

DEPARTMENT COMPREHENSIVE EXAMINATION #80

March 31 - April 1, 1997

Comprehensive Examination for Spring 1997

PART I

General Instructions

This Comprehensive Examination for Spring 1997 (#80) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 31, and lasts three hours. The second part (Problems 3-4) will be handed out on the same day, at 1:30 pm, and also lasts three hours. The third and fourth parts (Problems 5-6 and Problems 7-8) will be administered in the same way on Tuesday, April 1, 1997.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

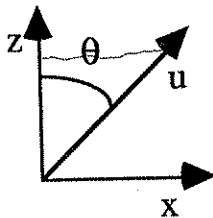
If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulae and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one page from your solutions. "Scratch" work will not be graded.

A beam of identical neutral particles with spin \vec{S} (but no other angular momentum) travels along the y -axis. The beam passes through a Stern-Gerlach spin analyzing magnet that is aligned along the z -axis, and is then detected downstream. The particles have a gyromagnetic ratio g .

- a) Draw a diagram of the experiment and describe how the Stern-Gerlach spin analyzer works.
- b) What important properties concerning the spin characteristics of the particles can be learned from the results of this experiment?
- c) Consider the case where the particles have spin $1/2$ and are prepared before reaching the Stern-Gerlach analyzer in the spin up state $|+\rangle_{\hat{u}}$ along the direction \hat{u} (in the $x-z$ plane) as shown below. What are the possible results of the experiment and what are the probabilities of each of these results? You may want to know that

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$



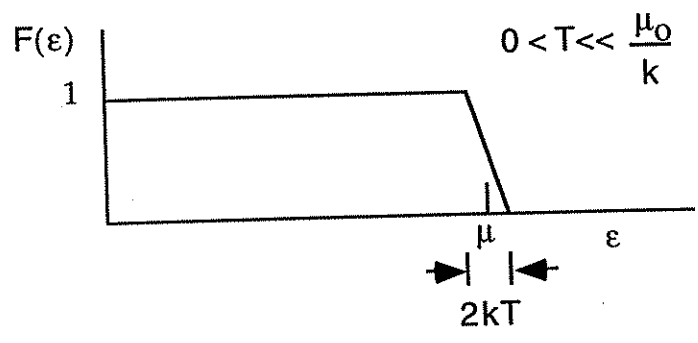
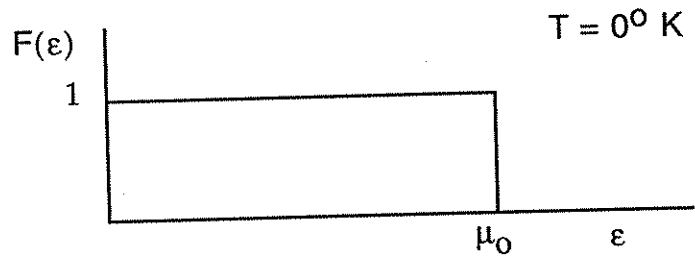
Consider a free electron gas (no interactions) to be governed by a "Fiddled Fermi function" having the form illustrated by the following figures. The upper figure is for absolute zero ($T=0\text{K}$), while the lower figure is for a finite temperature, but one for which $T \ll \frac{\mu_0}{k}$, where μ_0 is the Fermi energy (chemical potential) at absolute zero. Show that the Fermi energy (chemical potential) μ has the following dependence on temperature

$$\mu = \mu_0 - \alpha T^2$$

where α is a temperature independent constant.

Hint: You might want to make use of integration by parts

$$\int u dv = uv - \int v du$$



Two identical long smooth cylinders are positioned horizontally at the same height, with their axes parallel. The distance between the two axes is $2D$. The cylinders rotate with the same angular speed ($\pm\omega$) in opposite directions, as shown in Fig. 1. A smooth board of uniform density, mass m and length $2L$ ($D < L \leq 2D$) is put on top of the two cylinders, as also shown in Fig. 1. Initially, the board is held motionless in such a position that the distance between its gravity center C and the symmetry plane of the cylinder system is Δx_0 . At $t = 0$ the board is released.

Find the board motion for $t > 0$, knowing that the coefficient of kinetic friction between the board surface and the cylinder surfaces is μ . Assume that the ω value is sufficiently large to assure that the friction between the board and both cylinder surfaces is always kinetic (i.e., $|\omega R| > |V_{\max}|$, where R is the cylinder radius, and V_{\max} is the maximum speed value that the board may attain).

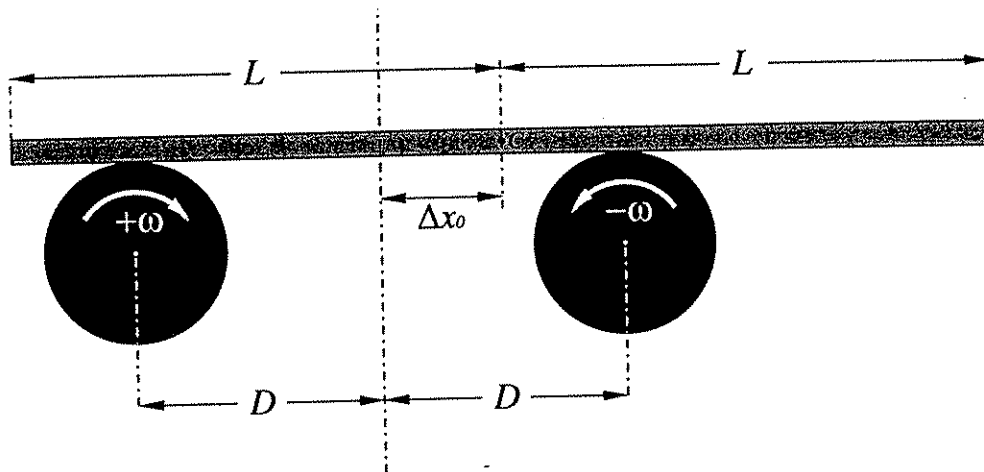


Figure 1. A board on top of two rotating cylinders — the position at $t = 0$.

The strange device shown in the figure below is a rotating spherical capacitor! The inner sphere of radius a is a permanent magnet having magnetization $\vec{M} = \hat{z}M$. It produces a field outside the sphere identical with that which would be produced by a point magnetic dipole,

$$\vec{B} = \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{r^3}$$

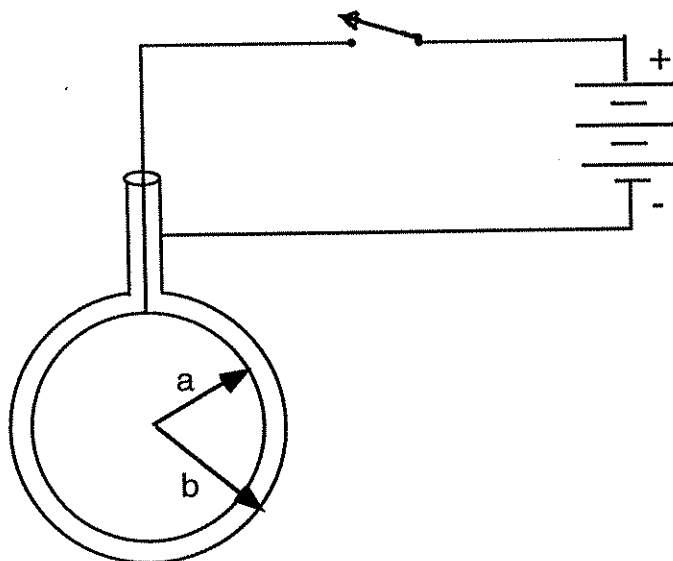
where \vec{m} is the total dipole moment,

$$m = M \left(\frac{4\pi a^3}{3} \right)$$

(This is a standard result. You don't have to derive it.) The inner sphere is made of a ferrite (a nonconducting material) but its surface is covered with a conducting film, connected at the point $\theta = 0$ by a thin wire along the z -axis to a battery. A charge $+Q$ coming in along the wire will flow only on the surface of the sphere and distribute itself uniformly there.

The outer shell of radius b is made of copper. It is connected to the negative terminal of the battery by a cylindrical lead, coaxial with the wire on the z -axis to the inner sphere. A charge $-Q$ coming in to the shell will also distribute itself uniformly on the inner surface of the outer shell.

If the two spheres are free to rotate without friction they will begin to rotate relative to one another as they are charged. Explain why this happens and calculate the total angular momentum that is transferred from one sphere to the other.



Consider a system of angular momentum \vec{J} , with $J = 1$. The Hamiltonian of the system is

$$H_0 = aJ_z + \frac{b}{\hbar} J_z^2$$

where a and b are positive, real constants with dimensions of an angular frequency.

- (1) What are the energy levels E_i and the eigenvectors $|i\rangle$ of the system? For what value of a/b is there degeneracy?
- (2) Now apply a perturbation described by $W = \omega J_x$, where $\omega \ll a$, and $\omega \ll b$.

Note that

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the corrections to the energy levels caused by this perturbation. Find the corrections to all levels to the same order in ω , but if all corrections are zero, then go to the next order in ω . Do this for the two cases:

- (i) $b = a$
- (ii) $b = 2a$.

Consider a dielectric solid consisting of N atoms, each having an electric dipole moment, p . The solid's heat capacity at constant electric field obeys a Debye law, given by

$$C_E = AT^3$$

and its electric susceptibility can be written as

$$\chi = \frac{Np^2}{\sqrt{3kT}}$$

there is a $\sqrt{}$ here - see solution

The equation of state is given by

$$P = \chi E$$

where P is the polarization of the solid.

Reducing an initially applied electric field, E_i , adiabatically to zero results in a reduction of the temperature of the solid, ΔT . Under the condition that $\Delta T \ll T_i$, where T_i is the initial temperature of the solid, show that

$$\Delta T \approx \frac{Np^2 E_i^2}{6AkT^4}$$

Consider a "cycloidal pendulum" depicted in Fig. 1. The apparatus consists of two cycloidal surfaces attached from below to a rigid horizontal plate. Each surface is described by the following set of parametric equations: equation set:

$$x = R(\alpha - \sin \alpha)$$

$$y = -R(1 - \cos \alpha)$$

A dimensionless mass m is attached to one end of a infinitesimally thin, massless, inextensible and ideally flexible string of the length $4R$; the other end is fastened to a point where the two surfaces meet.

Analyze the oscillatory motion of the mass in the xy plane — in particular, find out how the oscillation period depends on the amplitude of the oscillations.

Hints: 1. First prove that when the mass swings and the string wraps and unwraps itself on the cycloidal surfaces, the mass moves along a curve which is a cycloid corresponding to the same R value as that in the equations describing the two surfaces. To show that, you'll need to use the general formula for calculating the length of a curve defined by parametric equations $x = x(s)$ and $y = y(s)$:

$$L_{AB} = \int_{s_A}^{s_B} \sqrt{(\partial x / \partial s)^2 + (\partial y / \partial s)^2} ds,$$

where L_{AB} is the length of the curve portion between points A and B , s_A and s_B are the parameter values corresponding to these points.

The following trigonometric identities may also be helpful:

$$1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2}; \quad 1 - \cos \alpha = 2 \sin^2 \frac{\alpha}{2}; \quad \sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2};$$

$$\sin \alpha = \pm \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}; \quad \cos \alpha = \pm \frac{1}{\sqrt{1 + \tan^2 \alpha}}.$$

2. If you decide to use Lagrangian approach, take the path along the mass trajectory as the generalized coordinate.

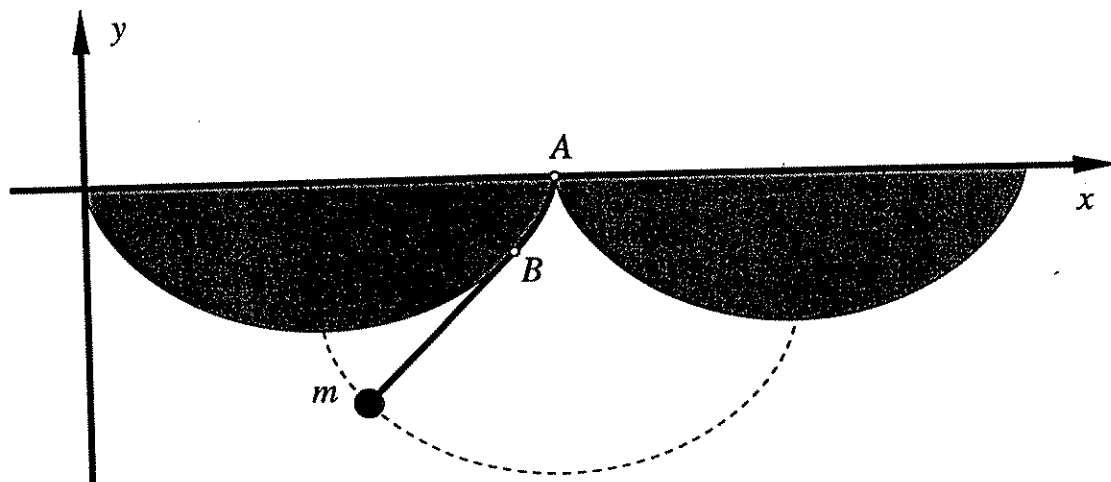
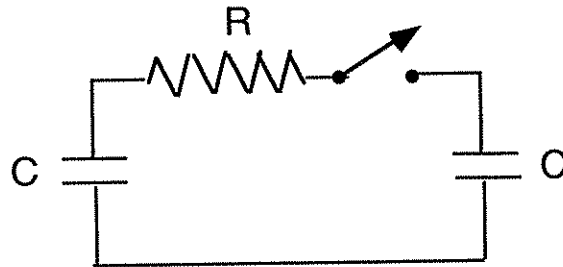
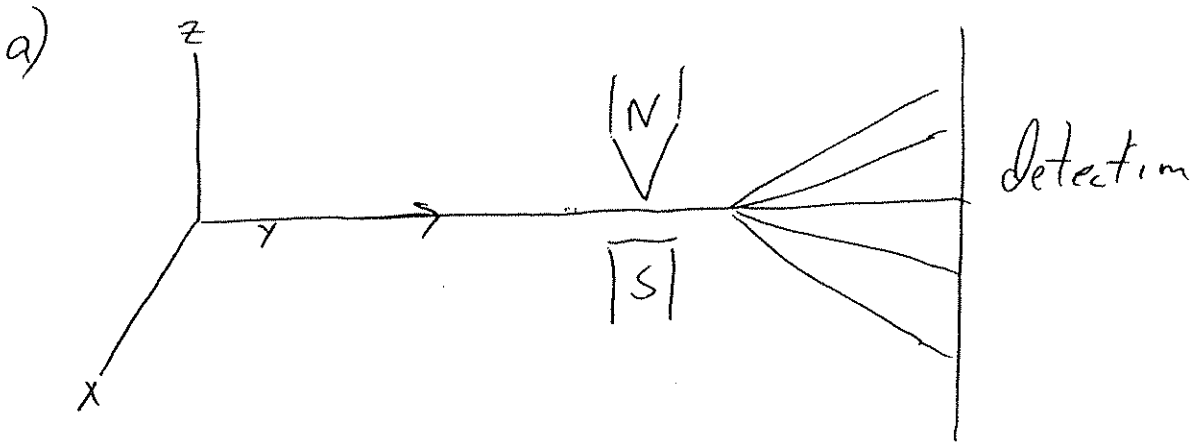


Figure 1. A cycloidal pendulum.

Consider the following simple circuit. Two identical capacitors are connected in series with a resistor. Initially, one of the capacitors is charged with charge Q , and the other is uncharged. At $t = 0$, the switch is closed and current begins to flow.



- (a) Describe what happens as $t \rightarrow \infty$. Calculate the energy stored in the system before the switch is thrown, and in the limit $t \rightarrow \infty$. What has happened to the missing energy?
- (b) Derive a differential equation for the current as a function of time. Solve it, and show that the energy turned to heat in the resistor is equal to the energy difference calculated in part a).
- (c) Now consider the limit $R \rightarrow \infty$. What becomes of the energy in this case?



A particle with spin \vec{S} and gyromagnetic ratio γ has a magnetic moment ~~the~~ $\vec{\mu}$

$$\vec{\mu} = \gamma \vec{S}$$

In a magnetic field, this particle has potential energy due to the interaction of the magnetic dipole moment with the external field:

$$W = -\vec{\mu} \cdot \vec{B}$$

If the field varies in space, then the particle will experience a force

$$\begin{aligned} \vec{F} &= -\nabla W \\ &= \nabla (\vec{\mu} \cdot \vec{B}) \\ &= \vec{\mu} \cdot \nabla \vec{B} \end{aligned}$$

since $\vec{\mu}$ indep of space.

A Stern-Gerlach magnet is designed with a gradient $\frac{\partial B}{\partial z}$ so that

$$\vec{F} = \mu_z \frac{\partial B}{\partial z} \hat{z}$$

$$\Rightarrow \vec{F} = \gamma S_z \frac{\partial B}{\partial z} \hat{z}$$

So the force on each particle is proportional to the projection S_z of the spin along the z -axis.

Thus particles are deflected as shown above according to their spin projection S_z .

b) We can learn 3 things:

- 1) That the spin is quantized. We see only discrete spots on the detection plane.
- 2) The value of the spin S is related to the # of observed spots, $2S+1$, since there are $2S+1$ possible values of m_s (i.e. $m_s = -S, \dots, 0, \dots, +S$)

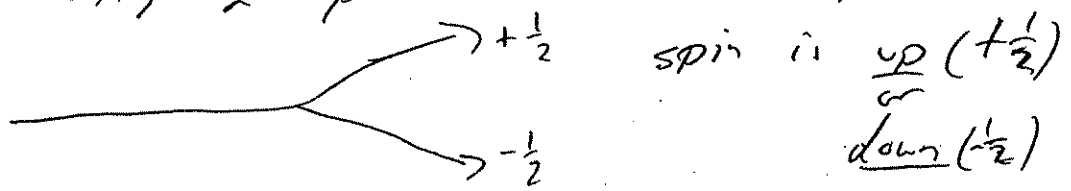
Thus from the # of spots we can learn S .

- 3) The intensity of each spot is proportional to the probability of measuring the value m_s appropriate to that spot.

Thus we learn the distribution of spin states m_s in the beam.

$$c) S = \frac{1}{2} \Rightarrow m_s = \pm \frac{1}{2}$$

\Rightarrow only 2 possible results at exp'l.



$$\Rightarrow a = \cos \frac{\theta}{2}, \quad b = \sin \frac{\theta}{2}$$

$$\Rightarrow |+\rangle_u = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle$$

$$\Rightarrow P_+ = |\langle + | \langle + \rangle_u|^2 = \cos^2 \frac{\theta}{2}$$

$$P_- = |\langle - | \langle + \rangle_u|^2 = \sin^2 \frac{\theta}{2}$$

$$\Rightarrow \begin{array}{l} \text{Prob of } m_s = +\frac{1}{2} \rightarrow \cos^2 \frac{\theta}{2} \\ \text{Prob of } m_s = -\frac{1}{2} \rightarrow \sin^2 \frac{\theta}{2} \end{array}$$

Before magnet, spin state is $|+\rangle_u$

$$S_u = \vec{S} \cdot \vec{u} = S_z \cos \theta + S_x \sin \theta$$

$$S_u |+\rangle_u = +\frac{\hbar}{2} |+\rangle_u$$

We want to know $|+\rangle_u$ in terms of $|+\rangle, |-\rangle$ referenced to z -axis

$$\Rightarrow \text{let } |+\rangle_u = a|+\rangle + b|-\rangle \quad \text{with } a^2 + b^2 = 1$$

$$S_u = \cos \theta \cdot \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sin \theta \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_u = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$|+\rangle_u = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow S_u |+\rangle_u = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{cases} a \cos \theta + b \sin \theta = a \\ a \sin \theta - b \cos \theta = b \end{cases}$$

$$\rightarrow b \sin \theta = a(1 - \cos \theta)$$

$$b^2 \sin^2 \theta = a^2 (1 - 2 \cos \theta + \cos^2 \theta)$$

$$\sin^2 \theta = a^2 2(1 - \cos \theta)$$

$$\Rightarrow a^2 = \frac{\sin^2 \theta}{2(1 - \cos \theta)}$$

$$\text{use } b^2 = 1 - a^2$$

$$\text{use } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$a^2 = \frac{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2}} = \cos^2 \frac{\theta}{2}$$

$$D(\vec{k}) d\vec{k} = 2 \frac{V}{(2\pi)^3}$$

$$D(\vec{k}) d\vec{k} \Rightarrow D(k) 4\pi k^2 dk = \frac{V}{\pi^2} k^2 dk$$

$$\text{But } D(\vec{k}) d\vec{k} = D(\epsilon) d\epsilon, \therefore D(\epsilon) = \frac{V k^2}{\pi^2} \frac{dk}{d\epsilon}$$

$$\text{and since } \epsilon = \frac{\hbar^2 k^2}{2m}, \therefore D(\epsilon) = \frac{\sqrt{2} V m^{3/2}}{\pi^2 \hbar^3} \epsilon^{1/2}$$

At $T=0$,

$$N = \int_0^{\mu_0} F(\epsilon) D(\epsilon) d\epsilon = \int_0^{\mu_0} D(\epsilon) d\epsilon = \frac{\sqrt{2} V m^{3/2}}{\pi^2 \hbar^3} \frac{2}{3} \mu_0^{3/2}$$

$$\text{and } \therefore \mu_0 = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3}, \text{ the usual expression}$$

At $T > 0$ K:

$$N = \int_0^{\infty} F(\epsilon) D(\epsilon) d\epsilon = - \int_0^{\infty} F'(\epsilon) \varphi(\epsilon) d\epsilon \quad (\text{integ. by parts})$$

$$\text{where } \varphi(\epsilon) = \int_0^{\epsilon} D(\epsilon') d\epsilon' = \frac{2}{3} \frac{V \sqrt{2} m^{3/2}}{\pi^2 \hbar^3} \epsilon^{3/2},$$

$$\text{and where } F'(\epsilon) = \begin{cases} 0, & 0 \leq \epsilon < \mu - kT \\ -\frac{1}{2kT}, & \mu - kT \leq \epsilon \leq \mu + kT \\ 0, & \epsilon > \mu + kT \end{cases}$$

$$\therefore N = \int_{\mu - kT}^{\mu + kT} \frac{1}{2kT} \frac{2}{3} \frac{V \sqrt{2} m^{3/2}}{\pi^2 \hbar^3} \epsilon^{3/2} d\epsilon$$

$$\text{so } \frac{N}{V} = \frac{2\sqrt{2} m^{3/2}}{15 kT \pi^2 \hbar^3} \left[(\mu + kT)^{5/2} - (\mu - kT)^{5/2} \right]$$

$$\approx \mu^{5/2} \left[\frac{5kT}{\mu} + \frac{15}{24} \left(\frac{kT}{\mu} \right)^3 \right]$$

Finally, simplifying yields

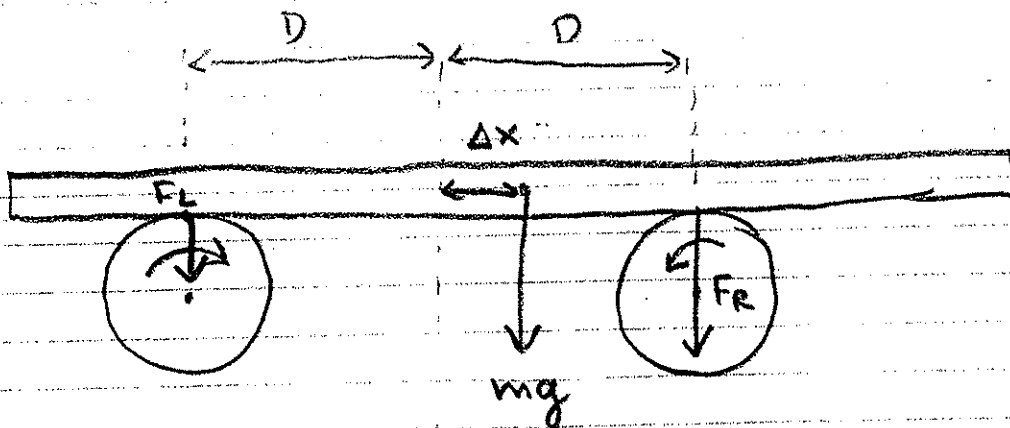
$$\mu_0^{3/2} = \mu^{3/2} + \frac{k^2 T^2}{8\mu^{1/2}}$$

^ 'old' or 'new' value, it doesn't matter.

$$\therefore \mu = \mu_0 - \frac{k^2 T^2}{12\mu_0} = \mu_0 - \alpha T^2.$$

Classical Mechanics

Problem 1 - solution:



The forces F_L and F_R must satisfy:

$$F_L + F_R = mg$$

$$F_L(D + \Delta x) = F_R(D - \Delta x)$$

So:

$$F_L(D + \Delta x) = (mg - F_L)(D - \Delta x)$$

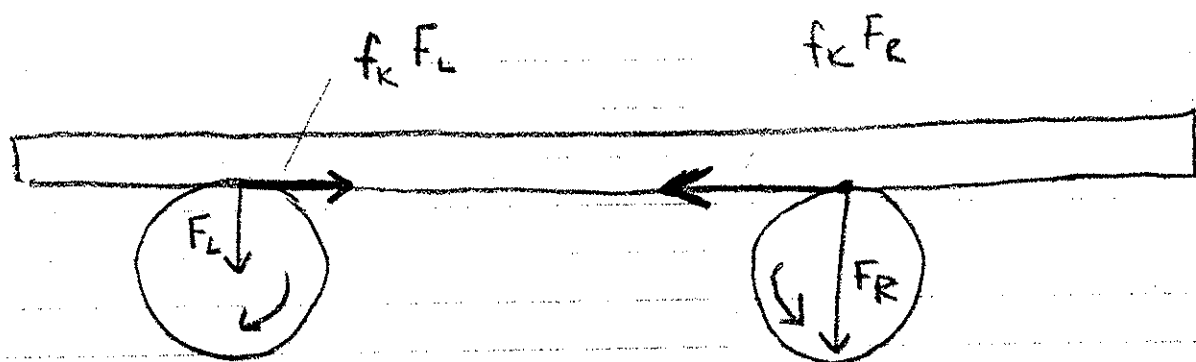
$$F_L D + \cancel{F_L \Delta x} = mg D - F_L D - mg \Delta x + \cancel{F_L \Delta x}$$

$$2F_L D = mg(D - \Delta x)$$

$$F_L = \frac{mg}{2} - mg \frac{\Delta x}{2D}$$

and:

$$F_R = \frac{mg}{2} + mg \frac{\Delta x}{2D}$$



The vector sum of the friction forces are:

$$\begin{aligned} \Delta F_f &= f_k F_L - f_k F_R = \\ &= f_k \left(\frac{mg}{2} - mg \frac{\Delta x}{2D} \right) - f_k \left(\frac{mg}{2} + mg \frac{\Delta x}{2D} \right) = \\ &= -2f_k mg \frac{\Delta x}{2D} = -\frac{f_k mg}{D} \Delta x \end{aligned}$$

The equation of motion for the board is:

$$m a_x = m (\ddot{\Delta x}) = \Delta F_f = -\frac{f_k mg}{D} \Delta x$$

i.e.,

$$(\ddot{\Delta x}) = -\frac{f_k g}{D} \Delta x$$

Which means that the board motion along the x direction is a harmonic oscillation with the frequency:

$$\omega = \sqrt{\frac{f_k g}{D}}$$

Taking into account the initial conditions:

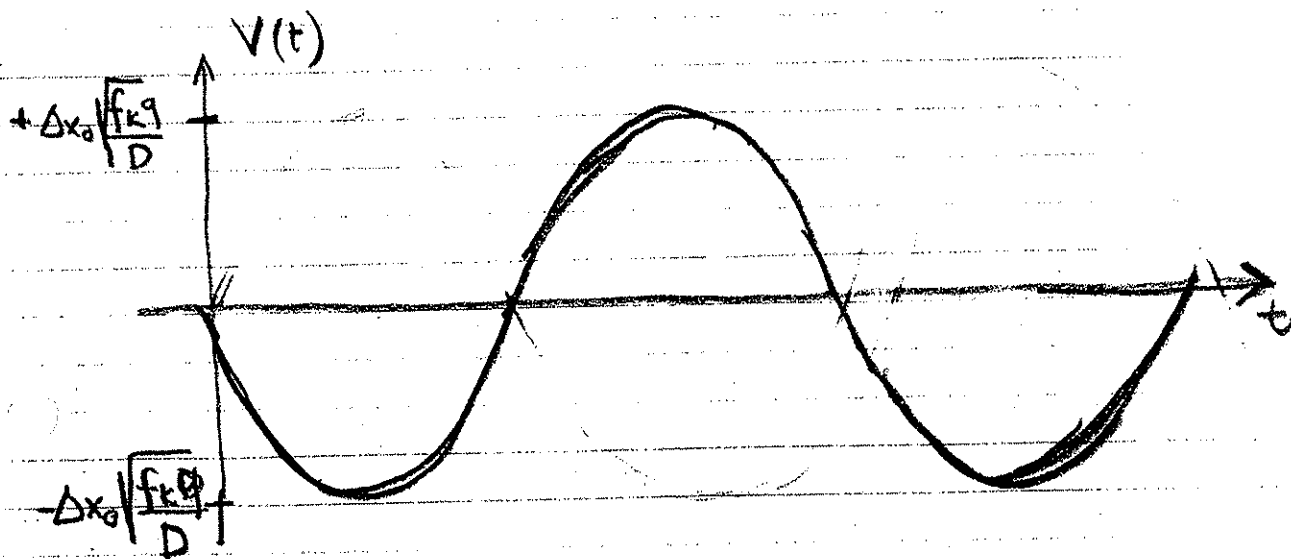
$\Delta x(t=0) = \Delta x_0$, we can write the position of the board's gravity center as a function of time:

$$\Delta x = \Delta x_0 \cos\left(\sqrt{\frac{fk_9}{D}} t\right) \quad (1)$$

However, the above solution is true only if the friction always remains kinetic — in other words, if the linear speed of the cylinder surfaces at any time t is larger than the board speed.

The board's velocity is:

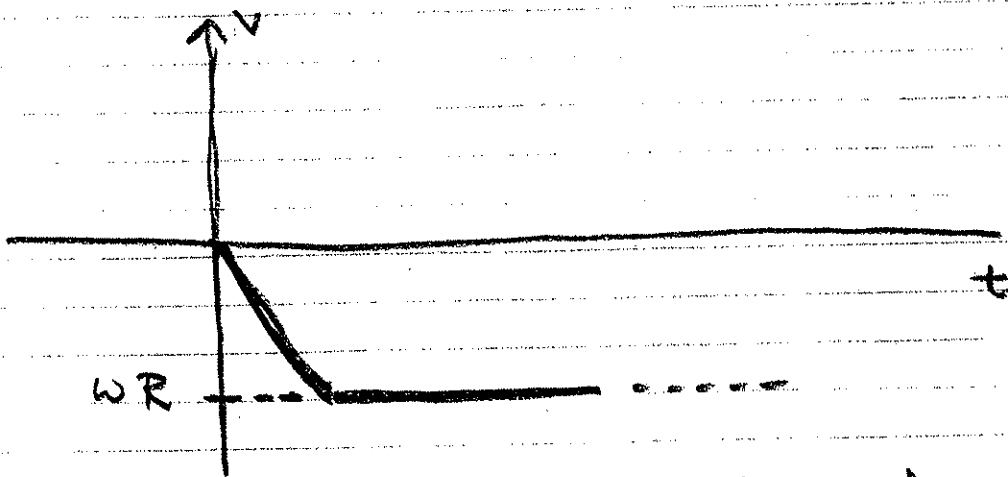
$$V(t) = \frac{d(\Delta x)}{dt} = -\Delta x_0 \sqrt{\frac{fk_9}{D}} \sin\left(\sqrt{\frac{fk_9}{D}} t\right) \quad (2)$$



It means that the linear velocity of the cylinder surfaces ωR (where R is the cylinder radius) must be:

$$\omega R > \Delta x_0 \sqrt{\frac{f_k g}{D}}$$

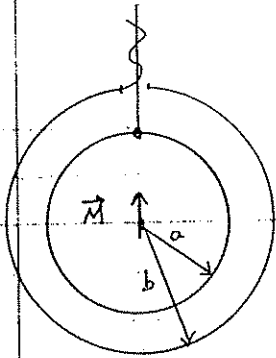
If this is not satisfied, the board will first start moving as described by Eqs. (1) and (2), but when $V(t)$ becomes equal ωR , the friction at the right-side cylinder suddenly changes from kinetic to static, and the board will move with constant speed $V = \omega R$:



However, since the gravity center will shift toward the left cylinder, the kinetic friction force F_k will keep growing, and the static friction force F_s will become smaller and smaller — at certain

moment F_L will take over. Then the motion will become harmonic again.

The momentum of the spheres



For a dipole $\vec{B} = \frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{r^3}$

$\hat{n} = \hat{e}_r$ $\vec{m} = \vec{e}_z = m(\hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta)$

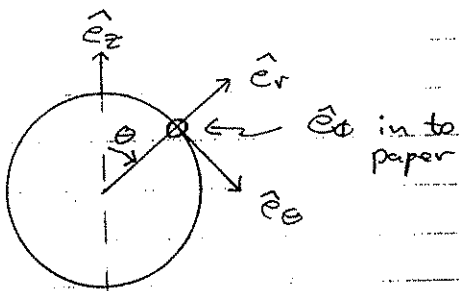
$\vec{B} = \frac{m}{r^3} (2\hat{e}_r \cos\theta + \hat{e}_\theta \sin\theta)$

$\vec{E} = \hat{e}_r Q/r^2$

$\vec{g} = \frac{1}{4\pi c} \vec{E} \times \vec{B}$

$= \frac{1}{4\pi c} \times \frac{Q}{r^2} \times \frac{m}{r^3} \hat{e}_r \times \hat{e}_\theta \sin\theta$

$\vec{g} = \hat{e}_\phi \frac{mQ \sin\theta}{4\pi cr^5}$



$\vec{l} = \vec{r} \times \vec{g} = \frac{mQ}{4\pi cr^4} \hat{e}_r \times \hat{e}_\phi \sin\theta$

$\hat{e}_r \times \hat{e}_\phi = -\hat{e}_\theta$

$l_z = \frac{mQ}{4\pi cr^4} \sin^2\theta$

$L_z = \int_0^{2\pi} \sin\theta d\phi \int_a^b r^2 dr \frac{mQ}{4\pi cr^4} \sin^2\theta$

$= \frac{2}{3} \left(\frac{1}{a} - \frac{1}{b} \right) \frac{mQ}{c}$

Now consider the charge flowing down the inner surface of the outer sphere. Let the total charge at any instant of time be $-\delta(t)$.

$\sigma = -\frac{\delta}{4\pi b^2}$

$$dg = \sigma da = -\frac{q}{4\pi b^2} \times b^2 \sin\theta d\theta d\phi$$

The total charge below the colatitude θ is

$$\int \sigma da = \int_0^{2\pi} d\phi \int_{\pi}^{\theta} \sin\theta d\theta \left[-\frac{q}{4\pi} \right]$$

$$= -q \left(\frac{1 + \cos\theta}{2} \right)$$

$$\therefore \vec{I} = -\vec{E} \frac{dq}{dt} \left(\frac{1 + \cos\theta}{2} \right) = \text{current flowing past the colatitude } \theta.$$

$$d\vec{F} = \frac{1}{c} \vec{J} \times \vec{B} dv = \frac{1}{c} \vec{R} \times \vec{B} da$$

$$r da = \frac{I}{2\pi b \sin\theta} \times b^2 \sin\theta d\theta d\phi = \frac{I b d\theta d\phi}{2\pi}$$

$$\vec{R} da = -\vec{E} \frac{dq}{dt} \left(\frac{1 + \cos\theta}{2} \right) \frac{b d\theta d\phi}{2\pi}$$

$$d\vec{F} = \frac{1}{c} [-\vec{E} \times \vec{e}_r] \frac{dq}{dt} \left(\frac{1 + \cos\theta}{2} \right) \left(\frac{2m \cos\theta}{b^3} \right) \frac{b d\theta d\phi}{2\pi}$$

$$= \vec{e}_\phi \left(\frac{1 + \cos\theta}{2\pi c b^2} \right) \left(\frac{dq}{dt} \right) m \cos\theta d\theta d\phi$$

$$d\vec{N} = \vec{r} \times d\vec{F} = -\vec{e}_\theta \left(\frac{1 + \cos\theta}{2\pi c b} \right) \left(\frac{dq}{dt} \right) m \cos\theta d\theta d\phi$$

$$dN_z = \left(\frac{1 + \cos\theta}{2\pi c b} \right) \left(\frac{dq}{dt} \right) m \cos\theta \sin\theta d\theta d\phi$$

$$N_z = \frac{2m}{3cb} \left(\frac{dq}{dt} \right) \int_0^{\pi} N_z(t) dt = \frac{2mQ}{3cb}$$

$$\text{Total Impulse} = \frac{2mQ}{3c} \left(\frac{1}{b} - \frac{1}{a} \right) \text{ exactly the opposite of the field}$$

$$c) H_0 = a J_z + \frac{b}{\hbar} J_z^2$$

$$J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow J_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow H_0 = \hbar \begin{pmatrix} a+b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a+b \end{pmatrix}$$

$$\Rightarrow E_1 = \hbar(a+b) \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |m_J = +1\rangle$$

$$E_2 = 0 \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |m_J = +0\rangle$$

$$E_3 = \hbar(-a+b) \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |m_J = -1\rangle$$

if $\underbrace{-a+b=0}$, then $E_3 = E_2 = 0 \Rightarrow$ degenerate
 $\Rightarrow \frac{a}{b} = 1$

\Rightarrow if $\frac{a}{b} = 1$ there is degeneracy.

$$b) W = W J_x = \frac{\hbar W}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For $b=a$ there is degeneracy and so we must use degenerate pert. theory within deg levels.

So can use non-deg pert theory for E_1 .

$$\Delta E_1 = \langle 1 | W | 1 \rangle = W_{11} = 0$$

For $E_2 + E_3$, must diagonalize $H = H_0 + W$ in subspace:

$$\Rightarrow H_0 = \begin{matrix} \text{12} & \text{13} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{matrix}, \quad W = \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow H = \frac{\hbar\omega}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

diagonalize:
$$\begin{vmatrix} -\lambda & \frac{\hbar\omega}{\sqrt{2}} \\ \frac{\hbar\omega}{\sqrt{2}} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \left(\frac{\hbar\omega}{\sqrt{2}}\right)^2 = 0$$

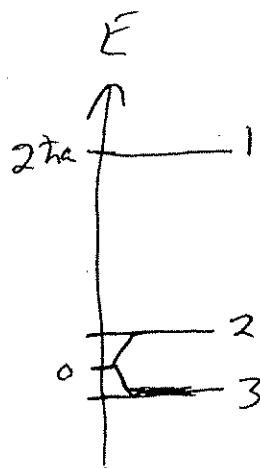
$$\Rightarrow \lambda = \pm \frac{\hbar\omega}{\sqrt{2}}$$

$$\Rightarrow \Delta E_2 = + \frac{\hbar\omega}{\sqrt{2}}$$

$$\Delta E_3 = - \frac{\hbar\omega}{\sqrt{2}}$$

\Rightarrow to 1st order in W :

$$\begin{aligned} E_1 &= 2\hbar\omega \\ E_2 &= \frac{\hbar\omega}{\sqrt{2}} \\ E_3 &= -\frac{\hbar\omega}{\sqrt{2}} \end{aligned}$$



c) if $b = 2a$ then no degeneracy: $H_0 = \hbar \begin{pmatrix} 3a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$
 \Rightarrow use non-deg. pert theory

$$\Delta E_i^{(1)} = \langle i | W | i \rangle = W_{ii} = 0 \quad \text{since } \langle J_x \rangle_{ii} = 0$$

\Rightarrow must use 2nd order:

$$\Delta E_i^{(2)} = \sum_{n \neq i} \frac{\langle n | W | i \rangle^2}{E_i^{(0)} - E_n^{(0)}}$$

For us: $E_1 = 3\hbar a$, $E_2 = 0$, $E_3 = \hbar a$

$$W = \frac{\hbar \omega}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \Delta E_1^{(2)} = \frac{(\hbar \omega / \sqrt{2})^2}{3\hbar a} = \frac{\hbar}{6} \frac{\omega^2}{a}$$

$$\Delta E_2^{(2)} = \frac{(\hbar \omega / \sqrt{2})^2}{-3\hbar a} + \frac{(\hbar \omega / \sqrt{2})^2}{-\hbar a} = -\frac{2\hbar}{3} \frac{\omega^2}{a}$$

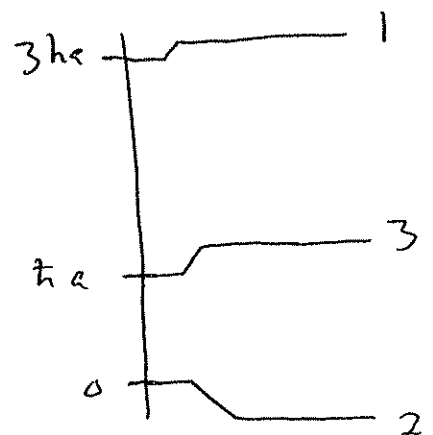
$$\Delta E_3^{(2)} = \frac{(\hbar \omega / \sqrt{2})^2}{\hbar a} = \frac{\hbar}{2} \frac{\omega^2}{a}$$

\Rightarrow to 2nd order in ω :

$$E_1 = 3\hbar a + \frac{\hbar}{6} \frac{\omega^2}{a}$$

$$E_2 = -\frac{2\hbar}{3} \frac{\omega^2}{a}$$

$$E_3 = \hbar a + \frac{\hbar}{2} \frac{\omega^2}{a}$$



$$C_E = AT^3$$

$$A: \frac{\text{Energy}}{K^4} \equiv \frac{J}{K^4}$$

$$\chi = \frac{Np^2}{3kT} \text{ actually}$$

$$\frac{Np^2}{V3kT}$$

omitted in statement of problem.

$$P' = \chi E = \frac{Np^2 E}{V3kT}$$

$$\left(\frac{\partial S}{\partial E}\right)_T = V \left(\frac{\partial P'}{\partial T}\right)_E$$

$$C_E dT = -TV \left(\frac{\partial P'}{\partial T}\right)_E dE = -TV \left(-\frac{Np^2 E}{3VkT^2}\right) dE$$

$$C_E dT = \frac{Np^2}{3kT} E dE$$

$$T AT^3 dT = \frac{Np^2}{3k} E dE$$

$$T^4 dT = \frac{Np^2}{3Ak} E dE$$

$$\Delta T \approx \frac{Np^2 E_i}{6AkT^4}$$

If you were to work with $\chi = \frac{Np^2}{3kT}$, as given, you would end up with

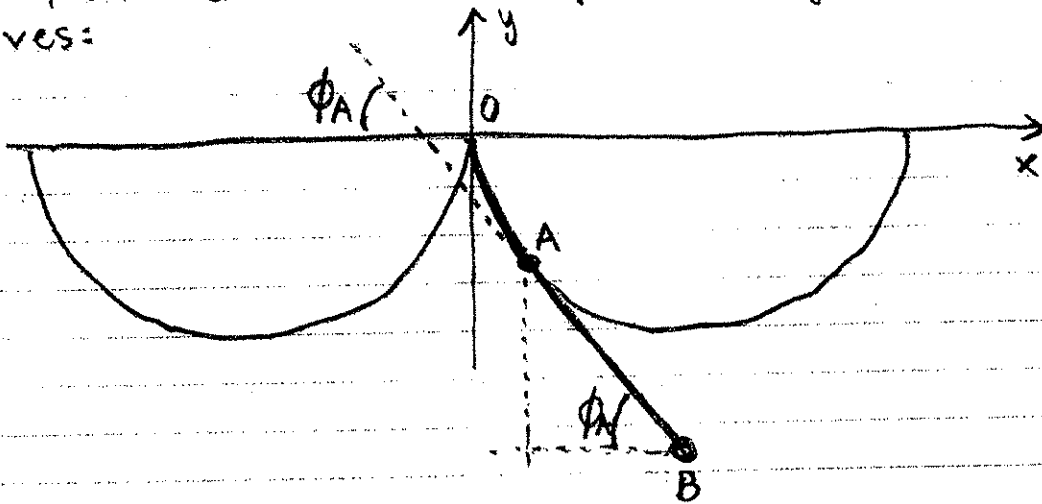
$$\Delta T \approx \frac{NVp^2 E_i}{6AkT^4}$$

which, of course, is not dimensionally correct.

Classical Mechanics - Second Problem

(1)

1. First, let's determine the path along which the mass m moves:



Suppose that the mass is at certain point B. The part OA of the string is wrapped on the cycloidal surface, while the part AB is stretched along a tangent line passing through the point A.

If ϕ_A is the angle the tangent line makes with the x axis, we can write:

$$x_B = x_A + AB \cos \phi_A \quad (1)$$

$$y_B = y_A - AB \sin \phi_A \quad (2)$$

where x_A, y_A and x_B, y_B are the x, y coordinates of the points A and B, respectively.

The length of the AB portion of the string can be written as the difference between the total length $4R$ and the length of the "wrapped" part OA:

$$AB = 4R - OA \quad (3)$$

Thus, the problem reduces to finding ϕ_A and OA.

Suppose that the values of the α parameter in the cycloid equations corresponding to points O and A are $\alpha = 0$ and α . The length of the cycloid between

(2)

O and A is: α_A

$$OA = \int_{\alpha_0}^{\alpha_A} \sqrt{(x'_\alpha)^2 + (y'_\alpha)^2} d\alpha \quad (4)$$

From the equations describing the cycloidal surface we can readily find:

$$x'_\alpha = R - R \cos \alpha$$

$$y'_\alpha = -R \sin \alpha$$

Hence: α_A

$$OA = \int_{\alpha_0}^{\alpha_A} R \sqrt{1 - 2 \cos \alpha + \cos^2 \alpha + \sin^2 \alpha} d\alpha = R \int_{\alpha_0}^{\alpha_A} \sqrt{2 - 2 \cos \alpha} d\alpha$$

$$= R \int_{\alpha_0}^{\alpha_A} \sqrt{4 \sin^2 \frac{\alpha}{2}} d\alpha = 2R \int_{\alpha_0}^{\alpha_A} \sin \frac{\alpha}{2} d\alpha = -4R \cos \frac{\alpha}{2} \Big|_0^{\alpha_A} = 4R \left(1 - \cos \frac{\alpha_A}{2}\right) \quad (5)$$

The tangent of the ϕ_A angle is equal to the function derivative at A:

$$\tan \phi_A = \left(\frac{dy}{dx}\right)_A = \left(\frac{y'_\alpha}{x'_\alpha}\right)_{\alpha=\alpha_A} = \frac{-R \sin \alpha_A}{R(1 - \cos \alpha_A)} = -\frac{\sin \alpha_A}{1 - \cos \alpha_A}$$

$$= -\frac{2 \sin \frac{\alpha_A}{2} \cos \frac{\alpha_A}{2}}{2 \sin^2 \frac{\alpha_A}{2}} = -\frac{\cos \frac{\alpha_A}{2}}{\sin \frac{\alpha_A}{2}} \quad (6)$$

Using the identities given in the "Hint" part, we can find $\sin \phi_A$ and $\cos \phi_A$:

$$|\sin \phi_A| = \frac{\tan \phi_A}{\sqrt{1 + \tan^2 \phi_A}} = \frac{\cos \frac{\alpha_A}{2} / \sin \frac{\alpha_A}{2}}{\sqrt{1 + \cos^2 \frac{\alpha_A}{2} / \sin^2 \frac{\alpha_A}{2}}} = \frac{\cos \frac{\alpha_A}{2} / \sin \frac{\alpha_A}{2}}{1 / \sin \frac{\alpha_A}{2}} = \cos \frac{\alpha_A}{2} \quad (7)$$

$$|\cos \phi_A| = \frac{1}{\sqrt{1 + \tan^2 \phi_A}} = \sin \frac{\alpha_A}{2} \quad (8)$$

From the original cycloid equation we can write:

$x_A = R(\alpha_A - \sin \alpha_A)$. Putting this into Eq. (1), and using also Eqs. (3), (5) and (8), we obtain:

$$\begin{aligned} x_B &= R(\alpha_A - \sin \alpha_A) + \left[4R - 4R \left(1 - \cos \frac{\alpha_A}{2} \right) \right] \sin \frac{\alpha_A}{2} \\ &= R\alpha_A - R\sin \alpha_A + 4R \cos \frac{\alpha_A}{2} \sin \frac{\alpha_A}{2} \\ &= R\alpha_A - R\sin \alpha_A + 2R\sin \alpha_A = R(\alpha_A + \sin \alpha_A) \end{aligned} \quad (9)$$

If we replace the α parameter by another angle, $\beta \equiv \alpha + \pi$, we can write:

$$x_B = R(\beta_A - \sin \beta_A) - \pi R \quad (9A)$$

Similarly, we can obtain the y_B coordinate:

$$\begin{aligned} y_B &= -R(1 - \cos \alpha_A) - \left[4R - 4R \left(1 - \cos \frac{\alpha_A}{2} \right) \right] \cos \frac{\alpha_A}{2} \\ &= -R + R\cos \alpha_A + 4R \cos^2 \frac{\alpha_A}{2} = -R + R\cos \alpha_A - 2R - 2R\cos \alpha_A \\ &= -3R + R\cos \alpha_A = -R(1 + \cos \alpha_A) - 2R \end{aligned} \quad (10)$$

Again, using the same trick with replacing α by $\beta - \pi$, we obtain:

$$y_B = -R(1 - \cos \beta_A) - 2R \quad (10A)$$

The curve described by Eqs. (9A) and (10A) is an identical cycloid as those describing the surfaces in the apparatus - the only difference is that this curve is shifted by one-half period in the x direction, and shifted by downward by $2R$ in the y direction.

4

3. Determining the motion.

For describing the path we now use Eq. (9) and (10) - here $\alpha = 0$ corresponds to the lowest point.

2a. Fully "orthodox" approach - Lagrange's method.

From Eqs. (9) and (10) - only we drop the subscript "A" and "B":

$$\dot{x} = R\dot{\alpha} + R\dot{\alpha}\cos\alpha$$

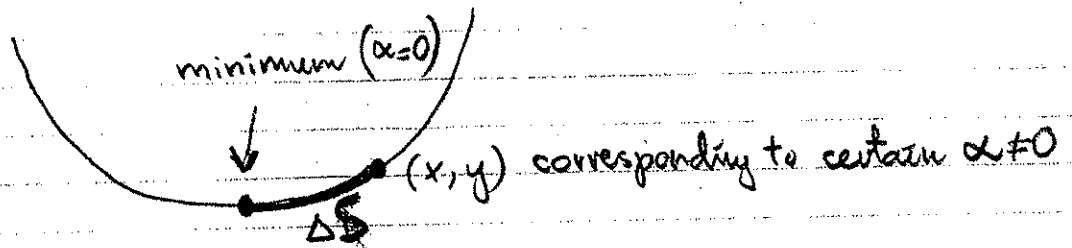
$$\dot{y} = R\dot{\alpha}\sin\alpha$$

$$\text{So: } T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{mR^2\dot{\alpha}^2}{2}[(1+\cos\alpha)^2 + \sin^2\alpha] =$$

$$\frac{mR^2\dot{\alpha}^2}{2}[1 + 2\cos\alpha + \cos^2\alpha + \sin^2\alpha] = \frac{mR^2\dot{\alpha}^2}{2}[2 + 2\cos\alpha] =$$

$$= \frac{mR^2\dot{\alpha}^2}{2} 4\cos^2\frac{\alpha}{2} = 2mR^2\dot{\alpha}^2\cos^2\frac{\alpha}{2} \quad (11)$$

In principle, one can use α as a "generalized coordinate", but it's a poor choice because it leads to a nasty equation. Instead, we will use ΔS .



Let's calculate ΔS in a similar way as in Eqs. (4) - (5), but using now Eq. (9) and (10):

$$\Delta S = \int_0^{\alpha} R\sqrt{(1+\cos\alpha)^2 + \sin^2\alpha} d\alpha = R \int_0^{\alpha} \sqrt{2+2\cos\alpha} d\alpha = R \int_0^{\alpha} 2\cos\frac{\alpha}{2} d\alpha$$

$$= 4R \int_0^{\alpha/2} \cos\beta d\beta = 4R \sin\frac{\alpha}{2} \quad (12)$$

So we can write the time derivative of ΔS :

$$\dot{\Delta S} = \frac{d}{dt} \Delta L = 4R \cdot \frac{\dot{\alpha}}{2} \cos\frac{\alpha}{2} = 2R\dot{\alpha}\cos\frac{\alpha}{2} \quad (13)$$

(5)

And from Eqs. (11) and (13) it follows that

$$T = \frac{m}{2} (\dot{\Delta S})^2 \quad (14)$$

Not surprising, of course $\forall \forall \forall$ (it would be surprising if we got something else)

Now, the potential energy is simply:

$$V = mg y = -mgR(1 + \cos \alpha)$$

(we used Eq. (10)). We can ignore the constant term $2R$.

But V can also be expressed in terms of ΔL using Eq. (12):

$$V = -mgR(1 + 1 - 2 \sin^2 \frac{\alpha}{2}) = -2mgR + 2mgR \sin^2 \frac{\alpha}{2} =$$

$$= \frac{mg}{8R} \cdot 16R^2 \sin^2 \frac{\alpha}{2} - 2mgR = \frac{mg}{8R} (\Delta S)^2 - 2mgR \quad (15)$$

Again, we can ignore the constant term, and write the Lagrangian:

$$L = T - V = \frac{m}{2} (\dot{\Delta S})^2 - \frac{mg}{8R} (\Delta S)^2 \quad (16)$$

Continuing:

$$\frac{\partial L}{\partial (\dot{\Delta S})} = m(\dot{\Delta S}); \quad \frac{d}{dt} \frac{\partial L}{\partial (\dot{\Delta S})} = m(\ddot{\Delta S}) \quad (17)$$

$$\frac{\partial L}{\partial (\Delta S)} = -\frac{mg}{4R} \Delta S$$

$$\text{So: } \frac{d}{dt} \frac{\partial L}{\partial (\dot{\Delta S})} - \frac{\partial L}{\partial (\Delta S)} = m(\ddot{\Delta S}) + \frac{mg}{4R} \Delta S = 0 \quad (18)$$

Ergo: $(\Delta S) = -\frac{g}{4R} \Delta S$ (19)

which is the equation of a harmonic oscillator with the frequency:

$$\omega = \sqrt{\frac{g}{4R}} \quad (20)$$

Eq.(19) is valid for any ΔS value, so that the frequency does not depend on the amplitude.

2.b. An "easy" method:

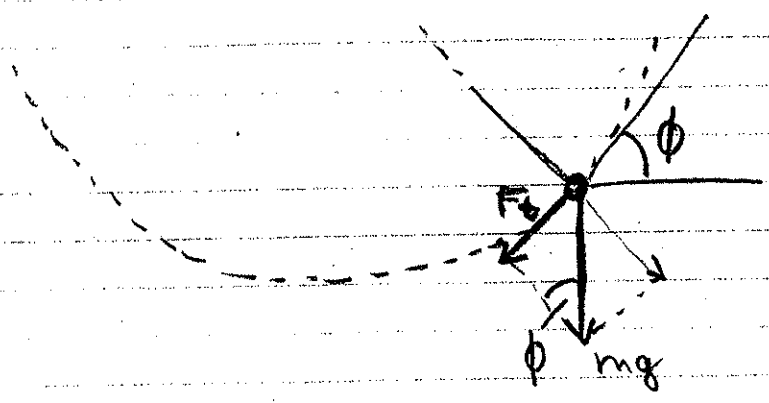
The tangential acceleration is:

$$a_t = \frac{d^2}{dt^2} (\Delta S) \quad (21)$$

From the 2nd Newton law:

$$m a_t = F_t \quad (22)$$

where F_t is the tangential force.



$$F_t = mg \sin \phi \quad (23)$$

But: $\tan \phi = \frac{y'_a}{x'_a} = \frac{R}{R}$

7

Using the trajectory equations (9) and (10), we obtain:

$$\tan \phi = \frac{R \sin \alpha}{R(1 + \cos \alpha)} = \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \cos^2 \frac{\alpha}{2}} = \tan \frac{\alpha}{2}$$

As follows from the above, $\phi = \frac{\alpha}{2}$

Ergo:

$$F_t = mg \sin \frac{\alpha}{2}$$

But from Eq. (12) it follows that:

$$\sin \frac{\alpha}{2} = \frac{\Delta S}{4R}$$

So:

$$F_t = -mg \frac{\Delta S}{4R} \quad (24)$$

The minus sign comes from the fact that F_t is always oriented toward the equilibrium point.

Combining Eqs. (21), (22) and (24) we get:

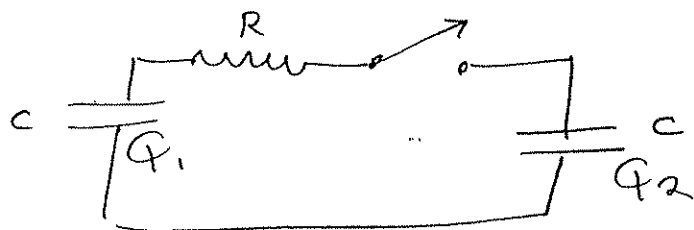
$$m \frac{d^2}{dt^2} (\Delta S) = -mg \frac{\Delta S}{4R}$$

Or:

$$(\ddot{\Delta S}) = -\frac{mg}{4R} \Delta S$$

Which is the same as Eq. (19).

Capacitor Problem:



(a) The two capacitors are equally charged with $Q_0/2$

$$E(\text{before}) - E(\text{after}) = \frac{Q_0^2}{2C} - 2 \times \frac{1}{2C} \left(\frac{Q_0}{2} \right)^2$$

$$= \frac{Q_0^2}{4C}$$

Presumably this is turned to heat in R .

(b) $\frac{Q_1}{C} - \frac{Q_2}{C} + IR = 0$

$$\frac{d}{dt} \left[Q_1 - (Q_0 - Q_1) + IRC = 0 \right]$$

$$\frac{dI}{dt} = - \frac{2}{RC} I$$

$$I(t) = \frac{Q_0}{RC} e^{-2t/RC}$$

$$\text{Energy} = \int_0^{\infty} I^2 R dt = R \left(\frac{Q_0}{RC} \right)^2 \int_0^{\infty} e^{-4t/RC} dt$$

$$= \frac{Q_0^2}{4C} \quad \text{as in part (a)}$$

(c) Oscillations cause energy loss through radiation

Radiation loss "looks like" pure resistance.