

Department of Physics Comprehensive Examination No. 84

September 28 and 29, 1998

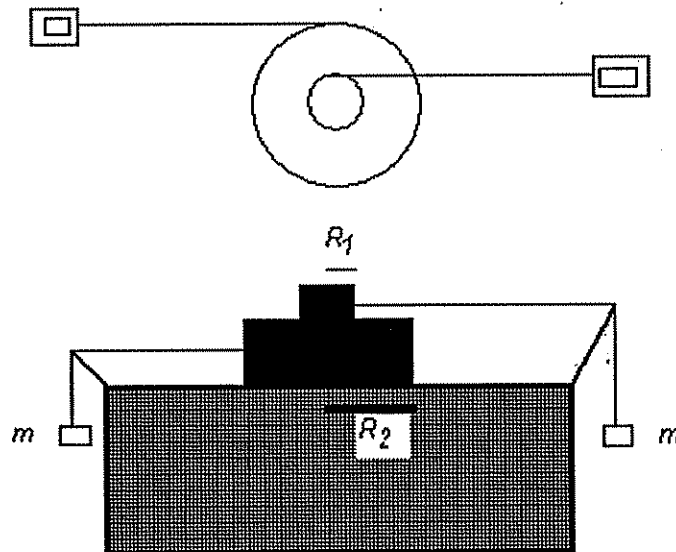
This Comprehensive Examination for Fall 1998 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Monday 28 September. Session Two (problems 3 and 4) begins at 1:30PM Monday 28 September. Session Three (problems 5 and 6) begins at 9:00 AM Tuesday 29 September. Session Four (problems 7 and 8) begins at 1:30PM Tuesday 29 September.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

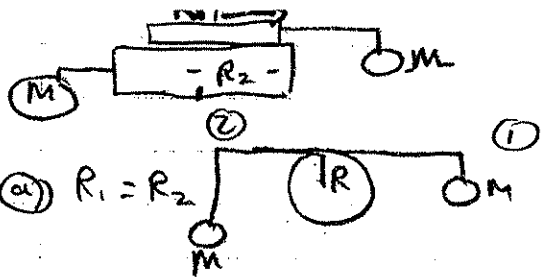
If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

A solid cylinder rests on a horizontal frictionless surface upon which it can rotate and slide. The cylinder has radius R_2 and has an attached hub on the top of it with radius R_1 (see diagram with top view and side view). The cylinder and hub together have a mass K and a moment of inertia I about a vertical axis. A string is wound about R_2 and a different string is wound around R_1 . The strings support two equal hanging masses m by means of massless, frictionless, fixed pulleys. The cylinder is released from rest with the axis vertical and the strings parallel.



- Describe the motion when R_1 equals R_2 .
- Describe the motion when R_1 does not equal R_2 .
- For the case $R_1 \neq R_2$, derive the complete set of differential equations that describe the accelerations of the various components as the system is set in motion by the action of gravity upon the masses m .
- Solve the equations of part (c).



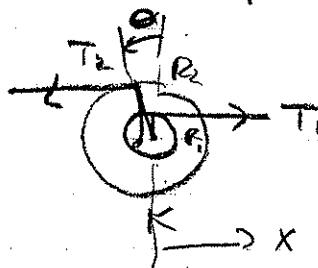
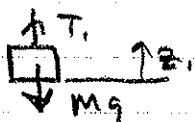
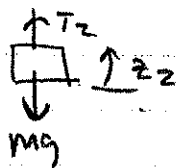
(#1) //

a) $R_1 = R_2$ No unbalanced torques or forces. System will slide with constant velocity if pushed, or rotate with constant frequency if twisted, or No Motion.

b) $R_1 \neq R_2 \Rightarrow a_1 = R_1 \ddot{\theta} \neq a_2 = R_2 \ddot{\theta} \Rightarrow T_1 \neq T_2$

If tensions are unequal, the plug slides to the right ($T_1 > T_2$ since rotate ccw so m_1 rises up)

(2)



$\theta \propto x$
are two indep variables

If plug rotates θ m_1 rises by $R_1 \theta$, m_2 falls by $R_2 \theta$
If plug moves x to right, m_1 falls x , m_2 rises x

$z_1 = R_1 \theta - x$ $z_2 = x - R_2 \theta$

4 eqns
4 unknowns

$F = ma$ for Plug

1) $T_1 - T_2 = K \ddot{x}$

2) $T_2 - mg = m \ddot{z}_2 = m \ddot{x} - m R_2 \ddot{\theta}$

3) $T_1 - mg = m \ddot{z}_1 = m R_1 \ddot{\theta} - m \ddot{x}$

4) $\sum \tau = I \ddot{\theta}$
 $T_2 R_2 - T_1 R_1 = I \ddot{\theta}$

z about what axis

d) 2) $T_2 = mg + m \ddot{x} - m R_2 \ddot{\theta}$

3) $T_1 = mg - m \ddot{x} + m R_1 \ddot{\theta}$

1) $-2m \ddot{x} + (m R_1 + m R_2) \ddot{\theta} = K \ddot{x}$

$\ddot{\theta} = \frac{(K + 2m)}{m R_1 + m R_2} \ddot{x}$

$$4) R_2 m g + R_2 m \ddot{x} - m R_2^2 \ddot{\theta} - m g R_1 + m \ddot{x} R_1 - m R_1^2 \ddot{\theta} = I \ddot{\theta} \quad ($$

$$\ddot{x} (R_2 m + R_1 m) = \ddot{\theta} (I + m R_2^2 + m R_1^2) + m g (R_1 - R_2)$$

substitute for $\ddot{\theta}$

$$\ddot{x}_1 \left[R_2 m + m R_1 - \frac{(I + m R_2^2 + m R_1^2)(R_2 + 2m)}{m(R_1 + R_2)} \right] = m g (R_1 - R_2)$$

$$\ddot{x}_1 \left[\frac{m^2 (R_1^2 + R_2^2 + 2R_1 R_2) - I R - 2m^2 (R_1^2 + R_2^2)}{m(R_1 + R_2)} \right] =$$

$$- \ddot{x}_1 \left[\frac{m^2 (2R_1 R_2 - R_1^2 - R_2^2) - I R}{m(R_1 + R_2)} \right] = m g (R_1 - R_2)$$

$$\ddot{x}_1 = - \frac{m g (R_1 - R_2) (R_1 + R_2)}{m^2 (R_1 - R_2)^2 + I R} = \alpha$$

$$\dot{x}_1(t) = \alpha t + \dot{x}_1(0)$$

$$x_1(t) = \alpha t^2 / 2 + \dot{x}_1(0) t + x_1(0)$$

likewise for θ

$$\text{NB if } R_1 = R_2 \quad \ddot{x}_1 = 0 \quad \checkmark$$

$$c) \quad r = r_0 e^{\lambda t}$$

$$1) \quad \ddot{\theta}_1 + 2\lambda \dot{\theta}_1 + \frac{2k}{m} \theta_1 - \frac{2k}{m} \theta_2 + 2k\pi/m = 0$$

$$2) \quad \ddot{\theta}_2 + 2\lambda \dot{\theta}_2 + \frac{2k}{m} \theta_2 - \frac{2k}{m} \theta_1 - \frac{2k}{m} \pi = 0$$

assume $\theta_1 = \theta_{10} e^{i\omega t}$, $\theta_2 = \theta_{20} e^{i\omega t}$
 Ignore particular solution for character behavior

$$1) \quad \omega^2 \theta_{10} + 2\lambda \omega \theta_{10} + \frac{2k}{m} \theta_{10} - \frac{2k}{m} \theta_{20} = 0$$

$$2) \quad \omega^2 \theta_{20} + 2\lambda \omega \theta_{20} + \frac{2k}{m} \theta_{20} - \frac{2k}{m} \theta_{10} = 0$$

Need consistent solution ϕ

$$\begin{bmatrix} (\omega^2 + 2\lambda\omega + \frac{2k}{m}) & -\frac{2k}{m} \\ -\frac{2k}{m} & (\omega^2 + 2\lambda\omega + \frac{2k}{m}) \end{bmatrix} \begin{bmatrix} \theta_{10} \\ \theta_{20} \end{bmatrix} = 0$$

non trivial requires $\det || = 0$

$$(\omega^2 + 2\lambda\omega + \frac{2k}{m})^2 = (\frac{2k}{m})^2$$

$$\omega^2 + 2\lambda\omega + \frac{2k}{m} = \pm \frac{k}{m}$$

plus a) $\omega^2 = -2\lambda\omega$

$$\omega = 0 \text{ (translation)}, \quad \underline{\omega = -2\lambda}$$

minus b) $\omega^2 + 2\lambda\omega + \frac{3k}{m} = 0$

$$\omega = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 12k/m}}{2}$$

A one-dimensional harmonic oscillator with mass m , charge q , and natural frequency ω has the unperturbed Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

A small electric field \mathcal{E} is then applied to the system.

- a) Find the perturbed eigenenergies of the system to the lowest nonvanishing order in \mathcal{E} .
- b) Find the expectation value of the induced electric dipole moment in a stationary state of the perturbed system.

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

eigenstates are labeled $|n\rangle$ with eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega$

$$H_0|n\rangle = (n + \frac{1}{2})\hbar\omega |n\rangle$$

a) Electric field E gives rise to perturbation W :

$$W = -\vec{p} \cdot \vec{E} = -q\hat{x}E$$

perturbed energies are

$$E_n' = E_n + \langle n|W|n\rangle + \sum_{k \neq n} \frac{|\langle n|W|k\rangle|^2}{E_n - E_k}$$

From ref sheet

$$\langle n|x|k\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n+1} \delta_{n+1,k} + \sqrt{n} \delta_{n-1,k} \right]$$

so only adjacent states connected by perturbation.

$\Rightarrow \langle n|W|n\rangle = 0$ so no 1st order correction.

$$\Rightarrow E_n' = E_n + \sum_{k \neq n} q^2 E^2 \frac{|\langle n|x|k\rangle|^2}{(n + \frac{1}{2})\hbar\omega - (k + \frac{1}{2})\hbar\omega}$$

$$= (n + \frac{1}{2})\hbar\omega + q^2 E^2 \frac{\hbar}{2m\omega} \left[\frac{n+1}{(-\hbar\omega)} + \frac{n}{\hbar\omega} \right]$$

$$E_n' = (n + \frac{1}{2})\hbar\omega - \frac{q^2 E^2}{2m\omega^2}$$

b) The induced electric dipole moment is

$$p = q x$$

$$\Rightarrow \langle p \rangle = q \langle x \rangle = q \langle \psi | x | \psi \rangle$$

The stationary states are now $|n\rangle'$, which must be found to 1st order in \mathcal{E} , since we found energies to 2nd order

$$|n\rangle' = |n\rangle + \sum_{k \neq n} \frac{\langle k | W | n \rangle}{E_n - E_k} |k\rangle$$

$$= |n\rangle + \sum_{k \neq n} \frac{(-q\mathcal{E}) \langle k | x | n \rangle}{(n + \frac{1}{2})\hbar\omega - (k + \frac{1}{2})\hbar\omega} |k\rangle$$

$$= |n\rangle - q\mathcal{E} \cdot \sqrt{\frac{\hbar}{2m\omega}} \left(\sum_{k \neq n} \frac{\sqrt{k+1} \delta_{k+1,n} + \sqrt{k} \delta_{k-1,n}}{(n-k)\hbar\omega} |k\rangle \right)$$

$$= |n\rangle - q\mathcal{E} \sqrt{\frac{\hbar}{2m\omega}} \left[\frac{\sqrt{n} |n-1\rangle}{\hbar\omega} + \frac{\sqrt{n+1} |n+1\rangle}{-\hbar\omega} \right]$$

$$|n\rangle' = |n\rangle - q\mathcal{E} \sqrt{\frac{1}{2m\hbar\omega^3}} \left(\sqrt{n} |n-1\rangle - \sqrt{n+1} |n+1\rangle \right)$$

$$\Rightarrow \langle p \rangle = q \langle n | x | n \rangle'$$

$$= q \left[\langle n | x | n \rangle - q\mathcal{E} \sqrt{\frac{1}{2m\hbar\omega^3}} \left(\sqrt{n} \langle n-1 | x | n \rangle - \sqrt{n+1} \langle n+1 | x | n \rangle \right) \right]$$

$$\cdot \left(\sqrt{n} |n-1\rangle - \sqrt{n+1} |n+1\rangle \right)$$

x only has matrix elements between states differing by one quantum number. Thus we get

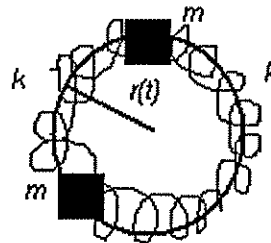
$$\langle p \rangle = q(-qE) \sqrt{\frac{1}{2m\hbar\omega^3}} \left[\sqrt{n} \langle n|x|n-1\rangle - \sqrt{n+1} \langle n|x|n+1\rangle \right. \\ \left. + \sqrt{n} \langle n-1|x|n\rangle - \sqrt{n+1} \langle n+1|x|n\rangle \right]$$

$$= -q^2 E \sqrt{\frac{1}{2m\hbar\omega^3}} \cdot \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n} \cdot \sqrt{n} - \sqrt{n+1} \cdot \sqrt{n+1} \right. \\ \left. + \sqrt{n} \cdot \sqrt{n} - \sqrt{n+1} \cdot \sqrt{n+1} \right]$$

$$\langle p \rangle = \frac{q^2 E}{m\omega^2}$$

independent of n .

Two point beads of mass m are connected by two identical springs of negligible mass, negligible unstretched length, and spring constant k . The springs are constrained to a circular hoop which has an externally-controlled time-varying radius $r(t)$. There are no gravitational or frictional forces present, and you may assume the beads never collide.

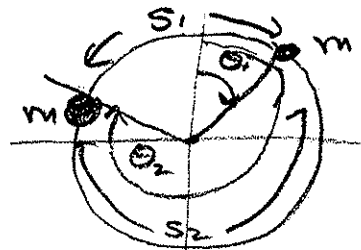


- Determine the Lagrangian for the system.
- Determine the Lagrangian equations of motion for the system.
- If the time dependence of the hoop is

$$r(t) = r(0) \exp(\lambda t),$$

deduce an algebraic equation which can be solved for the characteristic frequencies of the motion.

Solution CM1



a) $L = T - V$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (r^2 \dot{\theta}_1^2 + \dot{r}^2 + r^2 \dot{\theta}_2^2 + \dot{r}^2)$$

$$V = \frac{1}{2} k s_1^2 + \frac{1}{2} k s_2^2$$

$$s_2 = r(\theta_2 - \theta_1), \quad s_1 = 2\pi r - s_2 = r(2\pi - \theta_2 + \theta_1)$$

$$V = \frac{1}{2} k r^2 \left[(4\pi^2 + \theta_2^2 + \theta_1^2 - 4\pi\theta_2 + 4\pi\theta_1 - 2\theta_1\theta_2) + (\theta_2^2 + \theta_1^2 - 2\theta_1\theta_2) \right]$$

$$= \frac{1}{2} k r^2 \left[4\pi^2 + 2\theta_1^2 + 2\theta_2^2 - 4\theta_1\theta_2 - 4\pi\theta_1 + 4\pi\theta_2 \right]$$

$$L = \frac{1}{2} m (r^2 \dot{\theta}_1^2 + 2\dot{r}^2 + r^2 \dot{\theta}_2^2) - \frac{1}{2} k r^2 \left[4\pi^2 + 2\theta_1^2 + 2\theta_2^2 - 4\theta_1\theta_2 - 4\pi\theta_1 + 4\pi\theta_2 \right]$$

or leave out 2π :

$$= \frac{1}{2} m (r^2 \dot{\theta}_1^2 + 2\dot{r}^2 + r^2 \dot{\theta}_2^2) - \frac{1}{2} k r^2 \left[2\theta_1^2 + 2\theta_2^2 - 4\theta_1\theta_2 \right]$$

b) There are two degrees of freedom θ_1 & θ_2 .
 $r(t)$ is a parameter, the equivalent of an external field:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m r^2 \dot{\theta}_1, \quad \frac{\partial L}{\partial \dot{\theta}_2} = m r^2 \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} k r^2 [4\theta_1 - 4\theta_2 + 4\pi] = -2k r^2 \theta_1 + 2k r^2 \theta_2 - 2k r^2 \pi$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} k r^2 [4\theta_2 - 4\theta_1 - 4\pi] = -2k r^2 \theta_2 + 2k r^2 \theta_1 + 2k r^2 \pi$$

1) θ_1 : $2m r \dot{r} \dot{\theta}_1 + m r^2 \ddot{\theta}_1 + 2k r^2 \theta_1 - 2k r^2 \theta_2 + 2k r^2 \pi = 0$

$$r \ddot{\theta}_1 + 2 \dot{r} \dot{\theta}_1 + 2 \frac{k}{m} r \theta_1 - 2 \frac{k r}{m} \theta_2 + \frac{2k r \pi}{m} = 0$$

2) θ_2 : $2m r \dot{r} \dot{\theta}_2 + m r^2 \ddot{\theta}_2 + 2k r^2 \theta_2 - 2k r^2 \theta_1 - 2k r^2 \pi = 0$

$$r \ddot{\theta}_2 + 2 \dot{r} \dot{\theta}_2 + 2 \frac{k}{m} r \theta_2 - 2 \frac{k}{m} r \theta_1 - \frac{2k}{m} r \pi = 0$$

$$c) \quad \Gamma = \Gamma_0 e^{\lambda T}$$

$$1) \quad \ddot{\theta}_1 + 2\lambda \dot{\theta}_1 + \frac{2k}{m} \theta_1 - \frac{2k}{m} \theta_2 + 2k\pi/m = 0$$

$$2) \quad \ddot{\theta}_2 + 2\lambda \dot{\theta}_2 + \frac{2k}{m} \theta_2 - \frac{2k}{m} \theta_1 - \frac{2k}{m} \pi = 0$$

assume $\theta_1 = \theta_{10} e^{i\omega T}$, $\theta_2 = \theta_{20} e^{i\omega T}$

Ignore particular soltn

$$1) \quad \omega^2 \theta_{10} + 2\lambda \omega \theta_{10} + \frac{2k}{m} \theta_{10} - \frac{2k}{m} \theta_{20} = 0 \quad \text{for character behavior}$$

$$2) \quad \omega^2 \theta_{20} + 2\lambda \omega \theta_{20} + \frac{2k}{m} \theta_{20} - \frac{2k}{m} \theta_{10} = 0$$

Need consistent solution

q

$$\begin{bmatrix} (\omega^2 + 2\lambda\omega + \frac{2k}{m}) & -\frac{2k}{m} \\ -\frac{2k}{m} & (\omega^2 + 2\lambda\omega + \frac{2k}{m}) \end{bmatrix} \begin{bmatrix} \theta_{10} \\ \theta_{20} \end{bmatrix} = 0$$

non trivial requires $\det || = 0$

$$(\omega^2 + 2\lambda\omega + \frac{2k}{m})^2 = (\frac{2k}{m})^2$$

$$\omega^2 + 2\lambda\omega + \frac{2k}{m} = \pm \frac{k}{m}$$

plus a) $\omega^2 = -2\lambda\omega$

$$\omega = 0 \text{ (translation)}, \quad \underline{\omega = -2\lambda}$$

minus b) $\omega^2 + 2\lambda\omega + \frac{3k}{m} = 0$

$$\omega = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 12k/m}}{2}$$

On using Energy conservation in plane of Torque

$$E = mgz_1 + mgz_2 + \frac{1}{2}m\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 + \frac{1}{2}K\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \text{constant}$$

$$dE/dt = 0 = mg\dot{z}_1 + mg\dot{z}_2 + m\dot{z}_1\ddot{z}_1 + m\dot{z}_2\ddot{z}_2 + K\dot{x}\ddot{x} + I\dot{\theta}\ddot{\theta}$$

$$\dot{z}_1 = R_1\dot{\theta} - \dot{x} \quad \dot{z}_2 = \dot{x} - R_2\dot{\theta}$$

$$4) 0 = mg(R_1 - R_2)\dot{\theta} + m[(R_1\dot{\theta} - \dot{x})(R_1\ddot{\theta} - \ddot{x}) + (\dot{x} - R_2\dot{\theta})(\dot{x} - R_2\ddot{\theta}) + K\dot{x}\ddot{x} + I\dot{\theta}\ddot{\theta}]$$

yet $\ddot{\theta} = \frac{R_1 - R_2}{mR_1 + mR_2} \ddot{x} = \alpha \ddot{x}, \Rightarrow \dot{\theta} = \alpha \dot{x}$ (starts at rest)

$$4) 0 = mg(R_1 - R_2)\alpha\dot{x} + m\dot{x}\ddot{x} [(R_1\alpha - 1)(R_1\alpha - 1) + (1 - R_2\alpha)(1 - R_2\alpha) + K + I\alpha^2]$$

yet $\dot{x} \neq 0$ so can divide by \dot{x}

$$0 = mg(R_1 - R_2)\alpha + m\ddot{x} [R_1^2\alpha^2 + 1 - 2R_1\alpha + 1 + R_2\alpha^2 - 2R_2\alpha + K + I\alpha^2]$$

So again $\ddot{x} = \text{const}$ (presumably the same on

The following describes an experimental method of measuring the specific heat ratio $\gamma \equiv c_p/c_v$ of a gas. The gas, assumed ideal, is contained in a horizontal cylinder, closed off by a piston. The cylinder and the piston are made of materials that are relatively poor heat conductors. Hence, if the piston is moved fast enough, the process can be considered as adiabatic; however, if the piston is not moved for a sufficiently long time, the gas returns to thermal equilibrium with the surrounding environment.

The pressure and temperature of the ambient air (P_A and T_A , respectively) remain constant throughout the experiment. The pressure inside the cylinder P_i is monitored using a device that measures the *difference* between this pressure and the pressure outside:

$$\Delta P_i = P_i - P_A \quad (\text{e.g., a simple fluid manometer, as shown in Fig. 1}).$$

Initially, the system is in thermal and mechanical equilibrium with the surrounding environment ($T_0 = T_A$, $P_0 = P_A$). Then, the following steps are performed:

1. The piston is slightly displaced to produce a small increase in the gas pressure, and is clamped in such a position. When thermal equilibrium is reestablished, the pressure increase (ΔP_1 — see Fig. 1a) is recorded.
2. Next, the piston is unclamped, so that the gas starts expanding. The piston motion is slow enough that the gas always remains in *internal equilibrium*, but fast enough that the heat exchange with the outside is negligibly small. When the pressure inside the cylinder becomes exactly equal to the atmospheric pressure, the piston is stopped and fixed in that new position (Fig. 1b).
3. The system rests until it returns to thermal equilibrium (Fig. 1c). When it happens, the difference between the gas pressure and the atmospheric pressure (ΔP_3) is again recorded.

Analyze the relation between P and T in Step 2, and find the gas temperature at the moment the piston stops moving. Find the final pressure after the equilibrium is reached in Step 3 (P_3). Then show that the value of γ for the gas can be determined from pressure measurements only — i.e., there is no need to measure the gas volume or the gas temperature at any stage of the process. Finally, show that if the experiment is carried out in such a way that ΔP_1 is much smaller than the atmospheric pressure P_A , then it is not even necessary to know the P_A value with a very good accuracy.

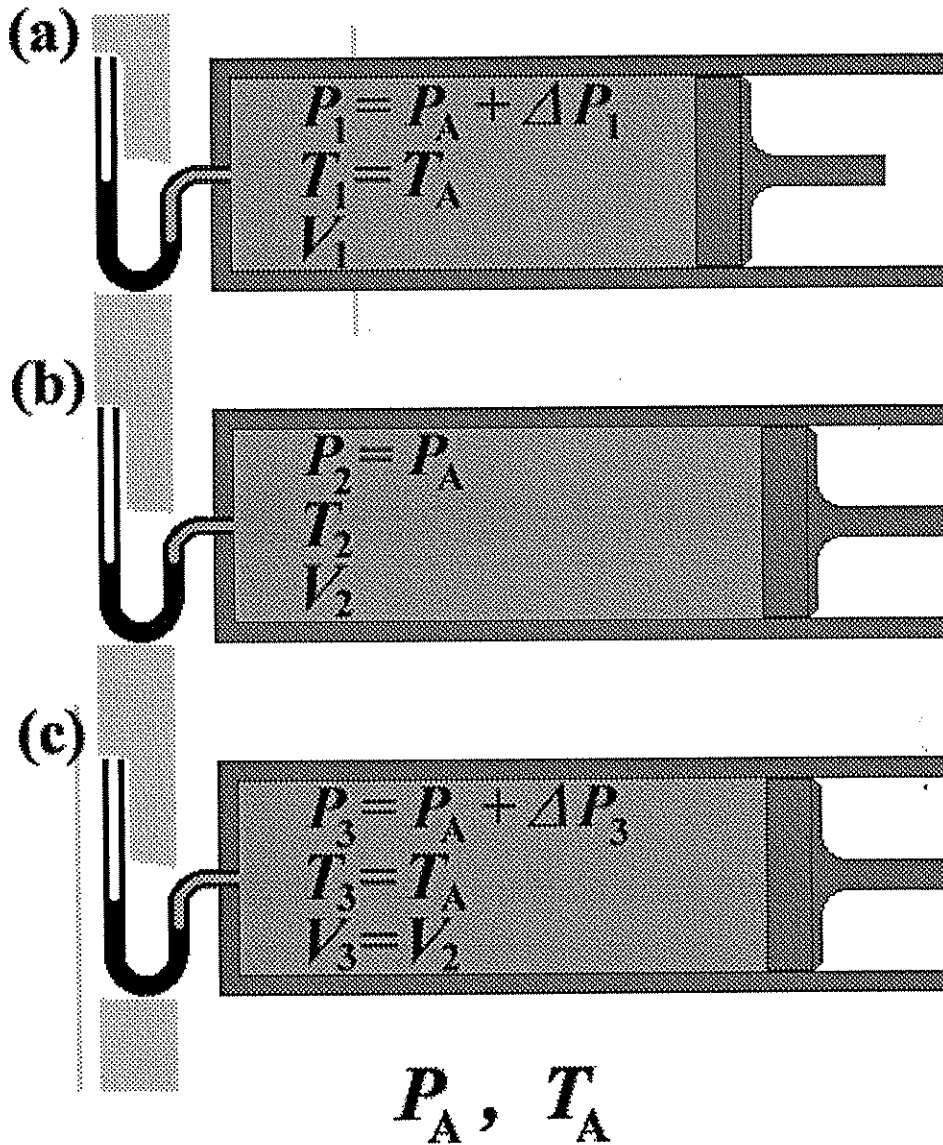


Figure 1. Three stages of the process described in the problem.

Solution

For ideal gas we have:

$$PV = NRT, \quad (1)$$

and

$$U = \kappa NRT \quad (2)$$

where P – pressure, V – volume, T – temperature, N – number of moles, R – gas constant ($= k_B N_A$, the Boltzmann constant times the Avogadro number), U – internal energy, and κ is $\frac{3}{2}$, $\frac{5}{2}$, or $\frac{6}{2}$ depending on whether the gas molecule has 3, 5 or 6 degrees of freedom).

If the gas is heated or cooled at constant volume, no work is done and the entire heat transfer is being used for changing U ; in other words, $dQ = dU$. Thus, the specific heat c_v , by definition $N^{-1}(dQ/dT)_V$, can be readily obtained from Eq. (2):

$$c_v = \frac{1}{N} \left(\frac{dU}{dt} \right)_V = \kappa R. \quad (3)$$

In a constant pressure process the volume changes and the work is no longer zero. Here we use the fundamental relation $dU = dQ - PdV$; since $P = \text{const.}$, from Eq. (1) we have $p dV = NR dT$. Hence,

$$dQ = dU + NR dT = \kappa NR dT + NR dT = (\kappa + 1)NR dT,$$

and

$$c_p \equiv \frac{1}{N} \left(\frac{dQ}{dt} \right)_P = (\kappa + 1)R. \quad (4)$$

From Eqs. (3) and (4) we obtain the two well-known formulae:

$$c_p - c_v = R \quad \text{and} \quad \gamma \equiv \frac{c_p}{c_v} = 1 + \frac{1}{\kappa}$$

Note: $c_p/c_v = 1 + \kappa^{-1}$ is really a very well known formula — it is absolutely O.K. if students who remember it use it without proof in this exam.

In adiabatic processes $dQ = 0$, so that $dU = -PdV$. Using Eq. (1), one can eliminate P : $dU = -NRT dV/V$. On the other hand, from Eq. (2) we have $dU = \kappa NR dT$. Combining these two forms, we obtain the equation:

$$\frac{dV}{V} = -\kappa \frac{dT}{T},$$

with a solution:

$$\ln \left(\frac{V'}{V''} \right) = \ln \left(\frac{T'}{T''} \right)^{-\kappa},$$

which may also be written in an equivalent form:

$$VT^\kappa = \text{const.}$$

However, we want rather an equation in terms of P and T ; once more using Eq. (1) and simple algebra, we obtain:

$$PT^{-\kappa-1} = \text{const.} \quad (5)$$

Applying this result to the adiabatic process taking place in Step 2, in which the pressure changes from P_1 to P_2 and the temperature from T_1 to T_2 , we can write:

$$\left(\frac{P_1}{P_2}\right) = \left(\frac{T_1}{T_2}\right)^{\kappa+1} \quad (6)$$

The process in Step 3 is a constant volume one, and from Eq. (1) we readily get:

$$\frac{P_3}{P_2} = \frac{T_3}{T_2}$$

However, it should be noted that T_3 is the same as T_1 , so that:

$$\frac{P_3}{P_2} = \frac{T_1}{T_2}; \quad (7)$$

hence, combining Eqs. (6) and (7) leads to:

$$\left(\frac{P_1}{P_2}\right) = \left(\frac{P_3}{P_2}\right)^{\kappa+1} \quad (8)$$

From this equation one can already obtain the solution for κ :

$$\kappa = \ln\left(\frac{P_1}{P_3}\right) - 1, \quad (9)$$

and then calculate γ by simply using the $1 + 1/\kappa$ formula. Hence, we have shown that the value of γ can be indeed determined from pressure measurements only.

However, the Eq. (8) can be still reduced to a more convenient form that includes not the absolute pressure values, but the measured pressure differences only. Let's note that $P_1 = P_A + \Delta P_1$, $P_2 = P_A$, and $P_3 = P_A + \Delta P_3$. Hence, the Eq. (8) takes the form:

$$\left(1 + \frac{\Delta P_1}{P_A}\right) = \left(1 + \frac{\Delta P_3}{P_A}\right)^{\kappa+1} \quad (10)$$

If $\Delta P_1 \ll P_A$, it automatically assures that $\Delta P_3 \ll P_A$ (note that ΔP_3 would be always smaller than ΔP_1). In such circumstances, we can use the approximation $(1 + \delta)^c \cong 1 + c\delta$ to obtain:

$$1 + \frac{\Delta P_1}{P_A} \cong 1 + (\kappa + 1) \frac{\Delta P_3}{P_A}, \quad (11)$$

which can be further processed:

$$\kappa = \frac{\Delta P_1}{\Delta P_3} - 1 = \frac{\Delta P_1 - \Delta P_3}{\Delta P_3} \quad (12)$$

and:

$$\gamma = 1 + \frac{1}{\kappa} = \frac{\Delta P_1}{\Delta P_1 - \Delta P_3}. \quad (13)$$

Suppose an atomic magnetic moment is capable of assuming any of the discrete values $g\mu_B m$ as its component along the direction of a magnetic field H (with $m = -S, -S+1, \dots, S-1, S$, where S is the spin number). Calculate the magnetization $M(H)$ of a solid containing N such atoms, assuming that there is no interaction of any kind between individual atomic moments.

Hints: $1 + q + q^2 + \dots + q^{n-1} = \frac{(q^n - 1)}{q - 1}$.

A good idea is to express the partition sum Z in terms of hyperbolic functions.

Solution

The magnetic energy of a single spin S_i with the magnetic quantum number m_i in external field H is $\epsilon_i = -g\mu_B H m_i$. The single-particle partition sum z_i is obtained by summing Boltzmann factors corresponding to all possible values of m_i :

$$z_i = \sum_{m_i=-S}^S \exp\left(\frac{g\mu_B H m_i}{kT}\right). \quad (1)$$

The overall partition function Z is the sum of the z_i contributions for all N magnetic spins in the specimen:

~~$$Z = \sum_{i=1}^N z_i, \quad (2)$$~~

$Z = \prod z_i$

which can be written as a multiple sum:

$$Z = \sum_{m_1=-S}^S \sum_{m_2=-S}^S \cdots \sum_{m_N=-S}^S \exp\left(\frac{g\mu_B H m_i}{kT}\right). \quad (3)$$

(m_1, m_2, \dots, m_N)

This sum may be factorized, and written in the form of a multiple product:

$$\begin{aligned} Z &= \sum_{m_1=-S}^S \exp\left(\frac{g\mu_B H m_1}{kT}\right) \sum_{m_2=-S}^S \exp\left(\frac{g\mu_B H m_2}{kT}\right) \cdots \sum_{m_N=-S}^S \exp\left(\frac{g\mu_B H m_N}{kT}\right) \\ &= \prod_{j=1}^N \left\{ \sum_{m_j=-S}^S \exp\left(\frac{g\mu_B H m_j}{kT}\right) \right\}. \end{aligned} \quad (4)$$

Since all single-particle factors in this product are identical, the partition function simplifies to:

$$Z = \left\{ \sum_{m_j=-S}^S \exp\left(\frac{g\mu_B H m_j}{kT}\right) \right\}^N. \quad (5)$$

Note: The above is a standard procedure — it's O.K. if the student skips the intermediate steps, and “jumps” at once from the Eq. (1) to Eq. (5).

It is easy to see that the terms in the sum in Eq. (5) represent a finite geometrical progression. Using the formula given in the hint, one can transform such sum into a more convenient expression:

$$\sum_{l=-n}^n q^l = q^{-n} \sum_{k=0}^{2n} q^k = \frac{q^{2n+1} - 1}{q^n(q - 1)} = \frac{q^{n+\frac{1}{2}} - q^{-n-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (6)$$

In particular, if $q = e^a$:

$$\sum_{l=-n}^n e^{la} = \frac{e^{(n+\frac{1}{2})a} - e^{(-n-\frac{1}{2})a}}{e^{\frac{1}{2}a} - e^{-\frac{1}{2}a}} = \frac{\sinh\left(a\frac{2n+1}{2}\right)}{\sinh\left(\frac{a}{2}\right)}. \quad (7)$$

By applying this operation to the Eq. (5), one can express the system partition function in the following simple form:

$$Z = \frac{\sinh\left(\frac{2S+1}{2} \times \frac{g\mu_B H}{kT}\right)}{\sinh\left(\frac{1}{2} \times \frac{g\mu_B H}{kT}\right)}. \quad (8)$$

If the sample magnetization is M , its total magnetic energy is $E_{\text{tot}} = -MH$. Using the standard statistical-mechanical recipe, one can calculate the mean magnetization $\langle M \rangle$:

$$\langle M \rangle = \frac{\sum_s M_s \exp\left(\frac{HM_s}{kT}\right)}{Z} \quad (9)$$

where the sum is over all accessible system states s . In a similar manner, Z can also be written as a sum over all states s , so that the above equation becomes:

$$\langle M \rangle = \frac{\sum_s M_s \exp\left(\frac{HM_s}{kT}\right)}{\sum_s \exp\left(\frac{HM_s}{kT}\right)}. \quad (10)$$

One can easily show that this expression can be written as the derivative of a simpler one:

$$\langle M \rangle = kT \frac{\sum_s \frac{M_s}{kT} \exp\left(\frac{HM_s}{kT}\right)}{\sum_s \exp\left(\frac{HM_s}{kT}\right)} = kT \frac{\frac{\partial}{\partial H} \sum_s \exp\left(\frac{HM_s}{kT}\right)}{\sum_s \exp\left(\frac{HM_s}{kT}\right)} = kT \frac{\partial}{\partial H} \left\{ \ln \left[\sum_s \exp\left(\frac{HM_s}{kT}\right) \right] \right\}. \quad (11)$$

Returning to the Z symbol for the partition function, we obtain the magnetization in a very compact form:

$$\langle M \rangle = kT \frac{\partial}{\partial H} \ln(Z). \quad (12)$$

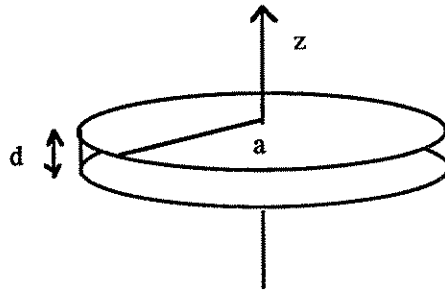
Since

$$\frac{d}{dx} \ln(\sinh x) = \frac{\cosh x}{\sinh x} = \coth x,$$

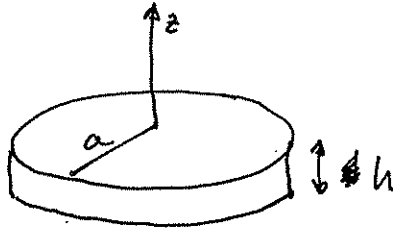
taking the derivative of the Eq. (8) is rather straightforward, and we obtain the final result in the form:

$$\langle M \rangle = Ng\mu_B \left[\frac{2S+1}{2} \coth\left(\frac{2S+1}{2} \times \frac{g\mu_B H}{kT}\right) - \frac{1}{2} \coth\left(\frac{1}{2} \times \frac{g\mu_B H}{kT}\right) \right]. \quad (13)$$

A metallic disk of thickness d , radius a , and conductivity σ is oriented perpendicular to the z -axis. It is subjected to a uniform and slowly-varying magnetic field $\vec{B}(\vec{r}, t) = B\hat{z} \cos \omega t$.



- Find the total current induced in the disk.
- Determine the power dissipated as heat in the disk.



$$\vec{B} = B \hat{z} e^{-i\omega t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = \frac{i\omega B}{c} \hat{z}$$

So in cylindrical coordinates $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_{\phi}) = \frac{i\omega B}{c} \Rightarrow E_{\phi} = \frac{\omega B \rho}{c^2}$

or, $\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} \Rightarrow E_{\phi} 2\pi \rho = -\frac{1}{c} \frac{\partial B}{\partial t} \pi \rho^2$

So $E_{\phi} = \frac{\omega B \rho}{c^2}$

Current density $\vec{J} = \sigma \vec{E} = \frac{\omega B}{c} \sigma \frac{\rho}{2} \hat{\phi}$

Total current = $I = \int \vec{J} \cdot d\vec{a} = h \int_0^a \vec{J} \cdot \hat{\phi} d\rho = \frac{h\omega B \sigma a^2}{c}$

Power = $\int \vec{E} \cdot \vec{J} dV = 2\pi h \int_0^a \frac{\omega^2 B^2 \sigma}{c^2} \frac{\rho^2}{4} \rho d\rho = \frac{\pi}{8} \frac{h\omega^2 B^2 \sigma a^4}{c^2}$

A beam of identical neutral particles with spin 1/2 (but no other angular momentum) is prepared in a definite spin quantum state $|\Psi\rangle$. The beam then passes through a Stern-Gerlach spin analyzing magnet, which can be oriented along any of the three coordinate axes: x , y , or z . (Don't worry about any technical problems related to which direction the beam is headed.) Two detectors placed behind the Stern-Gerlach analyzer count the number of particles that emerge with spin up and down, respectively, along the particular axis to which the analyzer is set. The results of the three experiments (one for each axis) are shown below.

Result	Axis		
	x	y	z
spin up \uparrow	180	260	200
spin down \downarrow	820	740	800

- Derive the eigenstates corresponding to spin up and spin down along the x and y axes, respectively, in terms of the eigenstates corresponding to spin up and spin down along the z axis.
- Determine the initial spin state $|\Psi\rangle$ of the particles in the beam in terms of the eigenstates corresponding to spin up and spin down along the z axis.

You may want to know that

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Label the eigenstates of the spin operators S_x, S_y, S_z by the eigenvalues $\pm \frac{\hbar}{2}$

$$\Rightarrow S_x |\pm\rangle_x = \pm \frac{\hbar}{2} |\pm\rangle_x$$

$$S_y |\pm\rangle_y = \pm \frac{\hbar}{2} |\pm\rangle_y$$

$$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$$

The results of the expts give us the probabilities of all possible measurements of the operators S_x, S_y, S_z . Thus we know that

$$P_{+z} = |\langle +|\psi\rangle|^2 = 0.20$$

$$P_{-z} = |\langle -|\psi\rangle|^2 = 0.80$$

$$P_{+x} = |\langle +|\psi\rangle|^2 = 0.18$$

$$P_{-x} = |\langle -|\psi\rangle|^2 = 0.82$$

$$P_{+y} = |\langle +|\psi\rangle|^2 = 0.26$$

$$P_{-y} = |\langle -|\psi\rangle|^2 = 0.74$$

We need to know how to write the $|\pm\rangle_x, |\pm\rangle_y$ eigenstates in terms of $|\pm\rangle$

Use given matrices:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{also know that } S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{and } |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{let } |\pm\rangle_x = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\text{also need } |a|^2 + |b|^2 = 1$$

This gives $|+\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} [|+\rangle + |-\rangle]$

$$|-\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} [|+\rangle - |-\rangle]$$

For S_y :

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -ib \\ ia \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

This gives $|+\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} [|+\rangle + i |-\rangle]$

$$|-\rangle_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} [|+\rangle - i |-\rangle]$$

Let the unknown state $|\Psi\rangle$ be written as

$$|\Psi\rangle = a |+\rangle + b |-\rangle$$

$$\Rightarrow |\langle + | \Psi \rangle|^2 = |a|^2 = 0.20 = \frac{1}{5}$$

Overall phase of $|\Psi\rangle$ is unimportant (not measurable)
so choose phase of a to be zero; that is
let a be real and positive.

$$\Rightarrow a = \frac{1}{\sqrt{5}}$$

$$|\langle - | \Psi \rangle|^2 = |b|^2 = 0.80 = \frac{4}{5}$$

Cannot choose phase of b yet, so call it β

$$\Rightarrow b = \frac{2}{\sqrt{5}} e^{i\beta}$$

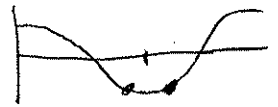
$$\Rightarrow |\Psi\rangle = \frac{1}{\sqrt{5}} [|+\rangle + 2 e^{i\beta} |-\rangle]$$

$$\begin{aligned}
 |\langle \psi | \psi \rangle|^2 &= \left| \frac{1}{\sqrt{2}} (\langle + | + \langle - |) \frac{1}{\sqrt{5}} (|+\rangle + 2e^{i\beta} |-\rangle) \right|^2 \\
 &= \frac{1}{10} |1 + 2e^{i\beta}|^2 \\
 &= \frac{1}{10} (1 + 4 + 2e^{i\beta} + 2e^{-i\beta}) \\
 &= \frac{1}{10} (5 + 4\cos\beta) = 0.18 = \frac{9}{50}
 \end{aligned}$$

$$\Rightarrow 5 + 4\cos\beta = \frac{9}{5}$$

$$\Rightarrow \boxed{\cos\beta = -\frac{4}{5}}$$

$$\Rightarrow \beta = 143^\circ \text{ or } 217^\circ$$



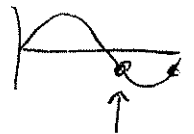
need more info:

$$\begin{aligned}
 |\langle \psi | \psi \rangle|^2 &= \left| \frac{1}{\sqrt{2}} (\langle + | - i \langle - |) \frac{1}{\sqrt{5}} (|+\rangle + 2e^{i\beta} |-\rangle) \right|^2 \\
 &= \frac{1}{10} |1 - 2ie^{i\beta}|^2 \\
 &= \frac{1}{10} (1 + 4 - 2ie^{i\beta} + 2ie^{-i\beta}) \\
 &= \frac{1}{10} (5 - 2i(2i\sin\beta)) \\
 &= \frac{1}{10} (5 + 4\sin\beta) = 0.26 = \frac{13}{50}
 \end{aligned}$$

$$\Rightarrow 5 + 4\sin\beta = \frac{13}{5}$$

$$\Rightarrow \boxed{\sin\beta = -\frac{3}{5}}$$

$$\Rightarrow \beta = 323^\circ \text{ or } 217^\circ$$



$$\Rightarrow \beta = 217^\circ = 1.2\pi$$

$$\Rightarrow \boxed{|\psi\rangle = \frac{1}{\sqrt{5}} [|+\rangle + 2e^{i\pi \cdot 1.2} |-\rangle]}$$

Rain scatters and absorbs microwave radiation. Consider a raindrop to be a dielectric sphere with a 1 mm radius and a dielectric constant $\epsilon = 40(1 + 4\pi i)$ at a frequency of 5 GHz. Unpolarized radiation at 5 GHz is scattered from a single drop with a differential cross section which depends upon the polarization of the scattered field and the direction of observation. Your task is to find the differential and total scattering cross sections.


- a. What approximations can be made to aid in finding a solution to this problem?
- b. Since electric dipole radiation from the raindrop is the central issue, what is the appropriate time-dependent electric dipole moment responsible for the scattered field?
- c. What is the scattered electric field?
- d. Determine the differential cross section of a raindrop for scattering of 5 GHz radiation for each polarization of the scattered field, parallel and perpendicular to the plane of scattering.
- e. Determine the total scattering cross section.

a. $\nu = 5 \text{ GHz} \Rightarrow \lambda = 6 \text{ cm}$ so $\lambda \gg$ radius of object
So make the small source and far field approximations

b. $\lambda \gg$ radius so use the total dipole moment for a dielectric sphere in a uniform field, $\vec{p} = \left(\frac{\epsilon - 1}{\epsilon + 2} \right) a^3 \vec{E}_0$

where $a \equiv$ radius and $\vec{E}_0 \equiv$ applied field.

This comes from analysis of a dielectric sphere in a uniform field

$\vec{E} = E_0 \hat{z}$ 

$$\Phi_{in} = \sum_l A_l r^l P_l(\cos\theta)$$

$$\Phi_{out} = \sum_l [B_l r^l + C_l r^{-(l+1)}] P_l(\cos\theta) \xrightarrow{r \rightarrow \infty} -E_0 r \cos\theta$$

so $B_1 = -E_0$

Tangential E continuity: $\frac{1}{a} \frac{\partial \Phi_{in}}{\partial \theta} \Big|_{r=a} = \frac{1}{a} \frac{\partial \Phi_{out}}{\partial \theta} \Big|_{r=a} \Rightarrow A_1 = -E_0 + \frac{C_1}{a^3}$

Normal D continuity: $-\epsilon \frac{\partial \Phi_{in}}{\partial r} \Big|_{r=a} = -\frac{\partial \Phi_{out}}{\partial r} \Big|_{r=a} \Rightarrow \epsilon A_1 = -E_0 - \frac{2C_1}{a^3}$

So $A_1 = \frac{-3}{2+\epsilon} E_0$, $C_1 = \frac{\epsilon-1}{\epsilon+2} a^3 E_0$, $B_1 = -E_0$

$$\Phi_{out} = -E_0 r \cos\theta + \left(\frac{\epsilon-1}{\epsilon+2} \right) \frac{a^3}{r^2} E_0 \cos\theta$$

hence $\vec{p} = \left(\frac{\epsilon-1}{\epsilon+2} \right) a^3 E_0 \hat{z}$

c. $\vec{E}_s = k^2 \frac{e^{ikr}}{r} (\hat{k} \times \vec{p}) \times \hat{k}$

This can be derived from the vector potential \vec{A} based upon the approximations in part (a). $\vec{A}_s(\vec{r}) = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{r}') d^3r'$

For a finite source, $\int \vec{J}(\vec{r}') d^3r' = -\int \vec{r}' (\vec{\nabla}' \cdot \vec{J}(\vec{r}')) d^3r'$

Charge conservation: $\vec{\nabla} \cdot \vec{J} = -\partial \rho / \partial t$ where $\rho = \rho e^{-i\omega t} \Rightarrow \vec{\nabla} \cdot \vec{J} = i\omega \rho$

so $\vec{A}_s(\vec{r}) = -\frac{e^{ikr}}{cr} \int \vec{r}' i\omega \rho(\vec{r}') d^3r' = -ik \vec{p} \frac{e^{ikr}}{r}$ since $k = \frac{\omega}{c}$

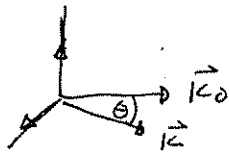
$$\text{Then } \vec{B}_s = \vec{\nabla} \times \vec{A} = (\hat{k} + \vec{p}) \frac{e^{ikr}}{r} k^2 \left(1 + \frac{i}{kr}\right) \rightarrow k^2 (\hat{k} + \vec{p}) \frac{e^{ikr}}{r}$$

$$\text{And } \vec{\nabla} \times \vec{B} = -i\omega \vec{E} \Rightarrow \vec{E}_s = -\hat{k} \times \vec{B}_s = k^2 \frac{e^{ikr}}{r} (\hat{k} + \vec{p}) \times \hat{k}$$

d. $\frac{d\sigma}{d\Omega} = \frac{\text{time-averaged power with polarization } \epsilon \text{ per unit solid angle}}{\text{incident power per unit area}}$

$$= \frac{c}{8\pi} |\hat{\epsilon}^* \cdot E_{sc}|^2 r^2 / |\hat{\epsilon}_0 \cdot \vec{E}_0|^2 c / 8\pi = \frac{k^4}{E_0^2} |\hat{\epsilon}^* \cdot \vec{p}|^2$$

$$= k^4 a^6 \left| \frac{e-1}{e+2} \right|^2 |\hat{\epsilon}^* \cdot \hat{\epsilon}_0|^2$$



Scattering plane is defined by \vec{k} and \vec{k}_0

So for \perp scattering $\hat{\epsilon}^* \cdot \hat{\epsilon}_0 = 1$
 and for \parallel " $\hat{\epsilon}^* \cdot \hat{\epsilon}_0 = \cos\theta$

So for an unpolarized incident field, $\frac{d\sigma}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{e-1}{e+2} \right|^2 (1 + \cos^2\theta)$

c) total scattering cross section $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

$$\sigma = \frac{k^4 a^6}{2} \left| \frac{e-1}{e+2} \right|^2 \left[4\pi + 2\pi \int_0^\pi \sin\theta \cos^2\theta d\theta \right], \text{ where } \int_0^\pi \sin\theta \cos^2\theta d\theta = -\frac{1}{3} \int d(\cos^3\theta) = \frac{2}{3}$$

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left| \frac{e-1}{e+2} \right|^2$$

