

Department of Physics Comprehensive Examination No. 85

January 4 and 5, 1999

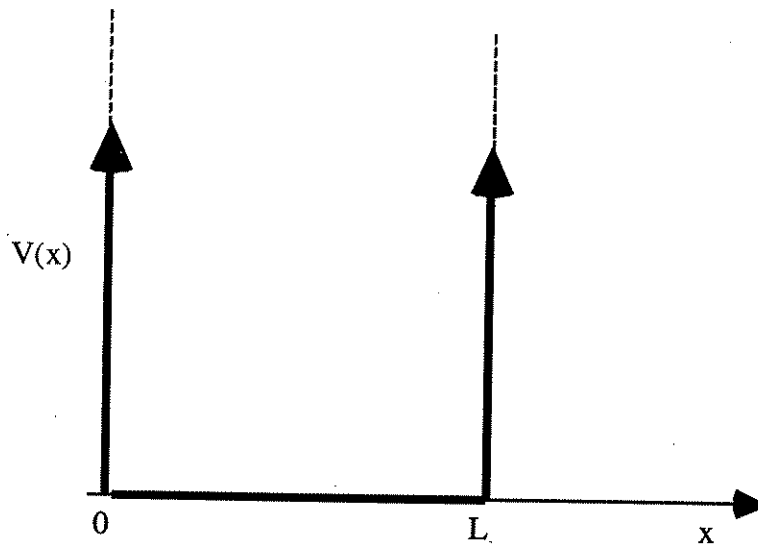
This Comprehensive Examination for Winter 1999 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Monday 4 January. Session Two (problems 3 and 4) begins at 1:30PM Monday 4 January. Session Three (problems 5 and 6) begins at 9:00 AM Tuesday 5 January. Session Four (problems 7 and 8) begins at 1:30PM Tuesday 5 January.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

Consider an infinite, one-dimensional square well potential as shown below.

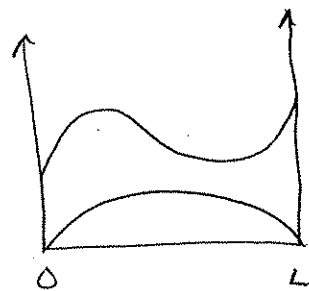


Two identical neutral particles with spin $1/2$ are placed in this potential. The interaction between the particles is described by a potential $V = \frac{4A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$, where A is a real constant that is much smaller than the ground state energy E_1 of a single particle in the well.

- Determine the eigenstates of the system, and the corresponding eigenenergies to first order in A .
- Explicitly write down the energies and the quantum state vectors of the 6 lowest states of distinct energy of this system.

The single particle states in the infinite well are

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$



The energies are

$$E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2} = n^2 E_1 \quad \text{w/} \quad E_1 = \frac{\hbar^2 \pi^2}{2mL^2} = \text{ground state.}$$

Since the spatial dependence of the states is not important in this problem, we will denote the states using Dirac notation:

$$H_0 |n\rangle = E_n |n\rangle$$

For 2 particles in the well, we will write the system ket as $|n_1, n_2\rangle$ where n_i labels the i th particle.

Since the particles have spin, we must include that in the total system description.

So let the total quantum state vector be written as

$$|\Psi\rangle = |n_1, n_2\rangle |\chi_1\rangle |\chi_2\rangle$$

where $|\chi_i\rangle$ is the spinor for the i th particle.

Since the particles are identical fermions (spin $\frac{1}{2}$), the total state vector must be ~~an~~ antisymmetric with respect to exchange of the 2 particles.

Before antisymmetrizing the state vector, it is useful to look at the perturbation:

$$V = \frac{4A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$$

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2 \vec{S}_1 \cdot \vec{S}_2$$

$$\Rightarrow \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2} (\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2)$$

$$= \frac{1}{2} [S(S+1)\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2]$$

$$\Rightarrow V = A [2S(S+1) - 3]$$

$$S = 0, 1 \quad \text{since } S_1 = S_2 = \frac{1}{2}$$

Since the potential is diagonal with respect to the total spin S , it is most useful to describe the spin part of the system using eigenstates of S, S_z rather than S_{1z}, S_{2z} .

Recall that $|S, m\rangle$ states are:

$$|0, 0\rangle = \frac{1}{\sqrt{2}} [|1+, -\rangle - |1-, +\rangle]$$

$$|1, 1\rangle = |1+, +\rangle$$

$$|0, 0\rangle = \frac{1}{\sqrt{2}} [|1+, -\rangle + |1-, +\rangle]$$

$$|1, -1\rangle = |1-, -\rangle$$

Note that the singlet state $|0, 0\rangle$ is antisymmetric with respect to exchange, while the triplet states $|1, M\rangle$ are symmetric. Since the total state vector must be antisymmetric, singlet spin must be associated with symmetric spatial states, and triplet states must go with antisymmetric spatial states.

$$\text{sym: } |n_1, n_2\rangle \rightarrow \frac{1}{\sqrt{2}} [|n_1, n_2\rangle + |n_2, n_1\rangle] \equiv |n_1, n_2\rangle_S$$

$$\text{or } |n_1, n_1\rangle \text{ for } n_1 = n_2$$

$$\text{antisym: } |n_1, n_2\rangle \rightarrow \frac{1}{\sqrt{2}} [|n_1, n_2\rangle - |n_2, n_1\rangle] \equiv |n_1, n_2\rangle_A$$

$$\text{or } 0 \text{ for } n_1 = n_2$$

So for $n_1 = n_2$, only singlet spin states allowed.

Allowed states of system are

$$|n_1, n_2\rangle / |00\rangle$$

$$|n_1, n_2\rangle_S |00\rangle \left. \vphantom{|n_1, n_2\rangle_S |00\rangle} \right\} n_1 \neq n_2$$

$$|n_1, n_2\rangle_A |1M\rangle$$

Energy perturbations are simply:

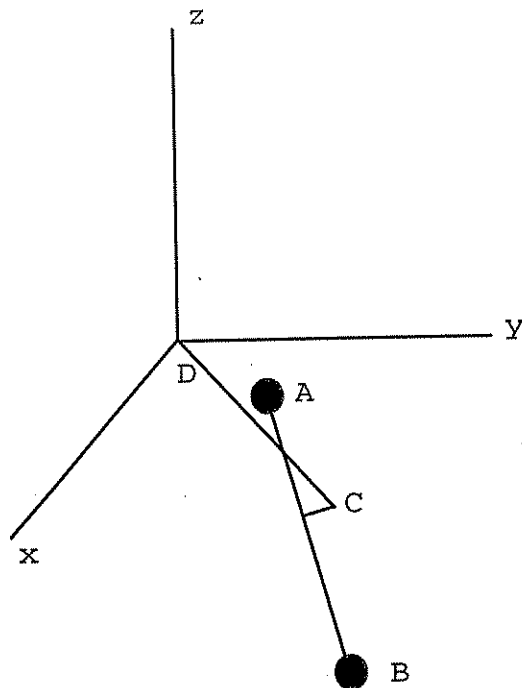
$$\Delta E = \langle \psi | V | \psi \rangle = A [2s(s+1) - 3]$$

Thus we get:

<u>Eigenstates</u>	<u>Energies</u>
$n_1 = n_2: n_1, n_1\rangle 00\rangle$	$2n_1^2 E_1 - 3A$
$n_1 \neq n_2 \left\{ \begin{array}{l} n_1, n_2\rangle_S 00\rangle \\ n_1, n_2\rangle_A 1m\rangle \end{array} \right.$	$(n_1^2 + n_2^2) E_1 - 3A$
	$(n_1^2 + n_2^2) E_1 + A$

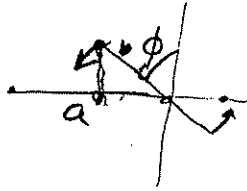
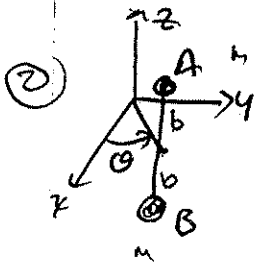
Lowest 6 states are:

State				<u>Energy</u>
<u>n_1</u>	<u>n_2</u>	<u>S</u>	<u>M</u>	
1	1	0	0	$2E_1 - 3A$
1	2	0	0	$5E_1 - 3A$
1	2	1	M	$5E_1 + A$
2	2	0	0	$8E_1 - 3A$
1	3	0	0	$10E_1 - 3A$
1	3	1	M	$10E_1 + A$



Consider the object in the figure. A weightless rod CD of length a rotates freely in the xy plane about D . The weightless rod AB of length $2b$ has two equal masses m fixed at its ends. The middle of AB is attached by a short peg to point C in such a way that rod AB rotates freely in a plane which includes the rod CD and the z axis.

- Determine the Lagrangian for this system.
- Deduce the Lagrangian equations of motion for this system.
- Suppose that initially the angular velocity of rotation of CD is ω , and that the rod AB is nearly parallel to CD . Discuss the resulting motion of the two rods.



$$v_A^2 = (b\dot{\phi})^2 + (a - b\sin\phi)^2 \dot{\theta}^2$$

$$v_B^2 = (b\dot{\phi})^2 + (a + b\sin\phi)^2 \dot{\theta}^2$$

a) $V = mgh - mgh = 0$ (const)

$$L = T - V = T = \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = \frac{1}{2}m(2b^2\dot{\phi}^2 + 2a^2\dot{\theta}^2 + 2b^2\sin^2\phi\dot{\theta}^2)$$

$$= m(b^2\dot{\phi}^2 + a^2\dot{\theta}^2 + b^2\sin^2\phi\dot{\theta}^2)$$

b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2ma^2\dot{\theta} + 2b^2m\sin^2\phi\dot{\theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2ma^2\ddot{\theta} + 2b^2m\sin^2\phi\ddot{\theta} + 4b^2m\sin\phi\cos\phi\dot{\phi}\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = 2mb^2\dot{\phi}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 2mb^2\ddot{\phi}, \quad \frac{\partial L}{\partial \phi} = 2mb^2\dot{\theta}^2\sin\phi\cos\phi$$

eqn
Mtn

① $a^2\ddot{\theta} + b^2\sin^2\phi\ddot{\theta} + 2b^2\sin\phi\cos\phi\dot{\phi}\dot{\theta} = 0$

② $\ddot{\phi} - \dot{\theta}^2\sin\phi\cos\phi = 0$

c) $\phi \approx \pi/2, \quad \dot{\theta} \approx \omega$

1) $a^2\ddot{\theta} + b^2\ddot{\theta} = 0 \Rightarrow \ddot{\theta} \approx \text{const}$ (rotates in circle)

2) $\phi \approx \frac{\pi}{2} + \sigma, \quad \sin\phi \approx 1, \quad \cos\phi \approx -\sigma$

$$\ddot{\sigma} + \omega^2\sigma = 0$$

$$\ddot{\sigma} = -\omega^2\sigma \Rightarrow \sigma = \sigma_0 \sin\omega t$$

$$\phi = \frac{\pi}{2} + \sigma_0 \sin\omega t \quad \left\{ \text{SHM about horizontal} \right.$$

Two atoms with identical spins $S_1 = S_2 = 1/2$ are coupled by a magnetic exchange interaction, described by the Hamiltonian:

$$H = \frac{A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$$

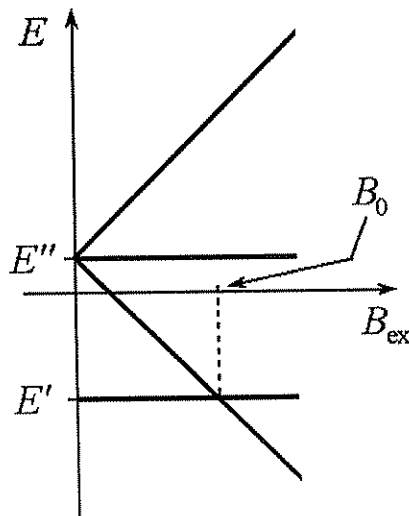
Such a pair may be in two possible quantum states, with the total spin either $S_T = 0$ or $S_T = 1$. One can readily show that in the absence of any other interactions the energy eigenvalues are:

$$E(S_T) = -3A + 2A S_T (S_T + 1)$$

The $S_T = 1$ state has a non-zero magnetic moment, the projection of which on the quantization axis z may take the values of $-g\mu_B S_{Tz}$ (where $S_{Tz} = -1, 0, \text{ or } +1$, μ_B is the Bohr magneton, and $g = 2$ is the so-called Lande factor). Thus in an external magnetic field B_{ex} the expression for energy eigenvalues becomes:

$$E(S_T, S_{Tz}, B_{ex}) = -3A + 2A S_T (S_T + 1) + g\mu_B B_{ex} S_{Tz}$$

where the last right-hand term represents the Zeeman splitting. The energy levels for a positive A (antiferromagnetic coupling between the spins) are schematically displayed in the plot below, in which $E = -3A$, $E' = +A$, and $B_0 = 4A/g\mu_B$.

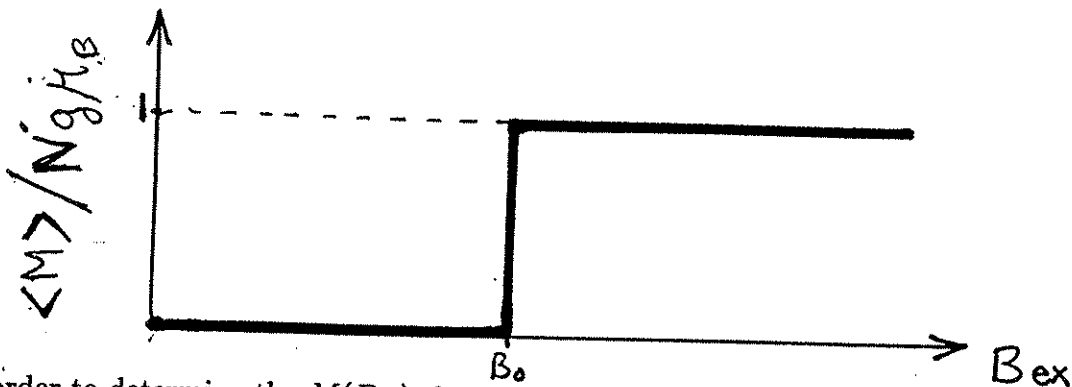


Consider a system consisting of N such spin pairs. There are no interactions of any kind between the pairs. First, answer the question: what would be the shape of the magnetization function $M(B_{ex})$ for such a system at $T = 0$? Then, derive the expression describing the $M(B_{ex})$ function at finite but low temperatures ($k_B T \ll A$).

Problem # — Solution

Comment: this problem is based on an existing experimental technique of determining anti-ferromagnetic coupling constants for isolated pairs of magnetic ions.

Below B_0 the ground state of the pair is the one with zero spin and zero magnetic moment. For $B_{\text{ex}} > B_0$ this situation changes, and now the state with $S_T = 1$, and with the magnetic moment projection $+g\mu_B$ is the ground state. Hence, at zero temperature the magnetization changes sharply at $B_{\text{ex}} = B_0$ from zero to $M = Ng\mu_B$.



In order to determine the $M(B_{\text{ex}})$ shape at finite temperatures, one has to consider the partition sum. Since the spin pairs do not interact with one another, the system partition function \mathcal{Z} is simply:

$$\mathcal{Z} = (\mathcal{Z}_{\text{s.p.}})^N, \quad (1)$$

where $\mathcal{Z}_{\text{s.p.}}$ is the partition sum for a single pair.

It is convenient to set the energy E' as zero. Then, the Boltzmann factor $\exp(-E/kT)$ for the zero-spin state will be 1. The energies of the three split states can be written as:

$$\begin{aligned} S_z = -1: \quad E_{(-)} &= 4A \left(1 + \frac{B_{\text{ex}}}{B_0}\right) \\ S_z = 0: \quad E_{(0)} &= 4A \\ S_z = +1: \quad E_{(+)} &= 4A \left(1 - \frac{B_{\text{ex}}}{B_0}\right) \end{aligned}$$

Note that if $kT \ll A$, the energies $E_{(-)}$ and $E_{(0)}$ will *always* be much larger than kT , and the terms in the partition sum corresponding to these states, $\exp[-E_{(-)}/kT]$ and $\exp[-E_{(0)}/kT]$, will always be much lower than 1. Only the term corresponding to $E_{(+)}$, $\exp[-E_{(+)}/kT]$, will become *comparable* to 1 with B_{ex} approaching B_0 from below, and *larger* than 1 for $B_0 > B_{\text{ex}}$. In view of this, there are only two terms that really matter in the partition sum, and one can write with a good approximation:

$$\mathcal{Z}_{\text{s.p.}} \cong 1 + \exp \left[-\frac{4A}{kT} \left(1 - \frac{B_{\text{ex}}}{B_0}\right) \right] \quad (2)$$

From now on, one can use different procedures to obtain the thermal average of the system magnetization $\langle M \rangle$. A highly 'professional' approach is to take advantage of the general formula, valid for any system in an external magnetic field:

$$\langle M \rangle = kT \frac{\partial}{\partial B_{\text{ex}}} \ln \mathcal{Z}. \quad (3)$$

(it is straightforward to derive this formula using the standard canonical formalism recipe — see, e.g., the solution of Problem # from the Fall 98 Comprehensive Exam).

Combining Eqs. 2, 3 and 4, and then plugging in Eq. 1, we obtain:

$$\langle M \rangle = kT \frac{N}{Z_{s.p.}} \frac{\partial Z_{s.p.}}{\partial B_{ex}} = kTN \frac{\frac{4A}{kTB_0} \exp \left[-\frac{4A}{kT} \left(1 - \frac{B_{ex}}{B_0} \right) \right]}{1 + \exp \left[-\frac{4A}{kT} \left(1 - \frac{B_{ex}}{B_0} \right) \right]} = Ng\mu_B \frac{\exp \left[-\frac{4A}{kT} \left(1 - \frac{B_{ex}}{B_0} \right) \right]}{1 + \exp \left[-\frac{4A}{kT} \left(1 - \frac{B_{ex}}{B_0} \right) \right]}$$

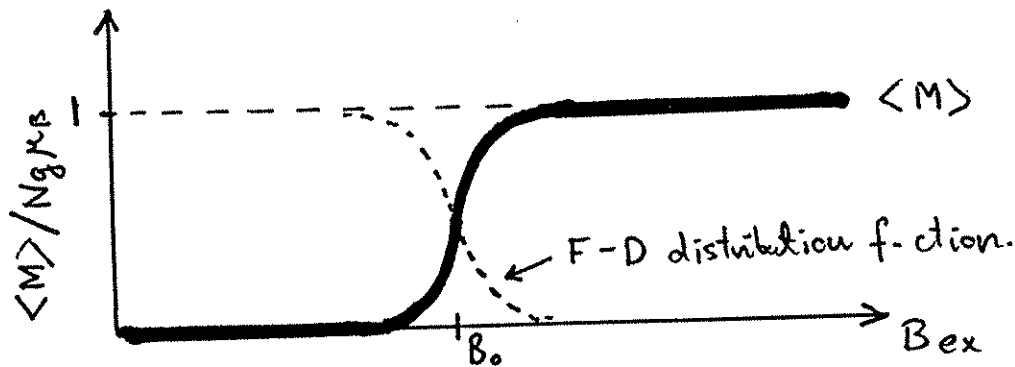
Again, using Eq. 1, we obtain:

$$\langle M \rangle = Ng\mu_B \frac{\exp \left[-\frac{g\mu_B}{kT} (B_0 - B_{ex}) \right]}{1 + \exp \left[-\frac{g\mu_B}{kT} (B_0 - B_{ex}) \right]} = Ng\mu_B \frac{1}{\exp \left[-\frac{g\mu_B}{kT} (B_{ex} - B_0) \right] + 1}. \quad (4)$$

The above equation can also be written in an equivalent form:

$$\langle M \rangle = Ng\mu_B \left\{ 1 - \frac{1}{\exp \left[\frac{g\mu_B}{kT} (B_{ex} - B_0) \right] + 1} \right\}. \quad (5)$$

It is easy to make a plot of $\langle M \rangle$ vs. B_{ex} if one notices that the second term in the braces has the same mathematical form as the well-known Fermi-Dirac distribution function.



An alternative method is not to use Eq. (4) and the partition sum for the entire system, but to calculate first the average contribution $\langle \mathcal{M}_{s.p.} \rangle$ of a *single pair* to the total magnetization. Using the standard formula for calculating thermal averages, we obtain:

$$\langle \mathcal{M}_{s.p.} \rangle = \frac{\sum_i \mathcal{M}_i \exp(-E_i/kT)}{Z_{s.p.}}$$

where \mathcal{M}_i is the projection of the pair magnetic moment on the field direction in the i -th state. There are four such states, but the two terms corresponding to the higher-energy states can be neglected (because the Boltzmann factors for them are very small). For one of the two remaining states — the one with zero spin — \mathcal{M}_i is zero. Thus, the sum in the denominator effectively reduces to a single term. By inserting appropriate expressions for \mathcal{M}_i and E_i , one obtains:

$$\langle \mathcal{M}_{s.p.} \rangle = \frac{g\mu_B \exp \left[-\frac{g\mu_B}{kT} (B_0 - B_{ex}) \right]}{1 + \exp \left[-\frac{g\mu_B}{kT} (B_0 - B_{ex}) \right]}. \quad (6)$$

The magnetization of the entire system is related to the single-pair magnetization simply as $\langle M \rangle = N \langle \mathcal{M}_{s.p.} \rangle$. Continuing, one obtains from Eq. (7) identical expressions as in Eqs. (5) and (6).

Consider a spin 1/2 particle with a magnetic moment.

- At time $t = 0$, we measure S_y and find the result $+\hbar/2$. What is the state vector $|\psi(0)\rangle$ immediately after the measurement?
- Immediately after this measurement, we apply a uniform time-dependent magnetic field in the positive z direction. The resultant Hamiltonian is

$$H(t) = \omega(t)S_z.$$

Assume that $\omega(t)$ is zero for $t < 0$ and for $t > T$ and increases linearly from 0 to ω_0 during the time $0 \leq t \leq T$. Solve the time-dependent Schrödinger equation to determine the state vector for all times $t > 0$.

- At a time $t > T$, we measure S_y . What are the possible results of that measurement and the corresponding probabilities for those results? For what values of T are we certain of the result of the measurement?

You may want to know that

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$a) \quad t=0 \quad : \quad S_y \rightarrow +\frac{\hbar}{2}$$

$$\Rightarrow |\psi(0)\rangle = |+\rangle_y$$

rewrite in terms of $|+\rangle, |-\rangle$ - eigenvectors of S_z

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} [|+\rangle + i |-\rangle]$$

$$b) \quad H(t) = w(t) S_z \quad \text{for } t > 0$$

Sch. eqn:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

$$H(t) = \frac{\hbar}{2} \begin{pmatrix} w(t) & 0 \\ 0 & -w(t) \end{pmatrix}$$

$$\text{let } |\psi(t)\rangle = a(t) |+\rangle + b(t) |-\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

with initial conditions $a(0) = \frac{1}{\sqrt{2}}, b(0) = i/\sqrt{2}$

$$\Rightarrow i\hbar \dot{a}(t) = \frac{\hbar}{2} w(t) a(t)$$

$$i\hbar \dot{b}(t) = -\frac{\hbar}{2} w(t) b(t)$$

$$\frac{\dot{a}(t)}{a(t)} = -i \frac{w(t)}{2}$$

$$\frac{d}{dt} [\ln a(t)] = -\frac{1}{2} w(t)$$

Now integrate:

$$\Rightarrow \ln a(t) - \ln a(0) = \int_0^t -\frac{i}{2} w(t) dt$$

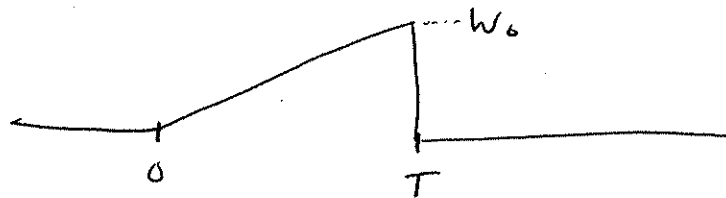
$$\Rightarrow a(t) = a(0) \exp \left[-\frac{i}{2} \int_0^t w(t) dt \right]$$

$$b(t) = b(0) \exp \left[\frac{i}{2} \int_0^t w(t) dt \right]$$

$$\text{let } \theta(t) = -\frac{1}{2} \int_0^t w(t) dt$$

$$\Rightarrow |\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left[e^{i\theta(t)} |+\rangle + i e^{-i\theta(t)} |-\rangle \right]$$

$w(t)$:



$$\Rightarrow w(t) = \frac{w_0 t}{T} \quad 0 \leq t \leq T$$

$$\Rightarrow \text{for } t < T: \quad \theta(t) = -\frac{1}{2} \int_0^t \frac{w_0 t}{T} dt = -\frac{1}{2} \frac{w_0}{T} \frac{t^2}{2}$$

$$\Rightarrow \theta(t) = \frac{-w_0 t^2}{4T} \quad 0 \leq t \leq T$$

~~$\theta(t)$~~

For $t > T$, $\omega(t) = 0$

$$\Rightarrow \theta(t > T) = -\frac{1}{2} \int_0^T \frac{\omega_0 t}{T} dt + \int_T^t \cancel{\theta} dt$$

$$\boxed{\theta(t > T) = -\frac{\omega_0 T}{4}} = \theta(T)$$

c) measure S_y : can get $\pm \frac{\hbar}{2}$

$$|\psi(t > T)\rangle = \frac{1}{\sqrt{2}} \left[e^{i\theta(T)} |+\rangle + i e^{-i\theta(T)} |-\rangle \right]$$

$$\begin{aligned} P_{+\frac{\hbar}{2}} &= \left| \langle + | \psi(t) \rangle \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} \left[\langle + | - i \langle - | \right] \frac{1}{\sqrt{2}} \left\{ e^{i\theta(T)} |+\rangle + i e^{-i\theta(T)} |-\rangle \right\} \right|^2 \\ &= \left| \frac{1}{2} e^{i\theta(T)} + \frac{1}{2} e^{-i\theta(T)} \right|^2 \\ &= \cos^2 \theta(T) \end{aligned}$$

$$\boxed{P_{+\frac{\hbar}{2}} = \cos^2 \left(\frac{\omega_0 T}{4} \right)}$$

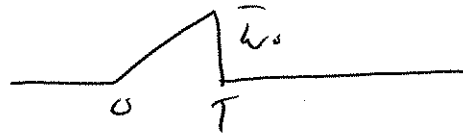
$$\boxed{P_{-\frac{\hbar}{2}} = \sin^2 \left(\frac{\omega_0 T}{4} \right)}$$

measurements are certain (i.e. $P_+ = 1, P_- = 0$
or $P_+ = 0, P_- = 1$) when

$$\frac{\omega_0 T}{4} = n \frac{\pi}{2} \quad n = 1, 3, 5, \dots$$

$$\Rightarrow T = \frac{n \pi}{2 \omega_0}$$

Area under pulse



$$\Rightarrow \frac{\omega_0 T}{2} = n \pi$$

Hence name π pulse

- a. The image of a mouse levitated in a strong magnetic field has appeared in various publications over the last year. Suppose you have a very small object to be levitated in the magnetic field of a single current loop of a radius a much larger than the size of the object. The object has mass M and is diamagnetic. The current loop is oriented horizontally, and the magnetic field in the center is pointed upward. The object is to be levitated just slightly above the center of the loop. In spherical coordinates, the magnetic field just above the center of the loop, where $r \ll a$ and θ is small, is approximately

$$B_r = C \left[1 - \frac{3}{2} \frac{r^2}{a^2} \right] \text{ and } B_\theta = -C\theta \left[1 - 6 \frac{r^2}{a^2} \right], \text{ where } C \text{ is a constant.}$$

Show that there is a stable, equilibrium position just above the center of the loop.

- b. The flat end of a long permanent magnet with magnetization $\vec{M} = M\hat{z}$ is placed very close to the surface of a material exhibiting a large permeability μ . The surface is the xy plane, and the cross sectional area of the magnet is A . Show that when the magnet is very close to the surface, the attractive force is proportional to AM^2 as $\mu \rightarrow \infty$.

a. Use either $U = -\vec{m} \cdot \vec{B}$ with $\vec{m} = m_0 \vec{B}$

$$\text{or } \vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B})$$

Ureate first: $U = -m_0 B^2$

$$B^2 = B_r^2 + B_\theta^2 = C^2 \left[1 - \frac{3r^2}{a^2} + \frac{9r^4}{4a^4} \right] + C^2 \theta^2 \left[1 - \frac{12r^2}{a^2} + \frac{36r^4}{a^4} \right]$$

$$B^2 \approx C^2 \left[1 + \theta^2 - \frac{3r^2}{a^2} \right]$$

$$U = -m_0 C^2 \left[1 + \theta^2 - \frac{3r^2}{a^2} \right]$$

$$\vec{F} = -\vec{\nabla} U \text{ where } \vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial U}{\partial r} = -m_0 C^2 \left(-\frac{6r}{a^2} \right) = 6m_0 C^2 \frac{r}{a^2}$$

$$\frac{1}{r} \frac{\partial U}{\partial \theta} = -\frac{2m_0 C^2 \theta}{r}$$

$$\text{So } \vec{F} = -6m_0 C^2 \frac{r}{a^2} \hat{r} + \frac{2m_0 C^2 \theta}{r} \hat{\theta}$$

F_r is ~~downward~~ upward for $m_0 < 0$ and F_θ is toward the center if $m_0 < 0$

Hence a stable position at some r_e .

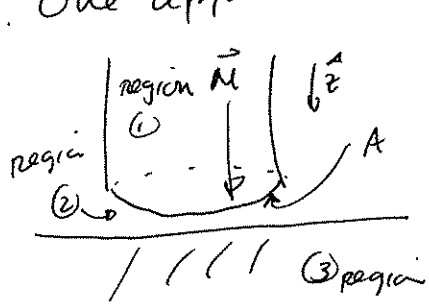
$$\text{Furthermore, } Mg = 6|m_0| \frac{C^2 r_e}{a^2} \Rightarrow r_e = \frac{a^2 Mg}{6|m_0| C^2}$$

$$\text{Now } \vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B}) = m_0 \vec{\nabla} B^2$$

$$\frac{\partial B^2}{\partial r} = C^2 \left(-\frac{6r}{a^2} \right) \text{ and } \frac{1}{r} \frac{\partial B^2}{\partial \theta} = \frac{C^2 \theta}{r}$$

$$\text{Then } \vec{F} = -6m_0 C^2 \frac{r}{a^2} \hat{r} + \frac{2m_0 C^2 \theta}{r} \hat{\theta}, \text{ as above.}$$

b. One approach is to use the stress tensor



$$\vec{H}_2 = 4\pi\sigma_m \hat{z} = 4\pi(\hat{z} \cdot \vec{M}) \hat{z} = 4\pi M \hat{z}$$

$$\vec{B}_2 = \vec{H}_2 \text{ since } \mu = 1 \text{ in air}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \text{ everywhere } \Rightarrow B_{\perp} \text{ is continuous}$$

$$\text{So in region (3), } \vec{B}_3 = \vec{B}_2 = 4\pi M \hat{z}$$

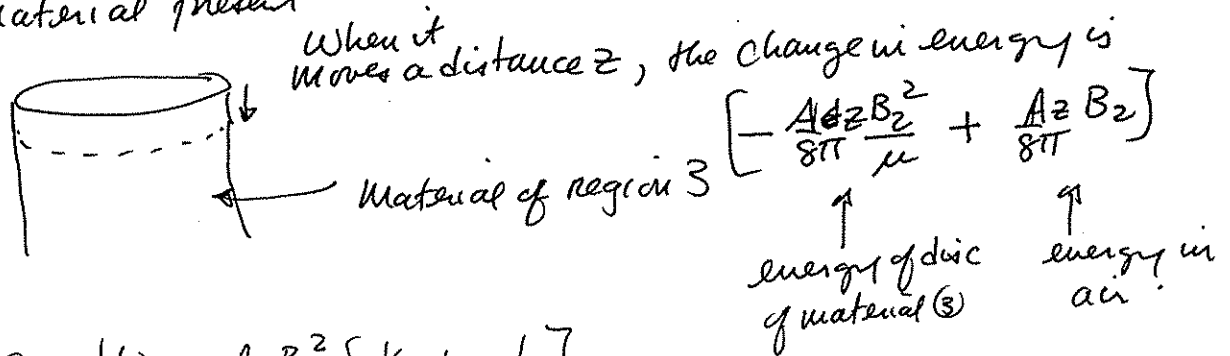
$$\text{Stress tensor } T_{ij} = \frac{1}{4\pi} \left[\epsilon E_i E_j + \frac{B_i B_j}{\mu} - \frac{1}{2} (E^2 + B^2) \delta_{ij} \right]$$

$$F_z = \int_{\text{area}} T_{zz} da = \frac{A}{4\pi} \left[\frac{B_3^2}{\mu} - \frac{B_3^2}{2} \right] = \frac{A}{4\pi} B_3^2 \left[\frac{1}{\mu} - \frac{1}{2} \right]$$

$$\text{So } F_z \rightarrow -\frac{1}{2} \frac{A B_3^2}{4\pi} \text{ as } \mu \rightarrow \infty$$

$$\text{or } F_z = -2\pi A M^2 \text{ as } \mu \rightarrow \infty \quad \text{Force on the material of region 3.}$$

The other approach is to find the energy with and without the material present



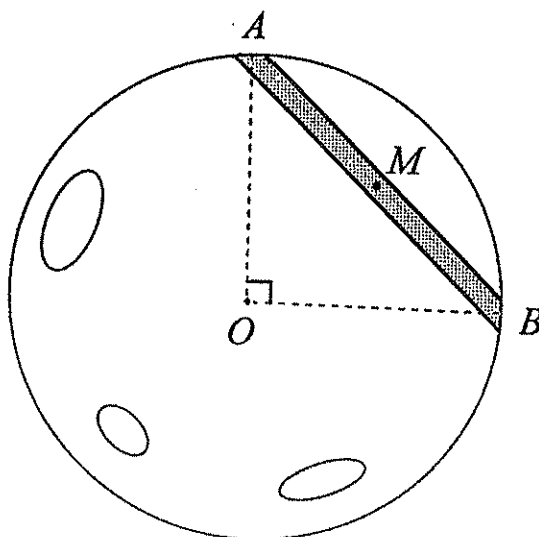
$$\text{So } \frac{dU}{dz} = \frac{A B_2^2}{8\pi} \left[\frac{1}{\mu} - 1 \right]$$

$$\text{and } F_z = -\frac{dU}{dz} = \frac{A B_2^2}{8\pi} \left[1 - \frac{1}{\mu} \right] \rightarrow -\frac{A B_2^2}{8\pi} = -2\pi M^2 A \text{ as } \mu \rightarrow \infty$$

Long, long time ago the Moon was inhabited by intelligent creatures, the Lunars. They built two cities: A at the north pole, and B at the equator. In order to travel faster between A and B , the Lunars dug a straight channel connecting the city centers. They sealed it at both ends and filled it with air. The air temperature in the channel was $T = 293$ K, and the pressure in the middle of the channel (point M at the figure) was $P_0 = 1$ atm.

Find the air pressure in the channel at the Moon surface.

The radius of the Moon is approximately 1750 km. The acceleration due to gravity on the surface of the Moon is $1/6 g$, where g is the gravity acceleration on Earth. Assume that the Moon can be treated with a good approximation as a sphere of a uniform density. Also, assume that the air in the channel obeys the ideal gas laws, and that it can be treated with a good approximation as a single-component gas with molecular weight $m = 29$.



Hint: You may want to use the chemical potential of an ideal gas, $\mu = kT \ln(n)$, where n is the concentration (there are other methods of solving this problem, of course – if you prefer not to use μ , such a solution will also be accepted).

Problem # — Solution

Suppose that the uniform density of the Moon is ρ . If a body with unit mass is at the distance $r \leq a$ from the Moon center, the force of gravity acting on it is:

$$F = G \frac{M_{\text{moon}}}{r^2} = G \frac{\frac{4}{3}\pi\rho a^3}{r^2} \quad (1)$$

However, if the object is located below the surface of the Moon ($r < a$), then only the mass *inside* the sphere with radius r contributes to the attractive force:

$$F = G \frac{\frac{4}{3}\pi\rho r^3}{r^2} = \frac{4}{3}G\pi\rho r \quad (2)$$

In other words, the force changes linearly with r — consequently, the gravitational potential is proportional to r^2 :

$$U(r) = Cr^2 \quad (3)$$

The acceleration due to gravity is equal to the force acting on a mass unit, and it is also equal to the derivative dU/dr . Hence, the acceleration g_m at the Moon surface is:

$$g_m(a) = \left(\frac{dU}{dr} \right)_{r=a} = 2Ca \quad (4)$$

But from the problem statement we also know that at the Moon surface $g_m = \frac{1}{6}g_E$. From that we can obtain the value of C , and the formula for the r -dependent gravitational potential inside the Moon becomes:

$$U(r) = \frac{g_E}{12a} r^2. \quad (5)$$

Having determined the $U(r)$ dependence, we can now handle the pressure problem. There are several possible ways of approaching this problem — the most convenient is to use the chemical potential μ . For a gas in thermal equilibrium μ is constant. Also, if the gas is located in a field with varying potential, its effective chemical potential is the sum of:

$$\mu_{\text{eff}} = kT \ln n + mU. \quad (6)$$

The first term, in which n is the concentration (i.e., the number of gas molecules per unit volume) is the 'thermal' part, and the second term is the potential energy of a single molecule (here m is the mass of a single molecule). Taking advantage of the $\mu_{\text{eff}} = \text{const.}$ condition, we can therefore write:

$$\ln \left[\frac{n(r_1)}{n(r_2)} \right] = \frac{mU(r_2)}{kT} - \frac{mU(r_1)}{kT} = \frac{mg_E}{12akT} (r_2^2 - r_1^2), \quad (7)$$

where $n(r_1)$, $n(r_2)$ are the concentrations at two different points in the tunnel, with distances r_1 , r_2 from the Moon center.

It is elementary to show that for an ideal gas at constant T the concentration n is proportional to the pressure. Hence, we can replace the ratio of concentrations in Eq. (7) by the ratio of pressures. Also, it is elementary to show that the distance of the point in the middle of the tunnel from the Moon center is $a/\sqrt{2}$. From Eq. (7) we obtain:

$$\ln \left[\frac{P_{\text{surface}}}{P_{\text{middle}}} \right] = \frac{mg_E}{12akT} (r_{\text{middle}}^2 - r_{\text{surface}}^2), = \frac{mg_E}{12akT} \left(\frac{a^2}{2} - a^2 \right) = -\frac{mg_E}{24kT} a \quad (8)$$

It is convenient to replace the Boltzmann constant with the gas constant R using:

$$k = \frac{R}{N_{\text{Avog.}}}, \quad (9)$$

where $N_{\text{Avog.}}$ is the Avogadro number. We obtain:

$$\ln \left[\frac{P_{\text{surface}}}{P_{\text{middle}}} \right] = -\frac{N_{\text{Avog.}} m g_E}{24RT} a. \quad (10)$$

But $N_{\text{Avog.}} m$ is simply the mass of a gas mole — 29 g in the present case. Accordingly,

$$\ln \left[\frac{P_{\text{surface}}}{P_{\text{middle}}} \right] = -\frac{(0.029 \text{ kg/mole}) g_E}{24RT} a. \quad (11)$$

At 293 K, the value of the expression at the right side is (the value of the R constant, 8.31 J/K·mole, may be readily obtained by multiplying the Boltzmann constant by the Avogadro number, which are both listed in the "FORMULAS AND DATA" sheet):

$$\ln \left[\frac{P_{\text{surface}}}{P_{\text{middle}}} \right] = -\frac{(0.029 \text{ kg/mole}) \times 9.81 \text{ m/s}^2 \times 1.75 \cdot 10^6 \text{ m}}{24 \times 8.31 \text{ J/K} \cdot \text{mole} \times 293\text{K}} = -8.52, \quad (12)$$

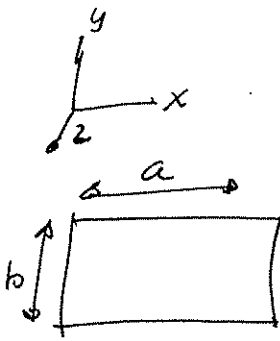
from which it follows that:

$$P_{\text{surface}} = P_{\text{middle}} \times e^{-8.52} = 10^{-3.7} P_{\text{middle}} \cong 0.0002 \text{ atm}. \quad (13)$$

Waveguides are used to constrain microwaves as they propagate from a magnetron source to an antenna. Imagine a rectangular waveguide of cross section 6 cm by 4 cm with metallic walls exhibiting a large conductivity. The waveguide is oriented along the z-axis, and the walls are situated at $x = 0$, $x = 6$ cm, $y = 0$ and $y = 4$ cm.

- What is the lowest frequency ν_0 that can be propagated through this waveguide without significant attenuation?
- For this mode of the waveguide, how does the z-component of the wavevector vary with frequency?
- For this mode of the waveguide, find the electric and magnetic fields.
- If you attempt to propagate radiation at the frequency $0.99 \nu_0$ in this mode, how far will it travel before the power is reduced by a factor of $1/e$?
- With the knowledge that the air inside the waveguide will exhibit dielectric breakdown at an electric field of 20 kV/cm, how much power can be sent along the waveguide at $\nu = 2 \nu_0$?
- When the conductivity σ is not infinite, power is lost into the walls. Derive the time-averaged power lost into the walls at $x = 0$ cm and $x = 6$ cm per unit length in the direction of propagation. Use the fact that at a wall the electric field parallel to the wall is

$$\vec{E}_{\text{parallel}} \approx \sqrt{\frac{\omega}{8\pi\sigma}} (1-i)\hat{n} \times \vec{H}_{\text{parallel}}$$



Consider a perfect conductor foil
at $x=0$ and $x=a$, $E_y = E_z = 0$

at $y=0$ and $y=a$, $E_x = 0$ and $E_z = 0$

$$\text{Then } E(x, y, z) = \left\{ \begin{aligned} & \tilde{E}_x \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \hat{x} \\ & + \tilde{E}_y \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \hat{y} \\ & + \tilde{E}_z \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \hat{z} \end{aligned} \right\} e^{i(kz - \omega t)} \quad (1)$$

$$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \left[-\tilde{E}_x \frac{m\pi}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - \tilde{E}_y \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \left(\frac{n\pi}{b} \right) + ik \tilde{E}_z \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right] e^{i(kz - \omega t)} = 0$$

$$\text{or } -\tilde{E}_x \frac{m\pi}{a} - \tilde{E}_y \frac{n\pi}{b} + ik \tilde{E}_z = 0$$

Also must satisfy $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c} \vec{B}$

$$\text{so } \vec{B} = \frac{c}{i\omega} \left[\left(\frac{n\pi \tilde{E}_z}{b} - ik \tilde{E}_y \right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \hat{x} \right.$$

$$\left. + \left(ik \tilde{E}_x - \frac{m\pi \tilde{E}_z}{a} \right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \hat{y} \right.$$

$$\left. + \left(\frac{m\pi \tilde{E}_y}{a} - \frac{n\pi \tilde{E}_x}{b} \right) \cos \left(\frac{m\pi x}{a} \right) \cos \left(\frac{n\pi y}{b} \right) \hat{z} \right] e^{i(kz - \omega t)} \quad (2)$$

Clearly, $\vec{\nabla} \cdot \vec{B} = 0$

Finally, $\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = -\frac{i\omega}{c} \vec{E}$

$$(\vec{\nabla} \times \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c^2} E_x = -\frac{i\omega}{c^2} E_x \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{i(kz - \omega t)}$$

$$\frac{\partial B_z}{\partial y} = \frac{c}{i\omega} \left[\frac{m\pi \epsilon_y}{a} - \frac{n\pi \epsilon_x}{b} \right] \cos \frac{m\pi x}{a} \left(\sin \frac{n\pi y}{b} \right) \frac{n\pi}{b} e^{i(kz - \omega t)}$$

$$\frac{\partial B_y}{\partial z} = \frac{c}{i\omega} \left[ik \epsilon_x - m\pi \frac{\epsilon_z}{a} \right] \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} ik e^{i(kz - \omega t)}$$

$$\text{So } \frac{c}{i\omega} \left\{ \left[\frac{m\pi \epsilon_y}{a} - \frac{n\pi \epsilon_x}{b} \right] \frac{n\pi}{b} + \left[ik \epsilon_x - m\pi \frac{\epsilon_z}{a} \right] ik \right\} = -\frac{i\omega}{c^2} \epsilon_x$$

$$-\frac{\omega^2}{c^2} \epsilon_x = \epsilon_y \frac{m n \pi^2}{ab} + \epsilon_x \left[-\frac{n^2 \pi^2}{b^2} - k^2 \right] + \epsilon_z \left[-i \frac{m \pi k}{a} \right]$$

$$\text{we } \epsilon_z = \frac{1}{ik} \left[\epsilon_x \frac{m\pi}{a} + \epsilon_y \frac{n\pi}{a} \right] - \epsilon_x \frac{m^2 \pi^2}{a^2} - \epsilon_y \frac{m n \pi^2}{ab}$$

$$-\frac{\omega^2}{c^2} \epsilon_x = \epsilon_x \left[-\frac{n^2 \pi^2}{b^2} - \frac{m^2 \pi^2}{a^2} - k^2 \right] +$$

$$\text{So } \frac{\omega^2}{c^2} = \frac{n^2 \pi^2}{b^2} + \frac{m^2 \pi^2}{a^2} + k^2$$

Minimum freq. occurs for $k=0$ for each pair (m,n)

$$\frac{\omega_{mn}^2}{c^2} = \frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2}$$

Note: both m and n cannot be zero.

$$\text{Lowest frequency for } m=1 \quad n=0 \Rightarrow \frac{\omega_{10}^2}{c^2} = \frac{\pi^2}{a^2}$$

$$\gamma_{mn}^2 = \frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}$$

EM general
3/4

$$\mu \epsilon \frac{\omega_{mn}^2}{c^2} = \gamma_{mn}^2 + k^2$$

a) So cutoff freq is $\omega_{mn} = \frac{c}{\sqrt{\mu \epsilon}} \sqrt{\gamma_{mn}^2}$ or $\gamma_{mn} = \frac{\sqrt{\mu \epsilon}}{c} \omega_{mn}$

b) In general, ω_{mn}
then $k_{mn}^2 = \mu \epsilon \frac{\omega^2}{c^2} - \gamma_{mn}^2$

lowest for $m=1, n=0$

$$\text{So } k_{mn}^2 = \mu \epsilon \frac{\omega^2}{c^2} - \gamma_{mn}^2 = \frac{\mu \epsilon}{c^2} [\omega^2 - \omega_{mn}^2]$$

c) see above (1)+(2)

d) if $\omega = 0.99 \omega_{mn} = \omega_{10}$

$$\text{then } k = \frac{1}{c} \omega_{10} \sqrt{-0.01} = \frac{\omega_{10}}{c} 0.1 i$$

$$\text{then } e^{i k z} = e^{-\frac{\omega_{10} z}{10c}} \quad \text{and } \frac{1}{e} \text{ distance} = \frac{10c}{\omega_{10}}$$

$$\omega_{10} = \frac{c \pi}{a} = \frac{3.14 \cdot 3 \times 10^{10} \text{ cm}}{6 \text{ cm}} = \pi \times 0.5 \times 10^{10} = \pi \times 5 \times 10^9 \text{ Hz}$$

$$\text{or } \nu = 5 \text{ GHz}$$

$$\text{then } \frac{10c}{\omega_{10}} = \frac{10a}{\pi} \approx 3 \times 6 \text{ cm} = 18 \text{ cm}$$

~~EM~~

e) $\vec{S} = \frac{c}{8\pi} \vec{E} \times \vec{H}^*$ (time averaged) note: $\epsilon=1, \mu=1$

$\vec{E} = E_y \sin \frac{\pi x}{a} e^{i(kz - \omega t)}$

$\vec{B} = \left[\frac{c}{i\omega} (-ik) E_y \sin \frac{\pi x}{a} \hat{x} + E_y \frac{\pi}{a} \cos \frac{\pi x}{a} \hat{z} \right] e^{i(kz - \omega t)}$

$\vec{E} \times \vec{B}^* = \hat{x} E_y B_z^* - \hat{z} E_y B_x^*$

$S_z = \frac{c}{8\pi} E_y \sin \frac{\pi x}{a} \left(\frac{ck}{\omega} \right) E_y \sin \frac{\pi x}{a} = \frac{c^2 k}{8\pi \omega} E_y^2 \sin^2 \frac{\pi x}{a}$

Integrate over cross-sectional area

$\int \vec{S} \cdot d\vec{a} = \frac{c^2 k}{8\pi \omega} b E_y^2 \int_0^a \sin^2 \frac{\pi x}{a} dx = \frac{c^2 k b a E_y^2}{16\pi \omega}$
 $\int_0^a \sin^2 \frac{\pi x}{a} dx = \frac{a}{\pi} \int_0^\pi \sin^2 u du = \frac{a}{2}$

This is the transmitted power.

So $P_{max} = \frac{c^2 k a b}{16\pi \omega} E_{max}^2 \frac{1}{2}$, where the $\frac{1}{2}$ corrects for the time-averaging.

where $\omega = 2\omega_0$
 and $k^2 = \frac{1}{c^2} [4\omega_0^2 - \omega_0^2] = \frac{3\omega_0^2}{c^2}$

f) from part (e) $S_x = \frac{c}{8\pi} E_y B_z^* = 0$ at the walls $x=a, x=0$
 since $E_y = 0$ there

But now use, at the wall $x=a$, $E_y \hat{y} = \sqrt{\frac{\omega}{8\pi\sigma}} (1-i) (-\hat{x}) \times B_z \hat{z}$

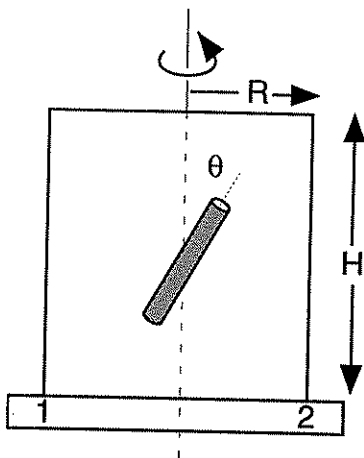
So $E_y = \sqrt{\frac{\omega}{8\pi\sigma}} (1-i) E_y \frac{\pi}{a} \cos \frac{\pi a}{a}$

Then $S_x = \frac{c}{8\pi} \sqrt{\frac{\omega}{8\pi\sigma}} (1-i) E_y^2 \left(\frac{\pi}{a}\right)^2 \left(\cos \frac{\pi a}{a}\right)^2$

Take only the real part and integrate over y to get $\frac{dP}{dz}$ at wall $x=a$

$\frac{dP}{dz} (x=a) = b \frac{c}{8\pi} \sqrt{\frac{\omega}{8\pi\sigma}} E_y^2 \left(\frac{\pi}{a}\right)^2$

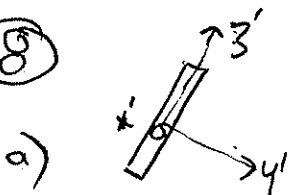
$\frac{dP}{dz} (x=0) = \frac{dP}{dz} (x=a) = 0$ $\frac{dP}{dz} \text{ (into both walls)} = \frac{bc}{4\pi} \sqrt{\frac{\omega}{8\pi\sigma}} E_y^2 \left(\frac{\pi}{a}\right)^2$



A right circular Styrofoam cylinder of negligible density has a cylindrically shaped iron bar embedded at its center (see diagram). The Styrofoam cylinder has radius R and height H . The geometrical centers of the bar and cylinder coincide, but the axis of the bar is inclined at an angle θ with respect to the vertical. The composite object is placed on a horizontal turntable that is rotating in the counter-clockwise direction as viewed from above, with a constant angular velocity ω .

- Give relative sizes of all the moments of inertia of the iron bar along its principal axes.
- What is the torque on the bar as would be measured in the laboratory reference frame?
- If some critical value of ω is exceeded, the composite object will begin to rise up on its edge. At the instant shown in the diagram, on which edge 1 or 2 will it rise?
- For what critical value of ω will the object begin to rise?
- What must be the relations among the moments of inertia of the iron bar along its principal axes for there to be no critical value for θ regardless of the value of ω ?

8



$$I_{x'} = I_{y'} \quad (\text{by symmetry}) = \int r^2 dm = \int (y'^2 + z'^2) dm$$

$$I_{z'} = \int r^2 dm = \int (x'^2 + y'^2) dm \ll I_{x'}$$

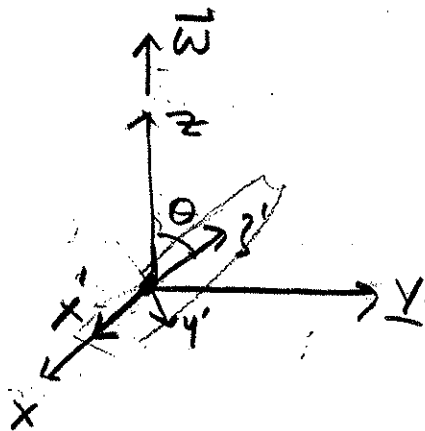
$$\underline{I_{x'} = I_{y'} \ll I_{z'}}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} [\mathbf{I}] \vec{\omega}$$

↑ tensor

$$\vec{L} = \mathbf{I} \vec{\omega}$$

Since $x'y'z'$ (fixed) system is com system, torque is the same as $\Sigma XYZ = \text{lab}$



Calc L in $x'y'z'$: $\vec{L} = \mathbf{I} \vec{\omega} = [\mathbf{I}] \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

$$\vec{L}_{x'y'z'} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} 0 \\ -\omega \sin\theta \\ \omega \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ -I_y \omega \sin\theta \\ I_z \omega \cos\theta \end{bmatrix}$$

This rotates in XYZ

LAB ΣYZ $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = (\omega_y L_z - \omega_z L_y) \hat{e}_x + (\omega_z L_x - \omega_x L_z) \hat{e}_y + (\omega_x L_y - \omega_y L_x) \hat{e}_z$

yet $\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$

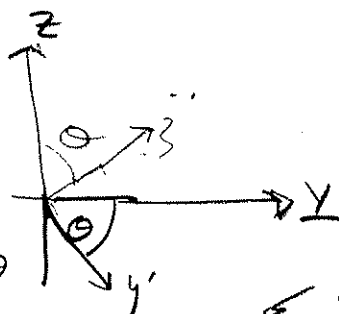
$$\Rightarrow \vec{\tau} = -\omega_z L_y \hat{e}_x$$

$$L_y = L_{z'} \sin\theta + L_{y'} \cos\theta$$

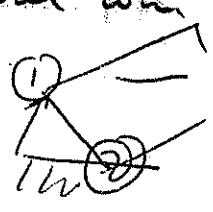
$$= I_{z'} \omega \cos\theta \sin\theta - I_{y'} \omega \sin\theta \cos\theta$$

$$= \omega \cos\theta \sin\theta (I_{z'} - I_{y'}) = -\omega \cos\theta \sin\theta (I_{y'} - I_{z'}) \approx 0$$

$$\vec{\tau} = -\omega_z L_y \hat{e}_x - \omega^2 m \cos\theta \sin\theta (I_{z'} - I_{y'}) \hat{e}_x$$

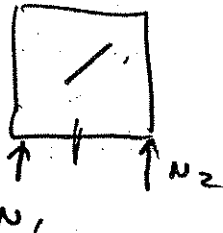


8 cont

c) Using centrifugal force reasoning, the bar will level itself out ~~out~~  So it tips on 2 (1 rises)

Independent of ω

d) Critical



$$N_1 + N_2 = mg$$

So max torque by table = mgR

$$mgR = \omega^2 r \cos \theta (I_{y'} - I_{z'})$$

$$N_1 \Rightarrow 0$$

$$\omega_{crit}^2 = \frac{gR}{r \cos \theta (I_{y'} - I_{z'})}$$

e) for $N \Rightarrow 0$ there can be no tipping

$N \Rightarrow 0$ when $I_{y'} = I_{z'}$ (like sphere)