Department of Physics Comprehensive Examination No. 82

January 5 and 6 1998

This Comprehensive Examination for Winter 1998 consists of eight problems each worth 20 points. The problems are grouped into four sessions, each of which lasts for three hours. Session One (problems 1 and 2) begins at 9:00 AM Monday 5 January. Session Two (problems 3 and 4) begins at 1:30PM Monday 5 January. Session Three (problems 5 and 6) begins at 9:00 AM Tuesday 6 January. Session Four (problems 7 and 8) begins at 1:30PM Tuesday 6 January.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it is possible to obtain partial credit, especially if you demonstrate conceptual understanding. Do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter, but not your name, is on the inside of the back cover of every bluebook. Be sure to remember your student letter for use in the remaining sessions of the examination.

If something is omitted from the statement of the problem or you feel there is an ambiguity, please ask your question quietly and privately, so as not to disturb the others. Only your bluebooks and the examination should be on the table before you. Any other items should be stored on the floor. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your notebooks for scratch work separated by at least one page from your solutions. Scratch work will not be graded.

OSU Physics Comprehensive Exam No. 82   5 - 6 January 1998   **Problem 1**

A cylinder of radius R and length h is made of material with uniform density ρ. A cylindrical hole of radius r is bored through the entire cylinder length. The bore axis is parallel to the cylinder axis, and the distance between the two axes is a (Fig. 1). The cylinder rests on a horizontal plane with its axis parallel to the plane. Find small motions of the cylinder about the position of equilibrium for the following cases:

(i) the cylinder rolls without slipping;

(ii) the plane is perfectly smooth (i.e., the cylinder can slide with no friction on the plane).

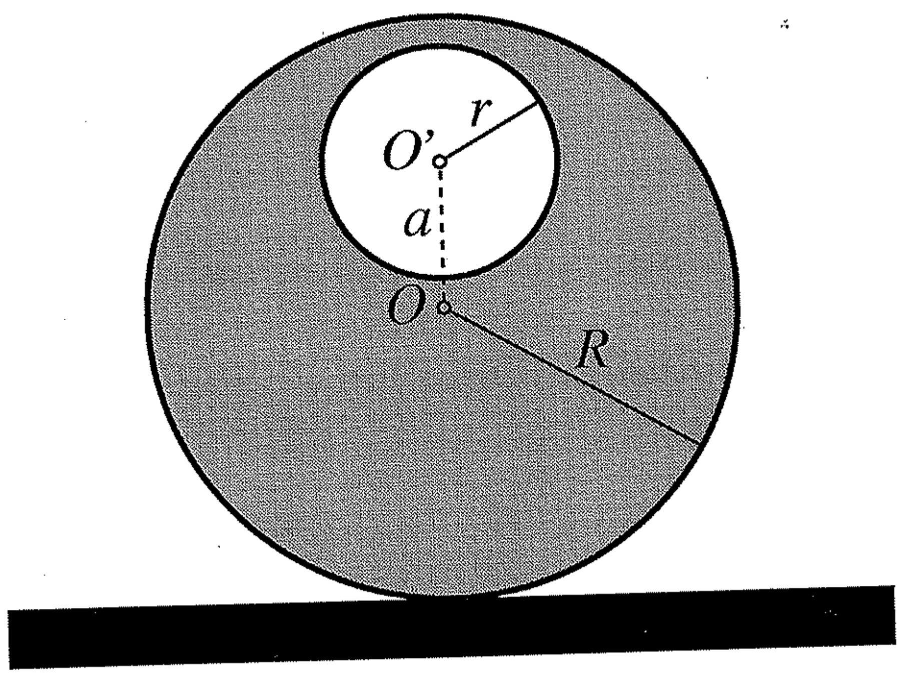


Figure 1. A cylinder with a bore resting on a plane. R and r are the cylinder and the bore radii, respectively, and $OO' = a$ is the distance between the cylinder axis and the bore axis.

Problem # — Solution:

First, one has to determine the position of the *mass center*. In the coordinate system defined in Fig. 1, due to the object symmetry, only the z-coordinate of the mass center — let's call it ζ — is not zero. We can find ζ from the general recipe for the mass center coordinates of a body with uniform density:

$$\zeta = \frac{1}{V_0}\iiint z dV \tag{1}$$

where V_0 is the volume of the body. In the present case, of course:

$$V_0 = \pi(R^2 - r^2)h. \tag{2}$$

An easy way of calculating ζ for the cyllinder with a bore in the present problem using Eq. (1) is to add and subtract an integral taken over the bore volume:

$$\zeta = \frac{1}{V_0} \iiint_{\text{bored cyl.}} zdV = \frac{1}{V_0}\left(\iiint_{\text{cyll. w. bore}} zdV + \iiint_{\text{bore}} zdV - \iiint_{\text{bore}} zdV \right). \tag{3}$$

The sum of the first two terms in parentheses can be simply written as a single integral over the entire volume of the cyllinder with radius R, so the Eq. (3) can be modified to:

$$\zeta = \frac{1}{V_0} \iiint_{\text{full cyl.}} zdV - \frac{1}{V_0} \iiint_{\text{bore}} zdV. \tag{4}$$

The first right-side term represents the position of the mass center of a "full" cyllinder. In the coordinate system we use this center is located exactly at the system origin, of course, so the first integral is zero. The remaining term can be rewritten as:

$$\zeta = -\frac{V_b}{V_0}\left(\frac{1}{V_b} \iiint_{\text{bore}} zdV \right). \tag{5}$$

where $V_b = \pi r^2 h$ is the bore volume. The expression inside the parenthese can be thought of as the z-coordinate of the mass center of an imaginary "cyllindrical rod" filling the bore. The mass center of such an object is located at the bore axis, of course — hence, the value of the expression within the parentheses is simply a. So, the z-coordinate of the mass center of the cyllinder with a bore is:

$$\zeta = -\frac{aV_b}{V_0} = -\frac{ar^2}{R^2 - r^2}. \tag{6}$$

The method used above represents the "most orthodox" calculation scheme. It is possible to find the mass center location in a much simpler way, by thinking of the cyllinder with a bore as of one with *two* symmetically located bores, of which one is empty, and one is filled with a cyllindrical "rod" exactly matching the bore shape. The mass of the two-bore cyllinder is $\rho h\pi R^2 - 2\rho h\pi r^2 = \rho h\pi(R^2 - 2r^2)$; due to the symmetric bore locations, the mass center is on the main cyllinder axis, i.e., its z coordinate is zero. The "rod" mass is $\rho h\pi r^2$, and, in the coordinate system we use, the z coordinate of its mass center is $-a$. Now one can find the position ζ of the common mass center in an elementary way writing the equation:

$$\rho h\pi(R^2 - 2r^2)\zeta = \rho h\pi r^2(\zeta - a), \tag{7}$$

which — not surprisingly — yields the same result for ζ as that in Eq. (6).
Second, we need to know the moment of inertia I of the cyllinder with a bore. Again, we start from the general expression for I of a body with uniform density ρ:

$$I = \rho \int\int\int \mathcal{R}^2(x, y, z)dV, \tag{8}$$

where $\mathcal{R}$ is the distance of the volume element dV from the axis of rotation. Since in the coordinate system we use the axis of rotation is the x axis, $\mathcal{R}^2 = y^2 + z^2$. Again, one can use the same trick as in calculations of the mass center position, and obtain I for the cyllinder with a bore by performing integration over the entire volume of the main cyllinder — which correspons to the moment of inertia of a cyllinder with the bore "filled" — and then taking away the contribution of the material filling the bore:

$$I = \rho \left(\iiint_{\text{full cyl.}} \mathcal{R}^2 dV - \iiint_{\text{bore}} \mathcal{R}^2 dV \right) = \rho \iiint_{\text{full cyl.}} \mathcal{R}^2 dV - \rho \iiint_{\text{bore}} \mathcal{R}^2 dV. \tag{9}$$

The first term in the righ-hand part is the moment of inertia $I_{\text{f.c.}}$ of a full cyllinder rotating about its *symmetry axis*. The integration can be readily performed by switching from Cartesian coordinates x, y, z to cyllindrical coordinates x, $\mathcal{R}$, ϕ:

$$I_{\text{f.c.}} = \rho \int_0^h \int_0^R \int_0^{2\pi} \mathcal{R}^2 dx(\mathcal{R}d\phi)d\mathcal{R} = \rho \times 2\pi \times h \times \frac{\mathcal{R}^4}{4} = \frac{\rho\pi\mathcal{R}^4 h}{2}. \tag{10}$$

In literature, this result is usually given in equivalent form, $\frac{1}{2}MR^2$, where M is the cyllinder mass.

The second term in Eq. (9) represents the moment of inertia of a cyllinder of radius r, rotating about an axis that is parallel to the cyllinder axis, and at a distance a from it. Here one can take adwantage of the well-known *parallel axis theorem*, from which it follows that the moment of inertia of such a cyllinder is $\frac{1}{2}Mr^2 + Ma^2 = \frac{1}{2}\rho h\pi r^2(r^2 + a^2)$. Inserting this result as well as Eq. (10) into Eq. (9), we obtain that the moment of inertia of the cyllinder with a bore is:

$$I_0 = \frac{1}{2}\rho h\pi(R^4 - r^4 - 2a^2r^2). \tag{11}$$

This is, of course, the moment of inertia *for rotation about the main cyllinder axis.* However, in this problem the cyllinder motion is more complicated: in addition to rotation, there is a translational component. In order to set up the Lagrangian, we will need the total kinetic energy. According to the well-known theorem, in the case of a combined rotational-translational motion, the total kinetic energy K is:

$$K = K_{\text{transl.}} + K_{\text{rot.}} = \frac{1}{2}MV_c^2 + \frac{1}{2}I^{(c)}\Omega^2, \tag{12}$$

where V_c is the linear velocity of the *mass center*, and $I^{(c)}$ is the moment of inertia for rotation about an axis going through the *mass center*, and Ω is the angular velocity. To obtain $I^{(c)}$ for the cyllinder in question, one can again take advantage of the parallel axis theorem which establishes the relation between I_0 and $I^{(c)}$:

$$I_0 = I^{(c)} + \mathcal{M}\zeta^2, \tag{13}$$

where:

$$\mathcal{M} = \rho h\pi(R^2 - r^2) \tag{14}$$

is the cyllinder mass. Accordingly,

$$I^{(c)} = I_0 - \mathcal{M}\zeta^2. \tag{15}$$

THE EQUATIONS OF MOTION

A. Rolling without slipping. In such a case, the cyllinder's mass center moves along a trajectory described by the following equations:

$$y_c = R\theta + \zeta \sin\theta \tag{16}$$
$$z_c = R + \zeta \cos\theta, \tag{17}$$

Here θ is the angle of rotation, with $\theta = 0$ corresponding to the equilibrium position. The cyllinder has one degree of freedom, and one can use the θ angle as the genralized coordinate.

The kinetic energy, which is the sum of the translational and the rotational motion components, can be written as:

$$K = K_{\text{tr}} + K_{\text{rot}} = \frac{\mathcal{M}}{2}(\dot{y}_c^2 + \dot{z}_c^2) + \frac{I^{(c)}\dot{\theta}^2}{2} = \frac{[I^{(c)} + \mathcal{M}(R^2 + \zeta^2 + 2R\zeta\cos\theta)]\dot{\theta}^2}{2}$$
$$= \frac{1}{2}\left[I^{(c)} + \mathcal{M}(R+\zeta)^2 - 4\mathcal{M}R\zeta\sin^2\frac{\theta}{2}\right]\dot{\theta}^2, \tag{18}$$

The potential energy is:

$$V = \mathcal{M}gz_c = \mathcal{M}g\zeta\cos\theta + \text{const.} \tag{19}$$

For small motions about the equilibrium position we use the approximations: $\sin(\theta/2) \cong \theta/2$, and $\cos\theta \cong 1 - \theta^2/2$. Taking the Lagrangian $L = K - V$, and neglecting all terms of the second or higher order in θ, $\dot{\theta}$ and $\ddot{\theta}$, one obtains:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = [I^{(c)} + \mathcal{M}(R+\zeta)^2]\ddot{\theta}, \tag{20}$$

and

$$\frac{\partial L}{\partial\theta} = \mathcal{M}g\zeta\theta. \tag{21}$$

Thus, the equation of motion is:

$$\ddot{\theta} = \frac{\mathcal{M}g\zeta}{I^{(c)} + \mathcal{M}(R+\zeta)^2}\theta, \tag{22}$$

where the definitions of ζ and $\mathcal{M}$ are given by the Eq. (6) and Eq. (14), respectively. Since ζ is a negative number, this is the equation of harmonic oscillations with angular frequency:

$$\omega = \sqrt{\frac{\mathcal{M}g|\zeta|}{I^{(c)} + \mathcal{M}(R+\zeta)^2}}. \tag{23}$$

If we plug Eqs. (6), (11), (14) and (15) into this formula, we obtain an elegant form containing only R, r, a and g:

$$\omega = \sqrt{\frac{gar^2}{\frac{3}{2}R^4 - \frac{1}{2}r^4 - r^2(R+a)^2}}. \tag{24}$$

B. "Zero-friction surface": Now the cyllinder has *two* degrees of freedom; as the generalized coordinates we choose the y_c coordinate, and the θ angle; note that they are no longer "coupled", as in the previous case. But the dependences of the z_c coordinate and of the potential energy on θ are still properly described by the Eqs. (17) and (19), respectively.

Let's write the Lagrangian as:

$$L = K + V = \frac{\mathcal{M}}{2}\dot{y}_c^2 + \frac{\mathcal{M}}{2}\dot{z}_c^2 + \frac{I^{(c)}\dot{\theta}^2}{2} + \mathcal{M}g\zeta\cos\theta + \text{const.} \tag{25}$$

Note that y_c shows up only in the first right-side term, and only as the time derivative; all other terms do not depend on y_c. Consequently, the partial derivative $\partial L/\partial y_c$ is always zero. Thus, the Lagrange equation for y_c reduces to:

$$\ddot{y}_c = 0, \tag{26}$$

which means that the mass center does not move in the y direction, or moves with constant velocity — however, the position of the mass center *does not oscillate* in the horizontal direction (this is what one expects, of course — in the total absence of friction, there is no force acting in the horizontal direction).

In the Lagrange equation for θ all terms associated with the translational motion are of the second or higher order and thus are neglected in the small θ limit. The only relevant terms in the equation come from the terms representing the rotation and the potential energy in the Lagrangian. The final equation for $\ddot{\theta}$ is:

$$\ddot{\theta} = \frac{\mathcal{M}g\zeta}{I^{(c)}}\theta, \tag{27}$$

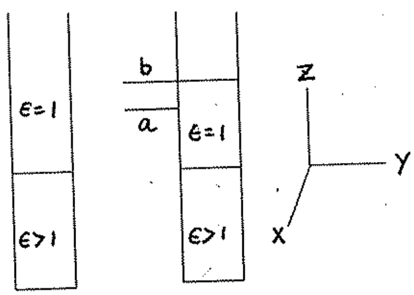
so θ oscillates with the frequency:

$$\omega = \sqrt{\frac{\mathcal{M}g|\zeta|}{I^{(c)}}}. \tag{28}$$

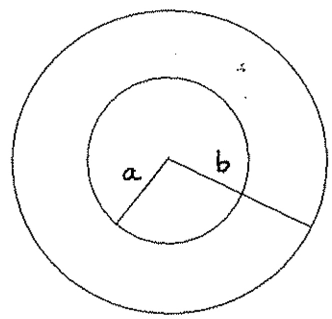
OSU Physics Comprehensive Exam No. 82 5 - 6 January 1998 **Problem 2**

A dielectric fluid ($\varepsilon > 1$) fills the space between two concentric, conducting cylinders for $z < 0$. The region $z > 0$ is open to the atmosphere ($\varepsilon = 1$). The cylinders are sufficiently long so that edge effects need not be considered. The inner cylinder is charged to the potential $\Phi = V > 0$, while for the outer cylinder, $\Phi = 0$.

side view



view from above



1. Find the potential and the electric field as functions of ρ, the radial distance from the center of the concentric system, for $z > 0$ and $z < 0$.

2. Show that the boundary conditions for E and D are satisfied at $z = 0$ for $a < \rho < b$.

3. Find the charge per unit length at the inner and outer surfaces of the dielectric.

4. The coaxial structure is placed in a large pool of the dielectric fluid such that it is perpendicular to the surface of the fluid. Find the height h to which the fluid will be raised within the space between the two cylinders. The density of the fluid is σ. Ignore surface tension.

Problem 2

1. $\Phi = A + B\ln\rho \qquad \Phi(a) = V = A + B\ln a \qquad \Phi(b) = 0 = B\ln b + A$

$A = -B\ln b \qquad V = -B\ln b + B\ln a \Rightarrow B = -V/\ln b/a$

$\Phi(\rho) = \frac{V}{\ln b/a}\ln b - \frac{V}{\ln b/a}\ln\rho = \frac{V}{\ln b/a}\ln b/\rho$

$\vec{E} = -\vec{\nabla}\Phi = -\hat{\rho}\frac{\partial}{\partial\rho}\Phi = \frac{V}{\ln b/a}\frac{1}{\rho}\hat{\rho}$ for $z>0$, $\frac{V}{\epsilon\ln b/a}\frac{1}{\rho}\hat{\rho}$ for $z<0$

2. normal $\vec{D}$ is continuous since $\hat{z}\cdot\vec{D}=0$

tangential $\vec{E}$ is continuous over a region encompassing the interface.

3. $\sigma = \vec{P}\cdot\hat{n} = \vec{P}\cdot(-\hat{\rho})$ where $\vec{P} = \chi\vec{E} = \frac{\epsilon-1}{4\pi}\vec{E}(\rho=a)$ at $\rho=a$

$\sigma = -\frac{(\epsilon-1)}{\epsilon 4\pi}\frac{V}{\ln b/a}\frac{1}{a} \Rightarrow \lambda = 2\pi a\sigma = -\frac{(\epsilon-1)}{\epsilon 2}\frac{V}{\ln b/a}$ is the charge/length

and at $\rho=b$ $\quad \sigma = \vec{P}\cdot\hat{\rho} = \frac{(\epsilon-1)}{\epsilon 4\pi}\vec{E}(b)\cdot\hat{\rho} = \frac{\epsilon-1}{\epsilon 4\pi}\frac{V}{\ln b/a}\frac{1}{b}$

$\lambda = 2\pi b\sigma = \frac{(\epsilon-1)}{\epsilon 2}\frac{V}{\ln b/a}$

4. $W = \frac{1}{8\pi}\int\vec{E}\cdot\vec{D}\,dv - \frac{1}{8\pi}\int|\vec{E}_0|^2\,dv = \frac{1}{8\pi}\left(\frac{1}{\epsilon}-1\right)\left(\frac{V}{\ln b/a}\right)^2\int_a^b\rho\,d\rho\int_0^{2\pi}d\varphi\int_0^h dz\frac{1}{\rho^2}$

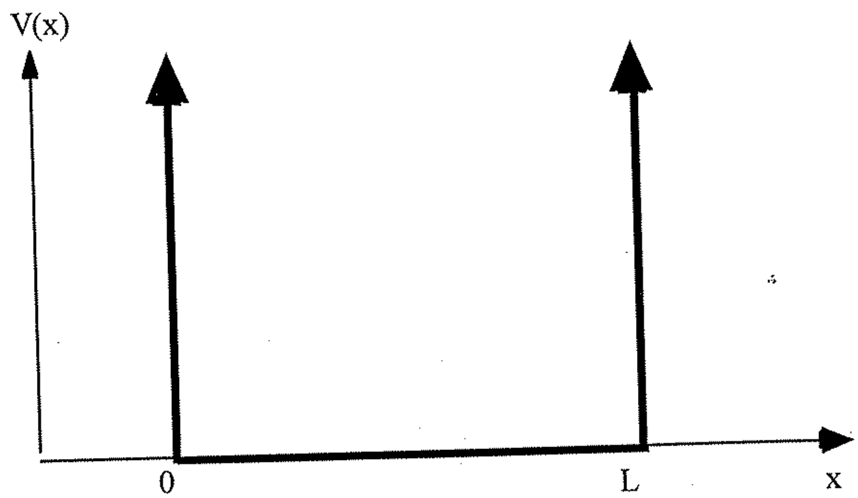
$= \frac{1}{8\pi}\left(\frac{1}{\epsilon}-1\right)\left(\frac{V}{\ln b/a}\right)^2\ln b/a\,2\pi h = \frac{h}{4}\left(\frac{1}{\epsilon}-1\right)\frac{V^2}{\ln b/a}$

Need to have $-W = g\int_0^h dz\int_a^b\rho\,d\rho\int_0^{2\pi}d\varphi\,\gamma z$, γ = density

$\frac{h}{4}\left(1-\frac{1}{\epsilon}\right)\frac{V^2}{\ln b/a} = \frac{h^2}{2}\gamma g\frac{(b^2-a^2)}{2}2\pi$

or $h = \frac{(1-\frac{1}{\epsilon})V^2}{\gamma g\ln b/a\,2\pi(b^2-a^2)}$

A particle of mass m is confined to the infinite square well potential shown below. The particle is in its lowest possible energy state. The system is thermally isolated.



a) What is the energy of this state?

b) The right-hand wall at $x = L$ is very slowly (*i.e.*, in a time long compared to the transit time across the well) moved to $x = 2L$. Calculate the expectation value of the energy in this new configuration and compare it to the classical result obtained by considering the force of the wall on the particle during the expansion.

c) Now assume that the wall is moved (again from $x = L$ to $x = 2L$) very quickly (*i.e.*, in a time short compared to the transit time across the well). Again calculate the expectation value of the energy in this new configuration. Then calculate the probability that the particle is in the lowest energy state of the new well.

a) Infinite square well,

solns: $\phi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x$ $\qquad k_n = \frac{n\pi}{L}$

$= \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$

$H = \frac{P^2}{2m}$ since $V = 0$ in well

$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ $\qquad H\psi = E\psi$

$\Rightarrow E_n = \frac{\hbar^2 k_n^2}{2m}$

$E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2}$

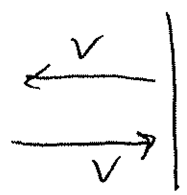
$\Rightarrow \boxed{E_1 = \frac{\hbar^2 \pi^2}{2mL^2}}$ lowest energy state

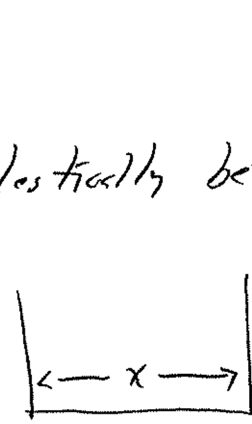
b) For an adiabatic change, the particle will remain in the $n=1$ state of the well, as the well changes. Thus the particle will end up in the lowest energy state of the new well.

$\Rightarrow \langle E \rangle = \langle \phi_1 | H' | \phi_1 \rangle = E_1' = \frac{\hbar^2 \pi^2}{2m(2L)^2}$ since new well is $2L$ wide

$\Rightarrow \boxed{\langle E \rangle = \frac{\hbar^2 \pi^2}{8mL^2}} = \frac{1}{4} E_1$

Classically, the particle bounces elastically between the walls.





$$\Rightarrow \text{Force of wall on ball} = \frac{\Delta p}{\Delta t} = \frac{-2mv}{\text{transit time}} = \frac{-2mv}{\frac{2x}{v}}$$

$$F = \frac{-mv^2}{x} \quad ; \quad x = \text{wall separation (variable)}$$

need to find $v(x)$; use $F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}$

$$F = mv\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-v}{x}$$

$$\Rightarrow v(x) = v_0\frac{L}{x} \qquad v_0 = v(L)$$

Use work-kinetic energy theorem to find new energy of ball

$$E_0 = \frac{1}{2}mv_0^2 \quad , \quad E' = \frac{1}{2}mv^2$$

$$E' - E_0 = \int_L^{2L} F\,dx = \int_L^{2L} -\frac{mv^2}{x}\,dx = -m\int_L^{2L} v_0^2\frac{L^2}{x^3}\,dx$$

$$E' - E_0 = -mv_0^2 L^2 \left[\frac{-1}{2x^2}\right]_L^{2L} = \frac{mv_0^2 L^2}{2}\left[\frac{1}{4L^2} - \frac{1}{L^2}\right]$$

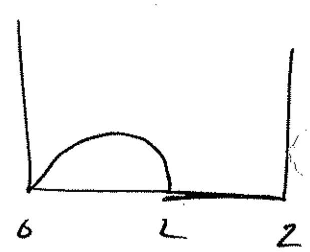
$$= -\frac{3}{8} mv_0^2$$

$$E' = E_0 - \frac{3}{8}mv_0^2 = \left(\frac{1}{2} - \frac{3}{8}\right) mv_0^2 = \frac{1}{8} mv_0^2$$

$$\boxed{E' = \frac{1}{4} E_0}$$ same as Q.M. result.

c) Move well quickly ⇒ spatial wave function unchanged.

$$\Rightarrow \psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} & 0 \le x < L \\ 0 & L \le x \le 2L \end{cases}$$



$$\langle E\rangle = \langle\psi | H' | \psi\rangle$$

$$= \int_0^{2L} \psi^*(x) \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) \psi(x)\, dx$$

$$= \int_0^L \phi_1(x) \cdot \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) \phi_1(x)\, dx$$ since $\psi(x)=0$ for $x>L$

but this is same as $\langle E\rangle$ in gnd state of initial well.

⇒ $\langle E\rangle$ unchanged

$$\boxed{\langle E\rangle = E_1 = \frac{\hbar^2\pi^2}{2mL^2}}$$

Probability of being in gnd state is

$$P_1 = |\langle \phi_1' | \psi \rangle|^2$$

$$= \left\{ \int_0^{2L} \sqrt{\frac{2}{2L}} \sin \frac{\pi x}{2L} \cdot \psi(x)\, dx \right\}^2$$

but $\psi(x) = 0$ for $x > L$

$$\Rightarrow P_1 = \left\{ \int_0^{L} \sqrt{\frac{2}{2L}} \sin \frac{\pi x}{2L} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}\, dx \right\}^2$$

$$= \left\{ \frac{\sqrt{2}}{L} \int_0^{L} \sin \frac{\pi x}{2L} \sin \frac{\pi x}{L}\, dx \right\}^2$$

let $y = \frac{\pi x}{2L}$

$$= \left\{ \frac{\sqrt{2}}{L} \frac{2L}{\pi} \int_0^{\frac{\pi}{2}} \sin y \sin 2y\, dy \right\}^2$$

$\sin 2y = 2 \sin y \cos y$

$$= \left\{ \frac{4\sqrt{2}}{\pi} \int_0^{\frac{\pi}{2}} \sin^2 y \cos y\, dy \right\}^2$$

$$= \left\{ \frac{4\sqrt{2}}{\pi} \left[\frac{\sin^3 y}{3} \right]_0^{\frac{\pi}{2}} \right\}^2$$

$$P_1 = \left\{ \frac{4\sqrt{2}}{3\pi} \right\}^2$$

$$\boxed{P_1 = \frac{32}{9\pi^2}}$$

Consider a **three-dimensional** harmonic oscillator of angular frequency, ω, whose allowed energies can be expressed in the form

$$E_j = \left(j + \frac{3}{2}\right)\hbar\omega \text{, where } j = 0, 1, 2, \ldots$$

(a) Show that the degeneracy, g_j, of the state with energy E_j can be written as $g_j = \frac{(j+2)!}{j!2!}$.

(b) Use the result obtained in part (a) to show that the canonical partition function of one such **three-dimensional** quantum oscillator at absolute temperature, T, is given by

$$Z_{qm} = \exp(-3x)[1 - \exp(-2x)]^{-3}$$

where $x = \frac{\hbar\omega}{2kT}$.

(c) Show that, for the case of a **one-dimensional** harmonic oscillator, the quantum and classical expressions for the mean internal energy, U, are the same at high temperatures ($T \to \infty$). To answer this question, you will need to make use of the classical **one-dimensional** partition function

$$Z_1 = \frac{1}{\hbar}\iint \exp\left(-\frac{p^2}{2mkT} - \frac{m\omega^2 q^2}{2mkT}\right) dp\,dq$$

THERMAL (G2)

(a) 

There are 3 ways to choose the first line. Then $(3-1+j)!$ ways to select the other lines and dots. However, there are $3!$ ways in which the lines can be permuted and $j!$ ways in which the dots can be permuted without producing a "different" arrangement.

$$\therefore \quad g_j = \frac{3(3-1+j)!}{3!\, j!} = \frac{(j+2)!}{2!\, j!}$$

(b) $$(Z)_{qm} = \sum_{j=0}^{\infty} g_j e^{-\beta E_j} \qquad \left(\beta = \frac{1}{kT}\right)$$

$$= e^{-3x} \sum_{j=0}^{\infty} \frac{(2+j)!}{j!\,2!} e^{-2jx},$$

where $x = \dfrac{\hbar\omega}{2kT}$.

Using the binomial theorem

$$\left(1-e^{-2x}\right)^{-3} = \sum_{j=0}^{\infty} \frac{(j+2)!}{2!\,j!}\, e^{-j2x} \,.$$ check it out!

Thus $$(Z)_{qm} = e^{-3x}\left(1-e^{-2x}\right)^{-3}.$$

(c) In 1D, $$(Z_1)_{qm} = e^{-x}\left(1-e^{-2x}\right)^{-1}.$$

$$(U_1)_{qm} = kT^2 \frac{\partial \ln Z_1}{\partial T} = -\frac{\hbar\omega}{2}\frac{\partial \ln Z_1}{\partial x}$$

$$\therefore (U_1)_{qm} = -\frac{\hbar\omega}{2}\left(-\frac{1}{x}\right) = \frac{\hbar\omega}{2x} = kT \;; \text{ for } \begin{matrix} x\to 0 \\ T\to\infty \end{matrix}$$

Classically, the integral for $(Z_1)_{class}$ becomes

$$(Z_1)_{class} = \frac{1}{h}\left(2\pi m kT \cdot \frac{2\pi kT}{m\omega^2}\right)^{1/2}$$

$$= \frac{kT}{h\nu}$$

$$(U_1)_{class} = kT^2 \frac{\partial}{\partial T}\ln Z = kT.$$

agreement with $(U_1)_{qm}$ at high T.

You could use equipartition to get
$$(U_1)_{class} = \tfrac{1}{2}kT + \tfrac{1}{2}kT = kT.$$

In a temperature range near absolute temperature, T, the tension force, F, of a stretched plastic rod is related to its length by the following expression (equation of state)

$$F = aT^2(L - L_0),$$

where a and L_0 are positive constants, L_0 being the unstretched length of the rod. When $L = L_0$, the heat capacity, C_L, of the rod (measured at constant length) is given by the relation $C_L = bT$, where b is a constant.

(a) When $T = T_0$ and $L = L_0$, the entropy is known to be $S(T_0, L_0)$. Find $S(T, L)$ at any other temperature, T, and length, L. Hint: It is most convenient to calculate first the change of entropy with temperature at the length, L_0, where the heat capacity is known.

(b) If one starts at $T = T_i$ and $L = L_i$ and stretches the thermally insulated rod quasi-statically until it attains the length, $L = L_f$, what is its final temperature, T_f?

THERMAL (UG)

(a) $T\,dS = dE - F\,dL$

via free energy Maxwell Relation $\left(\frac{\partial S}{\partial L}\right)_T = -\left(\frac{\partial F}{\partial T}\right)_L$

$$\therefore \left(\frac{\partial S}{\partial L}\right)_T = -2aT(L-L_0).$$

$$S(L_0,T) - S(L_0,T_0) = \int_{T_0}^{T} \frac{C_L}{T'}\,dT' = \int_{T_0}^{T} \frac{bT'}{T'}\,dT = b(T-T_0)$$

$$S(L,T) - S(L_0,T) = \int_{L_0}^{L} \left(\frac{\partial S}{\partial L}\right)_T dL = \int_{L_0}^{L} -2aT(L'-L_0)\,dL'$$

$$= -aT(L-L_0)^2$$

Thus, $S(L,T) = S(L_0,T_0) + b(T-T_0) - aT(L-L_0)^2.$

* While this has involved a line integral along a <u>specific</u> path in going from (L_0,T_0) to (L,T), the result is independent of the path taken since entropy is a state function.

(b) The process described is an adiabatic one, therefore $\Delta S = 0$. Thus $S(T_f, L_f) = S(T_i, L_i)$, and so

$$S(T_0, L_0) + b(T_f - T_0) - a T_f (L_f - L_0)^2$$

$$= S(T_0, L_0) + b(T_i - T_0) - a T_i (L_i - L_0)^2$$

which yields

$$T_f = T_i \left\{ \frac{b - a(L_i - L_0)^2}{b - a(L_f - L_0)^2} \right\},$$

Problem 6

Consider a particle bound in a double-well potential. We will treat this problem at its simplest level and neglect most details of the potential. We use $|\phi_A\rangle$ and $|\phi_B\rangle$ to denote two orthonormal states of this system, corresponding to localization in the two wells A and B. These two states constitute a basis (*i.e.*, we neglect all spatial variations of the states within the wells). When we neglect the possibility of the particle tunneling from one well to the other, then its energy is described by the Hamiltonian H_O whose eigenstates are $|\phi_A\rangle$ and $|\phi_B\rangle$ with the same eigenvalue E_O. The coupling between the states $|\phi_A\rangle$ and $|\phi_B\rangle$ is described by an additional Hamiltonian W defined by

$$W|\phi_A\rangle = a|\phi_B\rangle$$

where a is a positive real constant.

a) Write down the matrix representation of the full Hamiltonian $H = H_O + W$ in the basis defined by the states $|\phi_A\rangle$ and $|\phi_B\rangle$.

b) Calculate the energies and eigenstates of the full Hamiltonian $H = H_O + W$.

c) At time t=0 the particle is localized in well A. Find the wave function at subsequent times t (expressed in terms of the basis defined by the states $|\phi_A\rangle$ and $|\phi_B\rangle$).

d) Estimate the tunneling time for the particle in this system.

a) $H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$

$W|\phi_A\rangle = a|\phi_B\rangle$

$\Rightarrow \langle \phi_A|W|\phi_A\rangle = 0$

$\langle \phi_B|W|\phi_A\rangle = a$

$\Rightarrow W = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$

$\Rightarrow \boxed{H = \begin{pmatrix} E_0 & a \\ a & E_0 \end{pmatrix}}$

b) Find eigenvalues $\det|H - \lambda I| = 0$

$$\begin{vmatrix} E_0 - \lambda & a \\ a & E_0 - \lambda \end{vmatrix} = 0$$

$(E_0 - \lambda)^2 - a^2 = 0 \quad \Rightarrow E_0 - \lambda = \pm a$

$\lambda = E_0 \pm a$ eigenenergies

Find eigenstates:

$H|\psi_1\rangle = (E_0 + a)|\psi_1\rangle$ let $|\psi_1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\Rightarrow \begin{pmatrix} E_0 & a \\ a & E_0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (E_0 + a)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$\Rightarrow E_0 \alpha + a\beta = (E_0 + a)\alpha$$

$$\Rightarrow a\beta = a\alpha \quad \Rightarrow \alpha = \beta$$

$$\Rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}}|\phi_A\rangle + \frac{1}{\sqrt{2}}|\phi_B\rangle$$

$$H|\psi_2\rangle = (E_0 - a)|\psi_2\rangle$$

$$\Rightarrow \beta = -\alpha$$

$$\Rightarrow |\psi_2\rangle = \frac{1}{\sqrt{2}}|\phi_A\rangle - \frac{1}{\sqrt{2}}|\phi_B\rangle$$

Energies	Eigenstates
$E_0 + a$	$\frac{1}{\sqrt{2}}(\|\phi_A\rangle + \|\phi_B\rangle)$
$E_0 - a$	$\frac{1}{\sqrt{2}}(\|\phi_A\rangle - \|\phi_B\rangle)$

c) $$|\psi(t=0)\rangle = |\phi_A\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle$$

$$\Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\frac{(E_0+a)t}{\hbar}}|\psi_1\rangle + \frac{1}{\sqrt{2}} e^{-i\frac{(E_0-a)t}{\hbar}}|\psi_2\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-i\frac{(E_0+a)t}{\hbar}}\left[\frac{1}{\sqrt{2}}|\phi_A\rangle + \frac{1}{\sqrt{2}}|\phi_B\rangle\right]$$

$$+ \frac{1}{\sqrt{2}} e^{-i\frac{(E_0-a)t}{\hbar}}\left[\frac{1}{\sqrt{2}}|\phi_A\rangle - \frac{1}{\sqrt{2}}|\phi_B\rangle\right]$$

$$= \frac{1}{2} e^{-i\frac{E_0 t}{\hbar}}\left\{\left(e^{-i\frac{at}{\hbar}} + e^{i\frac{at}{\hbar}}\right)|\phi_A\rangle + \left(e^{-i\frac{at}{\hbar}} - e^{i\frac{at}{\hbar}}\right)|\phi_B\rangle\right\}$$

$$\boxed{|\psi(t)\rangle = e^{-i\frac{E_0 t}{\hbar}}\left\{\cos\left(\frac{at}{\hbar}\right)|\phi_A\rangle - i\sin\left(\frac{at}{\hbar}\right)|\phi_B\rangle\right\}}$$

d) To estimate tunneling time, find how long it takes particle to get from $A \to B$. From above

$|\psi(t=0)\rangle = |\phi_A\rangle$

i.e. $P_A = |\langle\phi_A|\psi\rangle|^2 = 1$ at $t=0$

find $P_B = |\langle\phi_B|\psi\rangle|^2$

$$= |\langle\phi_B|\psi(t)\rangle|^2$$

$$= \left|e^{-i\frac{E_0 t}{\hbar}}\left(-i\sin\left(\frac{at}{\hbar}\right)\right)\right|^2$$

$$P_B = \sin^2\left(\frac{at}{\hbar}\right)$$

$\Rightarrow P_B = 0$ at $t=0$ as expected

$P_B = 1$ at $\frac{at}{\hbar} = \frac{\pi}{2}$ (first time)

$$\Rightarrow \boxed{t = \frac{\pi\hbar}{2a}}$$

This is time to get to B well

$\Rightarrow$ good estimate of tunneling time.

A weight of mass M is attached to a vertical spring of force constant k and mass m. First, determine the frequency of vertical oscillations of the weight, neglecting the spring mass. Then, find how this frequency changes if the spring mass is small compared to the mass of the suspended object ($m \ll M$), but *not negligible.*

Hint: The condition $m \ll M$ is equivalent to the assumption that the spring *stretches proportionally* along its length. By "proportional stretch" we understand the following. Suppose that in the relaxed state the length of the spring is L_O, and in a stretched state it increases to $L_O + \Delta L$. Now consider a "mass element" dm located anywhere along the spring length. If in the relaxed state the distance between dm and the fixed spring end is z, then in the stretched state this distance increases to $z(1 + \Delta L / L_O)$.

Problem # — Solution:

Since the task is actually to find the finite spring mass *correction* to the oscillation frequency, in Step 1 the problem will be solved for a "perfectly massless" spring ($m = 0$), and the spring mass will be introduced in Step 2.

All forces in the system are strictly conservative. With the assumption that the spring is always proportionally stretched along its length, the system has only one degree of freedom — the motion of the suspended mass M as well as the motion of any mass element dm of the spring can be described in terms of a single coordinate. Thus, the Lagrange's method is an appropriate and convenient way of solving the problem.

Step 1. Let's denote the vertical axis as Z, with the positive direction upward. Let $z = 0$ mark the slack position of the spring.

If $m = 0$, the spring's only contribution to the Lagrangian is the elastic potential energy:

$$U_{\text{el.}} = \frac{1}{2}kz^2, \tag{1}$$

and the total potential energy is:

$$U_{\text{tot.}} = \frac{1}{2}kz^2 + Mgz. \tag{2}$$

The kinetic energy is:

$$K = \frac{1}{2}M\dot{z}^2 \tag{3}$$

So, one can write the Lagrangian:

$$L = K - U_{\text{tot}} = \frac{1}{2}M\dot{z}^2 - \frac{1}{2}kz^2 - Mgz. \tag{4}$$

Inserting this into the Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0, \tag{5}$$

after straightforward operations one obtains the equation of motion:

$$\ddot{z} = -\frac{k}{M}z - g \tag{6}$$

Putting $\ddot{z} = 0$, we obtain the z coordinate of the equilibrium point:

$$z_{\text{eq}} = -\frac{Mg}{k}. \tag{7}$$

As could be expected, this is the same result that one would obtain from equality of the gravitational and elastic forces at the equilibrium point.

The solution for the oscillatory motion has the form:

$$z = z_{\text{eq}} + A\sin(\omega t), \tag{8}$$

with the frequency ω equal:

$$\omega = \sqrt{\frac{k}{M}}. \tag{9}$$

Let's now consider how the Lagrangian changes if we take into account the finite spring mass. Note that in this new situation there is no need to change anything in the terms describing the kinetic energy and the potential energy of the mass M. Also, one can use the expression for the elastic potential energy in the previous form — however, now it should be clearly stated that

$z = 0$ marks the *zero stretch* position of the spring's lower end. Note that it's no longer the same as the "slack position", because even with the mass M removed, the spring will be somewhat stretched by its own weight. Therefore, in order to avoid any further confusion, let's define the $z = 0$ position as *the point located at the distance L_0 below the upper "fixed" spring end,* where L_0 is the length of the spring in a completely relaxed state (e.g., when it rests on a horizontal plane).

In addition to these three "old" terms, the Lagrangian has to include two "new" terms, describing:

(a) the change in the potential energy, associated with the finite spring mass — when the mass M moves, the mass center of the spring also changes its position;

(b) the kinetic energy of the spring.

Based on the underlying assumption that the spring stretches *proportionally* along its length, one can readily find the potential energy term. Proportional stretch means that the the spring's mass center is always at one-half of its length. So, if at a given instant the mass M is at z, the spring length is $L_0 - z$, and its mass center is at $z + \frac{1}{2}(L_0 - z) = \frac{1}{2}(L_0 + z)$. The corresponding potential energy term is:

$$U_{\text{spr. mass}} = \frac{1}{2}(L_0 + z)mg. \tag{10}$$

It's slightly more complicated to evaluate the kinetic energy term. For a moment, let's switch to a different coordinate system z', with $z' = 0$ at the point of spring suspension, and the positive direction *downward* (so, e.g., in the unstretched state, the z' coordinate of the lower end of the spring is $z' = L_0$). Now consider a "mass element" dm of the spring located between the points $z'_{\text{n.s.}}$ and $z'_{\text{n.s.}} + dz'_{\text{n.s.}}$ in the unstretched spring (where the subscript 'n.s.' stands for 'no stretch'). Since the mass per unit length is m/L_0, we can write:

$$dm = \frac{m}{L_0}dz'_{\text{n.s.}} \tag{11}$$

If the spring is stretched so that its length becomes $L_0 + \Delta L$, the mass element considered will be relocated — according to the "proportional stretch" formula given in the *Hint* — to a new position z'_{s} (with the subscript 's' meaning 'stretched'):

$$z'_{\text{s}} = z'_{\text{n.s.}}(1 + \frac{\Delta L}{L_0}). \tag{12}$$

From the above it follows that if the lower end moves and ΔL changes in time, the speed of the mass element dm is:

$$\dot{z}'_{\text{s}} = z'_{\text{n.s.}}\frac{\dot{\Delta L}}{L_0}. \tag{13}$$

From the Eqs. (11) and (13) we obtain the corresponding kinetic energy:

$$dK = \frac{(\dot{z}'_{\text{s}})^2 dm}{2} = \frac{m\dot{\Delta L}^2}{2L_0^3}(z'_{\text{n.s.}})^2 dz'_{\text{n.s.}} \tag{14}$$

By integrating Eq. (14) over the entire spring, i.e., for the $z'_{\text{n.s.}}$ coordinate changing from 0 to L_0, we obtain the total kinetic energy contribution:

$$K_{\text{spr. mass}} = \int_{\text{all spring}} dK = \frac{m\dot{\Delta L}^2}{2L_0^3}\int_0^{L_0}(z'_{\text{n.s.}})^2 dz'_{\text{n.s.}} = \frac{m\dot{\Delta L}^2}{2L_0^3}\frac{L_0^3}{3} = \frac{m}{6}\dot{\Delta L}^2. \tag{15}$$

The time derivative $\dot{\delta L}$ is the same as $\dot{z}'_{\text{s}}$, of course; and if we return from the Z' system to the previously used Z system, we have $\dot{z}' = -\dot{z}$ (actually, the sign does not matter because of the square). So, we can write the spring kinetic energy in terms of z, i.e., the position of the mass M:

$$K_{\text{spr. mass}} = \frac{m}{6}\dot{z}^2. \tag{16}$$

We can now set up the Lagrangian by combining Eqs. (4), (10) and (16):

$$L = \frac{1}{2}\left(M + \frac{m}{3}\right)\dot{z}^2 - \frac{1}{2}kz^2 - \left(M + \frac{m}{2}\right)gz - \frac{1}{2}L_0 mg, \tag{17}$$

which leads to the following equation of motion:

$$\ddot{z} = -\frac{kz}{M + \frac{m}{3}} - \frac{M + \frac{m}{2}}{M + \frac{m}{3}}g. \tag{18}$$

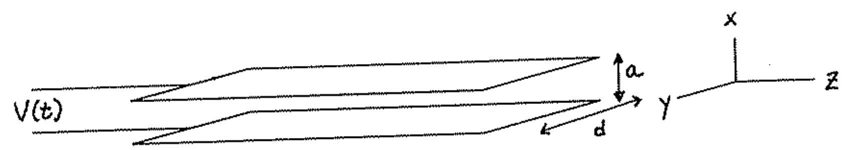
Note that this equation has the same form as Eq. (6) in the case of $m = 0$, only the coefficients are different. In the same way as we did before, we obtain the new equilibrium position:

$$z_{\text{eq}} = \frac{g}{k}\left(M + \frac{m}{2}\right), \tag{19}$$

and the new oscillation frequency:

$$\omega = \sqrt{\frac{k}{M + \frac{m}{3}}}. \tag{20}$$

Propagation of high frequency signals over a circuit board requires consideration of the impedance of a transmission line. In the structure pictured below, a very long transmission line of width d consists of two perfectly conducting plates separated by distance $a = 1\ mm$. The space between the plates is filled with a uniform dielectric (ε,μ). Edge effects are to be ignored. A time-dependent signal $V(t) = V_0 e^{i\omega t}$ at 1 GHz is applied at the left edge and propagates a great distance to the right.



a. What are the solutions for **E** and **B** as functions of x and z?

b. What is the current $I(z)$ flowing in one of the plates?

c. Find the impedance, the capacitance per unit length in the z direction and the inductance per unit length for this transmission line.

d. What is the power passing through this structure?

Problem 8

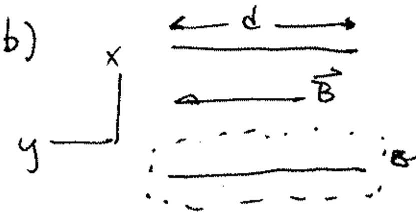
a) Since $\lambda \gg a$, TEM mode $\vec{E} = E\hat{x}$ $\vec{B} = B\hat{y}$ with time dependence $e^{i\omega t}$

Satisfies: tangential $E = 0$ at metal surface; normal E is discontinuous

tangential B is discontinuous, normal $B = 0$:

$E = \mathcal{E} e^{i\beta z}$, $\mathcal{E} = V_0/a$, $B = \mathcal{B} e^{i\beta z}$, $\sqrt{\epsilon\mu}\,\mathcal{E} = \mathcal{B}$

b)



$\int \vec{\nabla} \times \vec{H} \cdot d\vec{a} = \frac{4\pi I}{c} \Rightarrow \int \vec{H} \cdot d\vec{\ell} = \frac{4\pi I}{c}$

$Hd = 4\pi I/c$

or $I(z) = \frac{c\, d\, B(z)}{4\pi\mu}$

c) $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega \vec{B}}{c} \Rightarrow \frac{i\omega B}{c} = \frac{\partial E_x}{\partial z} = \frac{1}{a}\frac{\partial V(z)}{\partial z}$

So $\frac{i\omega B}{c} = \frac{i\omega}{c}\frac{4\pi\mu}{cd} I(z) = \frac{1}{a}\frac{\partial V(z)}{\partial z} = \frac{1}{a} i\omega L I \Rightarrow L = \frac{4\pi\mu a}{c^2 d}$

(per unit length in z direction)

$C = \epsilon d / 4\pi a$

In the medium $\vec{\nabla} \times \vec{H} = \frac{1}{c}\epsilon\frac{\partial \vec{E}}{\partial t} = \frac{i\omega\epsilon\vec{E}}{c}$ or $\frac{\partial H}{\partial z} = \frac{i\omega\epsilon}{c}E$

or $\frac{\partial}{\partial z} I(z) = i\omega C V(z)$ but $\frac{\partial}{\partial z} I(z) = \frac{c^2 d}{i\omega a 4\pi\mu}\frac{\partial^2 V}{\partial z^2}$

So $\frac{\partial^2 V}{\partial z^2} = \frac{i\omega a 4\pi\mu}{c^2 d} i\omega C\, V(z) = -\omega^2 \frac{\epsilon\mu}{c^2} V(z) \Rightarrow V(z) = e^{i\beta z} V_0$, $\beta = \frac{\omega\sqrt{\epsilon\mu}}{c}$

d) $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H} = \frac{c}{4\pi}\left(\frac{V_0}{a}\right)^2\sqrt{\frac{\epsilon}{\mu}}$