

Department of Physics Comprehensive Examination No. 83

March 30 and 31, 1998

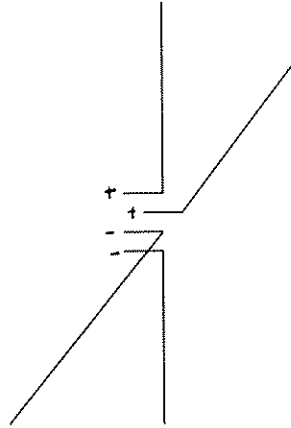
This Comprehensive Examination for Spring 1998 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Monday 30 March. Session Two (problems 3 and 4) begins at 1:30PM Monday 30 March. Session Three (problems 5 and 6) begins at 9:00 AM Tuesday 30 March. Session Four (problems 7 and 8) begins at 1:30PM Tuesday 30 March.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

A turnstile antenna consists of two center-fed dipoles of length L which are perpendicular and carry the complex current density $\vec{J}(\vec{r}, t) = Ie^{-i\omega t} \delta(z) [\delta(y)\hat{x} + i\delta(x)\hat{y}]$ for $-L/2 \leq x \leq L/2$ and $-L/2 \leq y \leq L/2$.



- Find the fields \vec{E} and \vec{B} propagating in the z direction in the long wavelength ($\lambda \gg L$) and far zone ($r \gg \lambda$) approximations. What is the polarization of the radiated wave?
- Find the fields \vec{E} and \vec{B} propagating in the xy plane in the long wavelength and far zone approximations. What is the polarization of the radiated wave?
- Compare the power radiated per steradian in the x direction to that radiated in the z direction.

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \vec{J}(\vec{r}', t') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dV'$$

far zone: $r \gg \lambda \Rightarrow \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{e^{ikr}}{r} e^{-ik\hat{r}\cdot\vec{r}'}$

long wavelength: $\lambda \gg l$ or $kl \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}'} \approx 1$

so $\vec{A}(\vec{r}, t) = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{r}', t) dV'$

but $\int \vec{J}(\vec{r}', t) dV' = -i\omega \int \vec{r}' \rho(\vec{r}') e^{-i\omega t} dV' = -i\omega \vec{p}(t) = -i\omega \vec{p} e^{-i\omega t}$

here $\vec{A} = -\frac{e^{ikr}}{cr} i\omega \vec{p} e^{-i\omega t} = -ik\vec{p} \frac{e^{i(kr-\omega t)}}{r}$

$\vec{B} = \vec{\nabla} \times \vec{A} = -ik \left[\frac{k}{r} - \frac{1}{r^2} \right] \hat{r} \times \vec{p} e^{i(kr-\omega t)}$

so $\vec{B} \rightarrow \frac{k^2}{r} e^{i(kr-\omega t)} \hat{r} \times \vec{p}$ as $r \rightarrow \infty$

$\vec{E} = \frac{1}{\epsilon_0} \vec{\nabla} \times \vec{B} = -k^2 \frac{e^{i(kr-\omega t)}}{r} \hat{r} \times (\hat{r} \times \vec{p}) = -\hat{r} \times \vec{B}$

$\int \vec{J}(\vec{r}') dV' = \int_{-l/2}^{l/2} dy \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dz I [\delta(z)\delta(y)\hat{x} + i\delta(z)\delta(x)\hat{y}]$
 $= \int_{-l/2}^{l/2} dy I [l\delta(y)\hat{x} + i\hat{y}] = Il(\hat{x} + i\hat{y})$

so $\vec{A}(\vec{r}, t) = \frac{e^{i(kr-\omega t)}}{cr} Il(\hat{x} + i\hat{y})$ so $\vec{p} = \frac{Il(\hat{x} + i\hat{y})}{-i\omega} = \frac{iIl}{\omega}(\hat{x} + i\hat{y})$

$\vec{B} = k^2 \frac{e^{i(kr-\omega t)}}{r} iIl \hat{r} \times (\hat{x} + i\hat{y}) = B \hat{r} \times (\hat{x} + i\hat{y})$

so for $\hat{r} = \cos\phi \hat{x} + \sin\phi \hat{y}$, $\hat{r} \times (\hat{x} + i\hat{y}) = -i\cos\phi \hat{z} - \sin\phi \hat{z}$

the $\vec{E} = -\hat{r} \times \vec{B} = i\cos\phi \hat{z} + \sin\phi \hat{z} (i\cos\phi + \sin\phi) \hat{y} \cos\phi + \sin\phi (i\cos\phi + \sin\phi) \hat{x}$
 linear polarization.

For $\hat{r} = \hat{z}$,

$$\vec{B} = B(\hat{y} - i\hat{x})$$

$$\vec{E} = -B(-\hat{x} - i\hat{y})$$

$$\begin{aligned} c. \frac{dP}{d\Omega} &= (\vec{S} \cdot \hat{r}) r^2 = \frac{c}{8\pi} \operatorname{Re} \left[(\vec{E} \times \vec{B}^*) \cdot \hat{r} \right] r^2 \\ &= \frac{c r^2}{8\pi} \operatorname{Re} \left[(\vec{B} \times \hat{r}) \times \vec{B}^* \cdot \hat{r} \right] = \frac{c k^4}{8\pi} |(\vec{r} \times \vec{p}) \times \hat{r}|^2 \\ &= \frac{c k^4}{8\pi} |\vec{p}|^2 \sin^2 \theta \quad \theta = \text{angle between } \vec{p} \text{ and } \hat{r} \end{aligned}$$

$$\text{So when } \hat{r} = \hat{z}, \quad \frac{dP}{d\Omega} = \frac{c k^4}{8\pi} p^2 \left| (\hat{z} \times (\hat{x} + i\hat{y})) \times \hat{z} \right|^2$$

$$\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} p^2 |\hat{x} + i\hat{y}|^2 = \frac{c k^4}{8\pi} p^2 \times 2$$

$$\text{When } \hat{r} = \hat{x}, \quad \frac{dP}{d\Omega} = \frac{c k^4}{8\pi} p^2 \left| \underbrace{(\hat{x} \times (\hat{x} + i\hat{y}))}_{i\hat{y}} \times \hat{x} \right|^2$$

$$\frac{dP}{d\Omega} = \frac{c k^4}{8\pi} p^2$$

A two-dimensional isotropic harmonic oscillator with mass m and resonance frequency ω has the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2.$$

A small perturbation $W = bxy$ is applied to the system, where b is a real, positive constant. Determine the energy corrections to the ground energy level and the second excited energy level (*i.e.*, the third level overall). In each case, find the corrections to the first nonvanishing order.

$$H = \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{P_y^2}{2m} + \frac{1}{2}m\omega^2 y^2$$

$$H = H_x + H_y$$

eigenvalues of H_x are $(n_x + \frac{1}{2})\hbar\omega$; $n_x = 0, 1, \dots$

" H_y are $(n_y + \frac{1}{2})\hbar\omega$ $n_y = 0, 1, \dots$

\Rightarrow eigenvalues of H are $(n_x + n_y + 1)\hbar\omega$

label eigenstates $|n_x n_y\rangle$

$\Rightarrow |00\rangle$ is ground state

$|10\rangle, |01\rangle$ are degenerate states of 1st excited ~~state~~ level.

$|20\rangle, |11\rangle, |02\rangle$ are degenerate states of 2nd excited level

$W = bxy$ is perturbation

From ref sheet: $\langle n_x | x | n_x' \rangle = \sqrt{\frac{\hbar}{m\omega}} \left[\sqrt{\frac{n_x+1}{2}} \int_{n_x+1, n_x'} \right]$

$$+ \sqrt{\frac{n_x}{2}} \int_{n_x-1, n_x'} \right]$$

\Rightarrow only states of n_x, n_y different from n_x', n_y' by ± 1 will have non-zero matrix elements of W .

Perturbed energy of non-degenerate state is given by ^{ground (00)}

$$E_{00}' = E_{00} + \langle 00|W|00 \rangle + \sum_{\substack{m_x \neq 0 \\ m_y \neq 0}} \frac{|\langle 00|W|m_x m_y \rangle|^2}{E_{00} - E_{m_x m_y}}$$

$\langle 00|W|00 \rangle = 0 \Rightarrow$ no 1st order correction

$$\langle 00|W|m_x m_y \rangle = \langle 00|W|11 \rangle \quad \text{only non-zero}$$

$$= \left(\sqrt{\frac{\hbar}{m\omega}} \cdot \sqrt{\frac{1}{2}} \right)^2 \cdot b^2$$

$$\Rightarrow E_{00}' = \hbar\omega + 0 + \frac{b^2 \left(\frac{\hbar}{2m\omega} \right)^2}{\hbar\omega - 3\hbar\omega}$$

$$E_{00}' = \hbar\omega - b^2 \frac{\hbar}{8m^2\omega^3}$$

$$E_{00}' = \hbar\omega \left[1 - \frac{1}{2} \left(\frac{b}{2m\omega^2} \right)^2 \right]$$

2nd excited level (111, 120, 102) is degenerate so we need to use degenerate pert. th.
Find matrix elements:

$$\langle 20|W|20 \rangle = \langle 11|W|11 \rangle = \langle 02|W|02 \rangle = 0$$

$$\langle 20|W|02 \rangle = 0 \quad \text{since } n_x, n_x' \text{ differ by 2}$$

$$\langle 11|W|20 \rangle = \frac{b\hbar}{m\omega} \cdot \sqrt{\frac{2}{2}} \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \frac{b\hbar}{m\omega}$$

$$\langle 11|W|02 \rangle = \frac{1}{\sqrt{2}} \frac{b\hbar}{m\omega}$$

\Rightarrow in degenerate subspaces, W is given by

$$W = \frac{b\hbar}{\sqrt{2}m\omega} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now diagonalize:

$$\det |W - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & \frac{b\hbar}{\sqrt{2}m\omega} & 0 \\ \frac{b\hbar}{\sqrt{2}m\omega} & -\lambda & \frac{b\hbar}{\sqrt{2}m\omega} \\ 0 & \frac{b\hbar}{\sqrt{2}m\omega} & -\lambda \end{vmatrix} = 0$$

$$(-\lambda) \left(\lambda^2 - \frac{1}{2} \frac{b^2 \hbar^2}{m^2 \omega^2} \right) - \frac{b\hbar}{\sqrt{2}m\omega} \left(-\lambda \frac{b\hbar}{\sqrt{2}m\omega} \right) = 0$$

$$-\lambda \left[\lambda^2 - \frac{b^2 \hbar^2}{m^2 \omega^2} \right] = 0$$

$$\Rightarrow \lambda = 0, \pm b \frac{\hbar}{m\omega}$$

\Rightarrow energies are:

$$3\hbar\omega + b \frac{\hbar}{m\omega}$$

$$3\hbar\omega$$

$$3\hbar\omega - b \frac{\hbar}{m\omega}$$

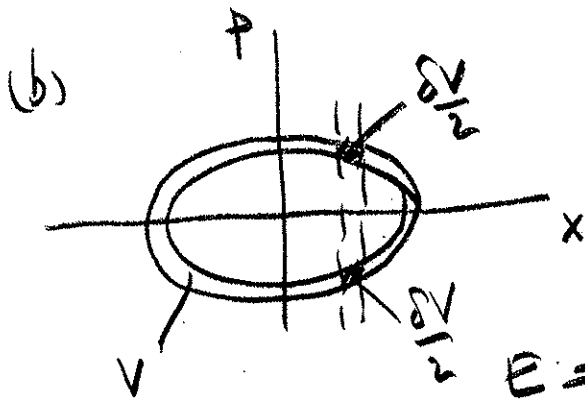
Consider an ensemble of classical one-dimensional oscillators.

- (a) Let the displacement x of an oscillator as a function of time t be given by $x = x_0 \cos(\omega t + \phi)$. Assume that the phase angle ϕ is equally likely to assume any value in its range $0 < \phi < 2\pi$. For any fixed time t , find the probability $P(x)dx$ that x lies between x and $x + dx$. Express $P(x)$ in terms of x_0 and x .
- (b) Consider the classical phase space for such an ensemble of one-dimensional oscillators, their energy being known to lie between E and $E + dE$. Determine $P(x)dx$ in terms of E and x .
- (c) Show that the result obtained in part (b) is the same as that obtained in part (a).

$$(a) P(x) dx = 2 \cdot \frac{d\psi}{2\pi} = \frac{d\psi}{\pi}$$

$$\left| \frac{d\psi}{dx} \right| = \frac{1}{x_0 \sin(\omega t + \varphi)} = \frac{1}{(x_0^2 - x^2)^{1/2}}$$

$$\therefore P(x) dx = \frac{dx}{\pi (x_0^2 - x^2)^{1/2}}$$



V is the volume of the phase space shell contained between surfaces on which $E = E$ and $E = E + \delta E$.

$$A_e = \pi p_{\max} x_{\max}; \text{ and } E = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$\therefore A_e = \pi \sqrt{2mE} \sqrt{\frac{2E}{m\omega^2}} = \frac{2\pi E}{\omega}, \text{ from which}$$

$$V = \delta A_e = \frac{2\pi}{\omega} \delta E$$

$$\text{Thus } P(x) dx = \frac{\delta V}{V} = \frac{2 \delta p dx}{2\pi \delta E}$$

But for any x , $p \delta p / m = \delta E$ and so

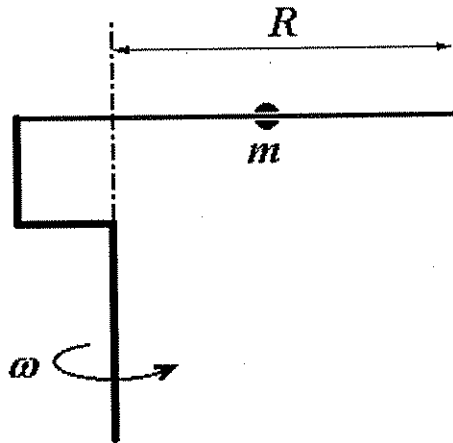
$$P(x) dx = \frac{m\omega \delta x}{\pi p} = \frac{m\omega}{\pi} \frac{dx}{(2mE - m^2\omega^2 x^2)^{1/2}}$$

(c) for $x = x_0 \cos(\omega t + \varphi)$, it follows that $E = \frac{1}{2} m\omega^2 x_0^2$ and so $P(x) dx = \frac{1}{\pi} \frac{dx}{(x_0^2 - x^2)^{1/2}}$ as in (a).

A smooth straight rod is attached to a rotating vertical crank, so that it rotates in the horizontal plane. The angular speed is $\omega = \text{const}$. The distance of the free end from the axis of rotation is R .

A bead m can slide on the rod without friction.

- Find the motion of the bead if it is initially held at a distance r_0 from the rotation axis (towards the free end) and then released at $t=0$.
- What is the bead's trajectory in the stationary system?
- How much time it will take the bead to slide off the free end?
- What is the speed of the bead in the stationary system immediately after it slides off the rod?
- What is the maximum possible value of this speed?



The molar latent heat of transformation in going from phase 1 to phase 2 at temperature T and pressure p is λ . Determine how the latent heat varies with temperature, that is, determine the total derivative, $\frac{d\lambda}{dT}$. Express your answer in terms of λ and the molar specific heat at constant pressure c_p , the coefficient of expansion α , and the molar volume v of each phase at the original temperature T and pressure p .

Note the Clausius-Clapeyron equation states that $\frac{dp}{dT} = \frac{\lambda}{T(v_2 - v_1)}$.

$$l = l(p, T)$$

$$\frac{dl}{dT} = \left(\frac{\partial l}{\partial p}\right)_T \frac{dp}{dT} + \left(\frac{\partial l}{\partial T}\right)_p$$

$$l = T(\alpha_2 - \alpha_1)$$

$$\left(\frac{\partial l}{\partial p}\right)_T = T \left[\left(\frac{\partial \alpha_2}{\partial p}\right)_T - \left(\frac{\partial \alpha_1}{\partial p}\right)_T \right]$$

$$\text{But } \left(\frac{\partial \alpha}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p = -\alpha v$$

$$\therefore \left(\frac{\partial l}{\partial p}\right)_T = -T (\alpha_2 v_2 - \alpha_1 v_1)$$

$$\left(\frac{\partial l}{\partial T}\right)_p = \alpha_2 - \alpha_1 + T \left[\left(\frac{\partial \alpha_2}{\partial T}\right)_p - \left(\frac{\partial \alpha_1}{\partial T}\right)_p \right]$$

$$= \alpha_2 - \alpha_1 + c_{p2} - c_{p1}$$

$$= \frac{l}{T} + c_{p2} - c_{p1}$$

and since we can use C-C for $\frac{dp}{dT}$, we get

$$\frac{dl}{dT} = c_{p2} - c_{p1} + \frac{l}{T} - \frac{l(\alpha_2 v_2 - \alpha_1 v_1)}{v_2 - v_1}$$

Consider a quantum mechanical system with a three-dimensional state space. In the basis defined by three orthonormal kets $|1\rangle$, $|2\rangle$, and $|3\rangle$ an observable Q is represented by the matrix

$$Q = q \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix},$$

where q is a real, positive constant. The Hamiltonian H of the system has three distinct eigenvalues $E_1 = a$, $E_2 = 2a$, and $E_3 = 3a$ (a is a real, positive constant) with corresponding eigenstates $|E_1\rangle = |1\rangle$, $|E_2\rangle = |2\rangle$, and $|E_3\rangle = |3\rangle$. At time $t = 0$, the state of the system is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{6}}|3\rangle.$$

- What is the matrix representation of the Hamiltonian H in the basis defined by $|1\rangle$, $|2\rangle$, and $|3\rangle$?
- Find the expectation value $\langle H \rangle$ and the r.m.s. deviation ΔH at time $t = 0$.
- At time t_0 (where $t_0 > 0$) the observable Q is measured. What are the possible results of that measurement and the corresponding probabilities?
- For the particular case when the result of the above measurement of Q is the lowest value possible, find the state of the system for times $t > t_0$.

$$a) \quad E_1 = a \quad |E_1\rangle = |1\rangle$$

$$E_2 = 2a \quad |E_2\rangle = |2\rangle$$

$$E_3 = 3a \quad |E_3\rangle = |3\rangle$$

\Rightarrow in $|1\rangle, |2\rangle, |3\rangle$ basis:

$$H = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$b) \quad \langle H \rangle = \langle \psi | H | \psi \rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{6}}|3\rangle$$

$$\Rightarrow \langle H \rangle = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \right) a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= a \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{3}} \\ \frac{3}{\sqrt{6}} \end{pmatrix}$$

$$= a \left[\frac{1}{2} + \frac{2}{3} + \frac{3}{6} \right]$$

$$\langle H \rangle = \frac{5}{3} a$$

$$\Delta H = \sqrt{\langle (H - \langle H \rangle)^2 \rangle}$$

$$= \sqrt{\langle (H^2 - 2H\langle H \rangle + \langle H \rangle^2) \rangle}$$

$$= \sqrt{\langle H^2 \rangle - 2\langle H \rangle^2 + \langle H \rangle^2}$$

$$\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$$

$$\langle H^2 \rangle = \langle \psi | H^2 | \psi \rangle$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \right) a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{3} \\ 1/\sqrt{6} \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \right) a^2 \begin{pmatrix} 1/\sqrt{2} \\ 4/\sqrt{3} \\ 9/\sqrt{6} \end{pmatrix}$$

$$= a^2 \left(\frac{1}{2} + \frac{4}{3} + \frac{9}{6} \right)$$

$$\langle H^2 \rangle = a^2 \frac{10}{3}$$

$$\Delta H = \sqrt{a^2 \cdot \frac{10}{3} - \left(\frac{5}{3} a \right)^2}$$

$$= a \sqrt{\frac{10}{3} - \frac{25}{9}}$$

$$\Delta H = a \frac{\sqrt{5}}{3}$$

$$c) \quad |\psi(0)\rangle = \frac{1}{\sqrt{2}} |1\rangle + \frac{1}{\sqrt{3}} |2\rangle + \frac{1}{\sqrt{6}} |3\rangle$$

$$\begin{aligned} \Rightarrow |\psi(t)\rangle &= \frac{1}{\sqrt{2}} e^{-i\frac{E_1}{\hbar}t} |1\rangle + \frac{1}{\sqrt{3}} e^{-i\frac{E_2}{\hbar}t} |2\rangle + \frac{1}{\sqrt{6}} e^{-i\frac{E_3}{\hbar}t} |3\rangle \\ &= \frac{1}{\sqrt{2}} e^{-i\frac{q}{\hbar}t} |1\rangle + \frac{1}{\sqrt{3}} e^{-i\frac{2q}{\hbar}t} |2\rangle + \frac{1}{\sqrt{6}} e^{-i\frac{3q}{\hbar}t} |3\rangle \end{aligned}$$

Now need eigenvalue, and eigenvectors of Q ,

$$Q = q \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$\det(Q - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} 2q - \lambda & 0 & 0 \\ 0 & -\lambda & iq \\ 0 & -iq & -\lambda \end{vmatrix} = 0$$

$$(2q - \lambda)(\lambda^2 - q^2) = 0$$

$$\Rightarrow \boxed{\lambda = 2q, \pm q \text{ are possible results}}$$

$$\text{let } q_1 = -q, \quad q_2 = q, \quad q_3 = 2q$$

$$\Rightarrow Q|q_i\rangle = -q|q_i\rangle$$

$$\text{let } |q_i\rangle = a|1\rangle + b|2\rangle + c|3\rangle$$

$$\Rightarrow q \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -q \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \left. \begin{aligned} 2a &= -a \Rightarrow a = 0 \\ 1c &= -b \end{aligned} \right\} |q_i\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

$$Q|q_2\rangle = q|q_2\rangle, \text{ change } -q \rightarrow q \text{ in above}$$

$$\Rightarrow |q_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$$

$$Q|q_3\rangle = 2q|q_3\rangle$$

$$\Rightarrow 2a = 2a$$

$$ic = 2b$$

$$-ib = 2c = 2\left(\frac{2b}{i}\right) = -i^2 4b \Rightarrow b = 0$$

$$\Rightarrow c = 0$$

$$\Rightarrow |q_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$P(-q) = |\langle q_1 | \psi(t) \rangle|^2 = \left| \frac{1}{\sqrt{2}} (0, 1, -i) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\frac{q}{\hbar}t_0} \\ \frac{1}{\sqrt{3}} e^{-i\frac{2q}{\hbar}t_0} \\ \frac{1}{\sqrt{6}} e^{-i\frac{3q}{\hbar}t_0} \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \left| \frac{1}{\sqrt{3}} e^{-i\frac{2q}{\hbar}t_0} - \frac{i}{\sqrt{6}} e^{-i\frac{3q}{\hbar}t_0} \right|^2$$

$$= \frac{1}{2} \cdot \frac{1}{6} \left| \sqrt{2} - i e^{-i\frac{q}{\hbar}t_0} \right|^2$$

$$= \frac{1}{12} (2 + 1 - i\sqrt{2} e^{-i\frac{q}{\hbar}t_0} + i\sqrt{2} e^{i\frac{q}{\hbar}t_0})$$

$$= \frac{1}{12} (3 - 2\sqrt{2} \sin \frac{q}{\hbar}t_0)$$

$$P(-q) = \frac{1}{4} - \frac{\sqrt{2}}{6} \sin \frac{q}{\hbar}t_0$$

$$P(1g) = |\langle g_2 | \psi(t) \rangle|^2 = \left| \frac{1}{\sqrt{2}} (0, 1, i) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\frac{a}{\hbar}t_0} \\ \frac{1}{\sqrt{3}} e^{-i\frac{2a}{\hbar}t_0} \\ \frac{1}{\sqrt{6}} e^{-i\frac{3a}{\hbar}t_0} \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \cdot \frac{1}{6} \left| \sqrt{2} + i e^{-i\frac{a}{\hbar}t_0} \right|^2$$

$$P(1g) = \frac{1}{4} + \frac{\sqrt{2}}{6} \sin \frac{a}{\hbar} t_0$$

$$P(2g) = |\langle g_3 | \psi(t) \rangle|^2 = \left| (1, 0, 0) \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\frac{a}{\hbar}t_0} \\ \frac{1}{\sqrt{3}} e^{-i\frac{2a}{\hbar}t_0} \\ \frac{1}{\sqrt{6}} e^{-i\frac{3a}{\hbar}t_0} \end{pmatrix} \right|^2$$

$$P(2g) = \frac{1}{2}$$

note that $P(1g) + P(2g) = 1$ as required

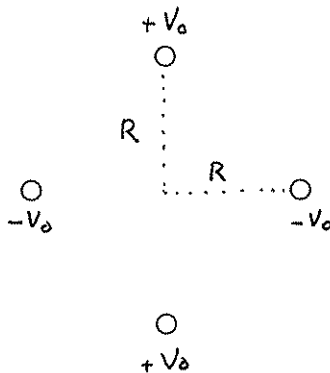
d) result of measurement is $-g$ (lowest value)

$$\Rightarrow |\psi(t_0)\rangle = |g_1\rangle$$

$$= \frac{1}{\sqrt{2}} |2\rangle + \frac{i}{\sqrt{2}} |3\rangle$$

$$\Rightarrow |\psi(t > t_0)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{2a}{\hbar}(t-t_0)} |2\rangle + \frac{i}{\sqrt{2}} e^{-i\frac{3a}{\hbar}(t-t_0)} |3\rangle$$

The behavior of an ion in a RF-driven quadrupole mass spectrometer or a quadrupole ion trap can be analyzed by determining the potential and electric field for the electrostatic case. An approximation to this structure consists of four parallel, infinitely long, very thin wires with the applied potentials indicated in the figure below.



- Find the electrostatic potential $\Phi(\vec{\rho})$ for $\rho \leq R$ in both polar and cartesian coordinates.
- What is the electric field in cartesian coordinates?
- A quadrupole mass spectrometer functions by applying a time-varying potential $V_0 e^{i\omega t}$ to the wires which effects the trajectory of a positively-charged ion moving approximately along the axis of the structure. Describe physically how the potential you calculated could allow a $+1$ ion of mass m_1 to travel along the axis while allowing another $+1$ ion of mass m_2 to wander out between the wires.

$$a). \quad \Phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} \rho^{-n} [C_n \cos n\varphi + D_n \sin n\varphi] \\ + \sum_{n=1}^{\infty} \rho^{+n} [C_n \cos n\varphi + A_n \sin n\varphi]$$

$$BC: \quad \Phi(\rho=R, \varphi) = V_0 (\delta(\varphi) - \delta(\varphi - \pi/2) - \delta(\varphi - \pi) \\ - \delta(\varphi - 3\pi/2)) \\ = V(\varphi)$$

$$b_0 = 0 \text{ since } \ln \rho \rightarrow -\infty \text{ as } \rho \rightarrow 0$$

$$\text{and } \{C_n, S_n\} = 0 \text{ since } \rho^{-n} \rightarrow \infty \text{ as } \rho \rightarrow 0$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} V(\varphi) d\varphi = 0$$

$$C_n = \frac{1}{\pi R^n} \int_0^{2\pi} V(\varphi) \cos n\varphi d\varphi = \frac{1}{\pi R^n} [\cos n\varphi - \cos n\pi/2 + \cos n\pi \\ - \cos 3\pi/2]$$

$$\text{So } C_2 = \frac{4}{\pi R^2} \quad C_6 = \frac{4}{\pi R^6} \quad C_{10} = \frac{4}{\pi R^{10}}, \text{ etc}$$

$$S_n = \frac{1}{\pi R^n} \int_0^{2\pi} V(\varphi) \sin n\varphi d\varphi = \frac{1}{\pi R^n} [\sin n\varphi - \sin n\pi/2 + \sin n\pi - \sin n3\pi/2] \\ = 0 \text{ for all } n$$

$$\text{So } \Phi = C_2 \rho^2 \cos 2\varphi + C_6 \rho^6 \cos 6\varphi + \dots$$

$$\approx \frac{4}{\pi R^2} \rho^2 \cos 2\varphi$$

$$\text{Use } \cos 2\varphi = \cos^2 \varphi - \sin^2 \varphi \Rightarrow \rho^2 \cos 2\varphi = x^2 - y^2$$

$$\text{So } \Phi = \frac{4}{\pi R^2} (x^2 - y^2)$$

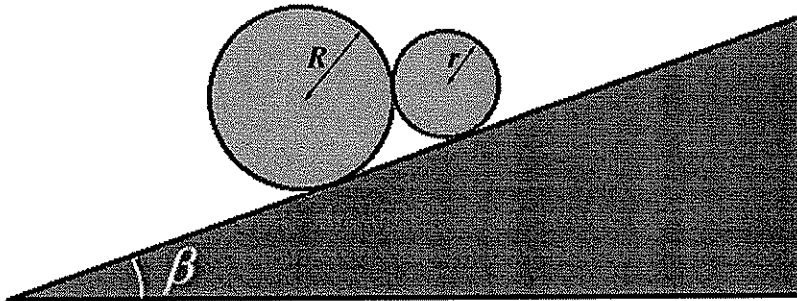
$$b. \quad \vec{E} = -\nabla \Phi = \frac{8}{\pi R^2} (x\hat{x} - y\hat{y})$$

c. Choose $m_1 > m_2$. Then at some frequency, m_1 will move helically along the axis, but m_2 will move in a widening helix, thereby escaping.

Two spheres with radii R and r ($R > r$), made of the same material of a uniform density, are glued together by a firm glue so that their surfaces are in contact. The mass of the glue is negligibly small. This two-sphere object rolls without slipping on a plane inclined at an angle β to the horizontal.

Assuming that r/R is larger than this minimum value, find the frequency of small oscillations of the object about its equilibrium position.

Hints: Use the line going through the points where the spheres touch the plane as the axis of instantaneous rotation. Note that the point where the line passing through the centers of the two spheres intersects the plane remains fixed.



The two-sphere system on the inclined plane in the equilibrium position – side view.

Equilibrium position - side view:

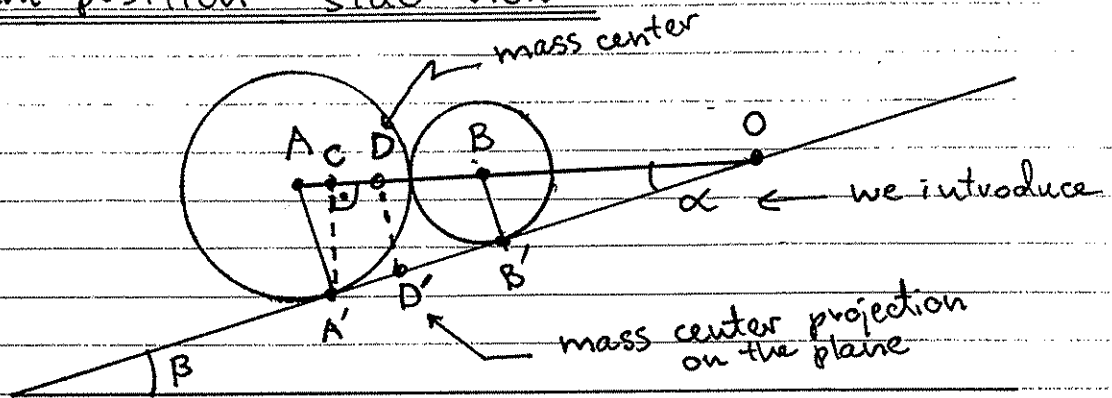


Fig. 1

ABO is the line going through the sphere centers (A & B), and intersecting the plane at O. The O point does not move when the object rolls.

Points A', B' are the points of contact with the plane.

The A'C line is perpendicular to the axis ABO.

Note that when the object rolls, the points on the larger sphere that are in contact with the plane do not lie on an equatorial circle, but on a smaller circle with radius A'C.

In the equilibrium position, the A'C line is vertical. When the object rolls, the A'C line turns. Let's call the angle between A'C and the vertical direction φ (in other words, φ is the angle of rotation about the ABO axis).

View from the above — non-equilibrium position.

(2)

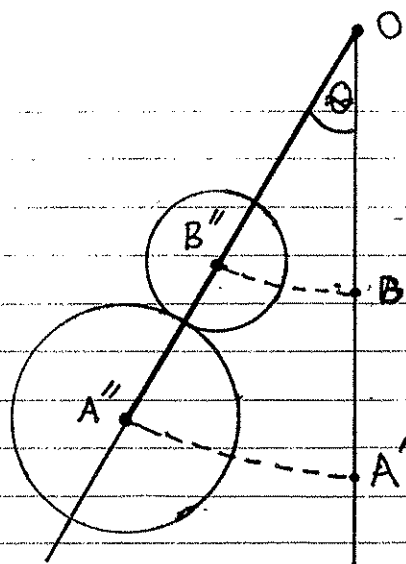


Fig. 2.

A', B' are the points of contact in the equilibrium position;

A'', B'' are the points of contact in the new non-equilibrium position.

As suggested in the hint, the angle between the OA' and OB'' lines should be taken as the generalized coordinate.

Step # 1 is to find the relation between the θ angle and the angle of rotation φ .

The length of the $A'A''$ arc is:

$$A'A'' = OA' \cdot \theta \quad (1)$$

Also, this length can be expressed in terms of the φ angle:

$$A'A'' = A'C \cdot \varphi \quad (2)$$

From the similarity of the $AA'O$ and ACA' triangles in Fig. 1

$$\frac{A'C}{AC} = \frac{A'O}{AA'} \Rightarrow \frac{A'C}{A'O} = \frac{AC}{AA'} \quad (3)$$

3.

By drawing a line parallel to $A'B'O$ and passing through B in Fig. 1, one can readily see that:

$$\frac{AC}{AA'} = \frac{R-r}{R+r} \quad (4)$$

Combining Eqs. 1-4 we obtain:

$$\varphi = \frac{R+r}{R-r} \dot{\theta} \quad (5)$$

which also allows us to express the angular speed of rotation of the object, $\omega = \dot{\varphi}$, in terms of the time derivative of θ :

$$\omega = \dot{\varphi} = \frac{R+r}{R-r} \dot{\theta} \quad (6)$$

Step #2: The kinetic energy. Following the suggestion from the hint, we use the line ~~of~~ passing through the points of contact as the instantaneous axis of rotation.

The moment of inertia with respect to this axis can be readily obtained using the parallel axis theorem - with the aid of Fig. 1 it is easy to find that:

$$\begin{aligned} I_{\text{Inst.}} &= \frac{2}{5} MR^2 + MR^2 + \frac{2}{5} mr^2 + mr^2 = \\ &= \frac{7}{5} (MR^2 + mr^2) \end{aligned}$$

where M, m are the masses of the two spheres. Expressing these masses through the density ρ :

$$M = \frac{4}{3} \pi \rho R^3 ; m = \frac{4}{3} \pi \rho r^3$$

we obtain:

$$I_{inst.} = \frac{28}{15} \pi g (R^5 + r^5) \tag{7}$$

The kinetic energy is:

$$K = I_{inst.} \frac{\omega^2}{2} \tag{8}$$

Combining (6) and (7) in (8) we obtain:

$$K = \frac{14}{15} \pi g \frac{(R^5 + r^5)(R+r)^2}{(R-r)^2} \dot{\theta}^2 \tag{8a}$$

Step #3: The potential energy.

First, we have to find the position of the mass center (D in Fig. 1). It's clear that:

$$M \cdot AD = m \cdot BD$$

But

$$AD = R+r - BD$$

so

$$BD = \frac{M(R+r)}{M+m} = \frac{\frac{4}{3} \pi g R^3 (R+r)}{\frac{4}{3} \pi g (R^3 + r^3)} = \frac{R^3 (R+r)}{R^3 + r^3}$$

From the BB'O triangle:

$$\frac{BB'}{BO} = \frac{R-r}{R+r} \text{ and } BB' = r \implies BO = r \frac{R+r}{R-r}$$

$$\text{So, } OD = BO + BD = (R+r) \left[\frac{r}{R-r} + \frac{R^3}{R^3 + r^3} \right] \tag{9}$$

We need the length of OD', where D' is the projection of mass center D on the plane. Denoting the angle between DO and D'O as α (Fig. 1), we have:



$$D'O = DO \cdot \cos \alpha \tag{10}$$

(5)

It's easy to show that.

$$\sin \alpha = \frac{R-r}{R+r}, \text{ so that } \cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{\frac{R^2 + r^2 + 2Rr - R^2 - r^2 + 2Rr}{(R+r)^2}} = \frac{2\sqrt{Rr}}{(R+r)} \quad (10)$$

From (9), (10) and (11):

$$D'O = 2\sqrt{Rr} \left[\frac{r}{R-r} + \frac{R^3}{R^3+r^3} \right] = 2\sqrt{Rr} \frac{R^4+r^4}{(R-r)(R^3+r^3)} \quad (12)$$

When the system rolls, its mass center moves along a circle with radius $D'O$, ~~which is~~ in a plane parallel to the plane on which the rolling takes place.

It is rather straight forward to show that the potential energy is

$$\begin{aligned} U &= -(M+m) \cdot g \cdot D'O \cdot \cos \theta \cdot \sin \beta \\ &= -\frac{4}{3} \pi g g (R^3+r^3) \cdot 2\sqrt{Rr} \left[\frac{r^4}{(R-r)} + \frac{R^4}{(R^3+r^3)} \right] \cos \theta \sin \beta = \\ &= -\frac{8}{3} \pi g g \sqrt{Rr} \frac{(R^4+r^4)}{(R-r)} \sin \beta \cos \theta = \\ &\cong -\frac{8}{3} \pi g g \sqrt{Rr} \frac{R^4+r^4}{R-r} \sin \beta \left(1 - \frac{\theta^2}{2}\right) = \text{(for small oscillations)} \\ &= \text{const} + \left(\frac{4}{3} \pi g g \sqrt{Rr} \frac{R^4+r^4}{R-r} \sin \beta \right) \cdot \theta^2 \quad (13) \end{aligned}$$

From (8a) and (13) it is straight forward to set up the Lagrange equation. After performing several straight forward operations, the equation reduces to:

$$\ddot{\theta} = - \left[\frac{2}{7} g \sqrt{Rr} \frac{(R-r)(R^4+r^4)}{(R^5+r^5)(R+r)^2} \sin \beta \right] \theta$$

So that the oscillation frequency is the square root of the expression

