{a}} \left[\kappa h + (l-h) \right]$$

$$\therefore \frac{\partial C}{\partial h} = \frac{2\pi\epsilon_0}{\ln \frac{b}{a}} (\kappa - 1).$$

Equating the two expressions for F gives

$$\rho g h \pi (b^2 - a^2) = \frac{V^2}{2} \cdot \frac{2\pi\epsilon_0}{\ln \frac{b}{a}} (\kappa - 1).$$

$$\text{Thus } h = \frac{\epsilon_0 (\kappa - 1) V^2}{\rho g (b^2 - a^2) \ln \frac{b}{a}}.$$