DEPARTMENT COMPREHENSIVE EXAMINATION #78

September 30 and October 1, 1996

Comprehensive Examination for Fall 1996

PART I – IV

General Instructions

This Comprehensive Examination for Fall 1996 (#78) consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 30, and lasts three hours. The second part (Problems 3-4) will be handed out on the same day, at 1:30 pm, and also lasts three hours. The third and fourth parts (Problems 5-6 and Problems 7-8) will be administrated in the same way on Tuesday, October 1, 1996.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit - especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulae and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

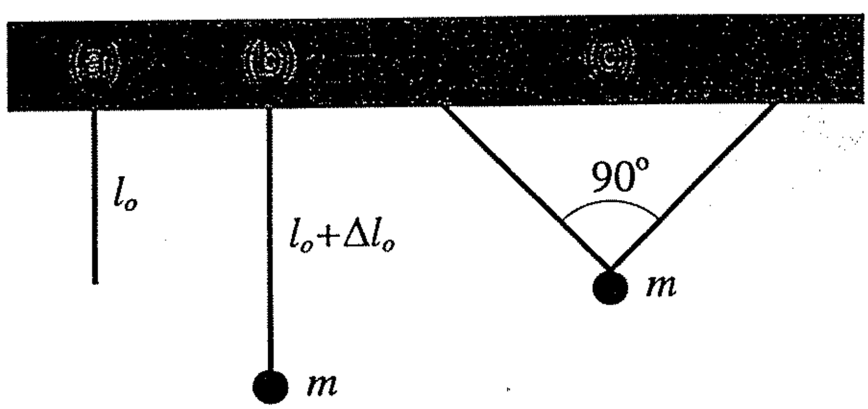
Use the last pages of your bluebooks for "scratch" work separated by at least one page from your solutions. "Scratch" work will not be graded.

The three part diagram for this problem appears at the end of the statement of the problem.

(A) An elastic string, the length of which in the unstretched state is l_0, hangs from a solid horizontal bar (a). When a weight with mass m is attached to the lower end, the length increases by Δl_0 (b). What is the frequency of small oscillations of the weight in the vertical direction?

(B) Now, another identical elastic string is fastened to the weight, and its other end is attached to the bar at such a point that in the state of equilibrium the two strings make a right angle with respect to each other (c). What is the frequency of small vertical oscillations of the weight?

Assume that the strings are massless, and that they perfectly obey Hooke's law.



#1

CLASSICAL MECHANICS, PROBLEM I — SOLUTION

I. Single string:
First, let's analyze the single-string oscillator [plot **(b)**]. The Hooke's law says that $T = -k\Delta l$, where T is the tension, Δl is the elongation, and k is the spring constant; so, from the equilibrium conditions $T_{\rm eq} = -mg$, $\Delta l_{\rm eq} = \Delta l_0$, we get the k value for the string:

$$k_0 = \frac{mg}{\Delta l_0}. \tag{1}$$

Let us introduce a vertical coordinate axis, pointing down, and let $z = 0$ be the equilibrium point. For small shifts Δz from the equilibrium position the string tension changes as:

$$T(\Delta z) = -k_0\Delta l = -k_0(\Delta l_0 + z) = -mg - \frac{mg}{\Delta l_0}z$$

The total force acting on the mass m is the sum of $T(z)$ and gravity mg. One can now set up the equation of motion:

$$\frac{d^2z}{dt^2} = \frac{1}{m}\left[T(z) + mg\right] = -\frac{g}{\Delta l_0}z,$$

which has the well-known form of the harmonic oscillator equation; so, the angular frequency we are looking for is, of course:

$$\omega_0 = \sqrt{\frac{g}{\Delta l_0}}. \tag{2}$$

II. Two strings:

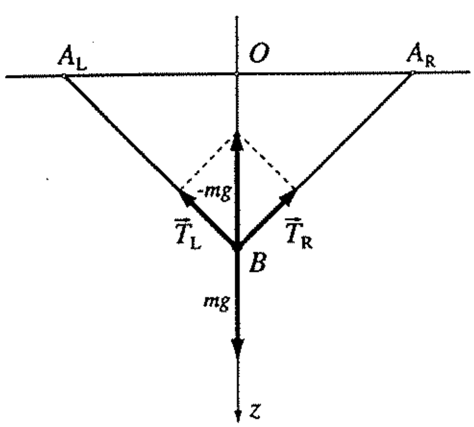


Figure 1S. Forces in the two-string system: equilibrium state (mass m not shown).

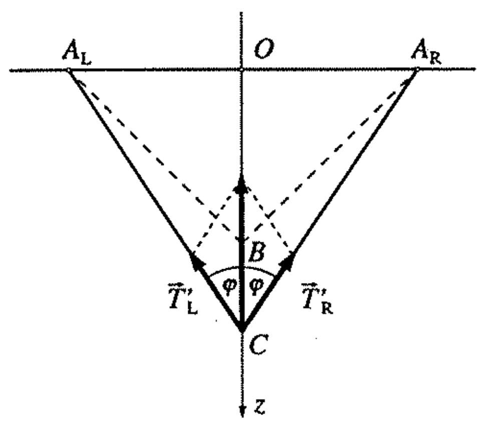


Figure 2S. The mass shifted from the equilibrium point (the mass and the gravity force not shown).

Let us now analyze the equilibrium state of the two-string system [plot **(c)**]. Let's make a more detailed graph, showing all relevant forces (Fig. S1). The gravity mg is balanced by the vector sum of the tension forces in the two strings, $\vec{T}_L$ and $\vec{T}_R$. Since both strings are identical, and the angle between them is 90°, one can write:

$$|\vec{T}_L| = |\vec{T}_R| = \frac{\frac{1}{2}mg}{\cos 45^\circ} = \frac{mg}{\sqrt{2}},$$

from which, using Eq. (1), one can get the elongations Δl_L and Δl_R of the strings:

$$\Delta l_L = \Delta l_R = \frac{1}{k_0} \times \frac{mg}{\sqrt{2}} = \frac{\Delta l_0}{\sqrt{2}},$$

so that the total length of each string (A_LB and A_RB in Fig. S1) is:

$$l_0 + \Delta l_L = l_0 + \Delta l_R = l_0 + \frac{\Delta l_0}{\sqrt{2}}.$$

Having this, one can find the distance $A_L A_R = 2a$ between the points where the strings are attached to the bar (points A_L and A_R):

$$a = OA_L = OA_R = A_L B/\sqrt{2} = \frac{l_0}{\sqrt{2}} + \frac{\Delta l_0}{2}. \tag{3}$$

Of course, the OB distance also is equal a.

As before, let's introduce a vertical coordinate axis z pointing downward, with $z = 0$ being the equilibrium point. The equation of motion of the mass m along the z axis can be written as:

$$m\frac{d^2 z}{dt^2} = mg + F(\Delta z), \tag{4}$$

where Δz is the displacement from the equilibrium point, and $F(\Delta z)$ is the sum of the string forces depending on this displacement. To find the expression for $F(\Delta z)$, let's make another plot (Fig. 2S), illustrating the situation for $\Delta z \neq 0$. Point C is the weight position, and, as in Fig. 1S, point B is the equilibrium position. The angle between the strings, 2φ, is no longer $90°$. The length of each string is now:

$$CA_L = CA_R = \sqrt{(OC)^2 + (OA_L)^2} = \sqrt{(a+\Delta z)^2 + a^2}.$$

Subtracting the unstretched length l_0 we get the elongation, and multiplying by the "spring constant" k given by Eq. (1), we get the tension in each string:

$$|\vec{T}'_L| = |\vec{T}'_R| = T' = \frac{mg}{\Delta l_0}\left[\sqrt{(a+\Delta z)^2 + a^2} - l_0\right], \tag{5}$$

The $F(\Delta z)$ force is the vector sum of the tension forces (the minus sign is because force pulls up):

$$F(\Delta z) = |\vec{T}'_R + \vec{T}'_L| = -2T'\cos\varphi \tag{6}$$

and the cosine value can be found from the COA_L triangle:

$$\cos\varphi = \frac{OC}{CA_L} = \frac{a+\Delta z}{\sqrt{(a+\Delta z)^2 + a^2}}. \tag{7}$$

Combining Eqs.(6) - (8), we obtain

$$\begin{aligned} F(\Delta z) &= -\frac{2mg}{\Delta l_0}\left[\sqrt{(a+\Delta z)^2 + a^2} - l_0\right] \times \frac{a+\Delta z}{\sqrt{(a+\Delta z)^2 + a^2}} \\ &= -\frac{2mg}{\Delta l_0}\left[a + \Delta z - l_0\frac{a+\Delta z}{\sqrt{(a+\Delta z)^2 + a^2}}\right]. \end{aligned}$$

This expression looks quite obnoxious; but we are interested in *small* scillation, so it may help to expand the $F(\Delta z)$ function into a power series:

$$F(\Delta z) = F(\Delta z = 0) + \Delta z\left(\frac{dF}{dz}\right)_{\Delta z = 0} + \frac{(\Delta z)^2}{2!}\left(\frac{d^2 F}{dz^2}\right)_{\Delta z=0} + \cdots$$

For small oscillations one can neglect the second- and higher-order terms; taking into account that $F(\Delta z = 0) = -mg$ (equilibrium condition), Eq.(4) can be written in a simple form:

$$\frac{d^2 z}{dt^2} = \frac{\Delta z}{m}\left(\frac{dF}{dz}\right)_{\Delta z = 0}. \tag{8}$$

What has to be done now is to take the derivative dF/dz. Since by our definition $\Delta z \equiv z$, we will skip the Δ symbol from now on.

$$\begin{aligned}
\frac{dF}{dz} &= -\frac{2mg}{\Delta l_0}\left\{1 - al_0\frac{d}{dz}\left[\frac{1}{\sqrt{(a+z)^2+a^2}}\right] - l_0\frac{d}{dz}\left[\frac{z}{\sqrt{(a+z)^2+a^2}}\right]\right\} \\
&= -\frac{2mg}{\Delta l_0}\left\{1 - al_0\frac{-\frac{1}{2}(2z+2a)}{[(a+z)^2+a^2]^{\frac{3}{2}}} - zl_0\frac{-\frac{1}{2}(2z+2a)}{[(a+z)^2+a^2]^{\frac{3}{2}}} - l_0\frac{1}{[(a+z)^2+a^2]^{\frac{1}{2}}}\right\} \\
&= -\frac{2mg}{\Delta l_0}\left\{1 + \frac{l_0(z+a)^2}{[(a+z)^2+a^2]^{\frac{3}{2}}} - \frac{l_0}{[(a+z)^2+a^2]^{\frac{1}{2}}}\right\}
\end{aligned}$$

What we need is the value of this derivative for $z = 0$ (see Eq.5):

$$\begin{aligned}
\left(\frac{dF}{dz}\right)_{z=0} &= -\frac{2mg}{\Delta l_0}\left[1 + \frac{2l_0a^2}{[2a^2]^{\frac{3}{2}}} - \frac{l_0}{[2a^2]^{\frac{1}{2}}}\right] = -\frac{2mg}{\Delta l_0}\left(1 + \frac{l_0a^2}{\sqrt{8a^6}} - \frac{l_0}{\sqrt{2a^2}}\right) \\
&= -\frac{2mg}{\Delta l_0}\left(1 + \frac{l_0}{2\sqrt{2}a} - \frac{l_0}{\sqrt{2}a}\right) = -\frac{2mg}{\Delta l_0}\left(1 - \frac{l_0}{2\sqrt{2}a}\right)
\end{aligned}$$

To get the final form of the equation, we need only to to cast in the value of a (see Eq. 3):

$$\begin{aligned}
\left(\frac{dF}{dz}\right)_{z=0} &= -\frac{2mg}{\Delta l_0}\left[1 - \frac{l_0}{2\sqrt{2}\left(\frac{l_0}{\sqrt{2}} + \frac{\Delta l_0}{2}\right)}\right] = -\frac{2mg}{\Delta l_0}\left(1 - \frac{l_0}{2l_0 + \sqrt{2}\Delta l_0}\right) \\
&= -\frac{2mg}{\Delta l_0}\left(\frac{2l_0 + \sqrt{2}\Delta l_0 - l_0}{2l_0 + \sqrt{2}\Delta l_0}\right) = -\frac{2mg}{\Delta l_0}\left(\frac{l_0 + \sqrt{2}\Delta l_0}{2l_0 + \sqrt{2}\Delta l_0}\right) \\
&= -\frac{mg}{\Delta l_0}\left(\frac{l_0 + \sqrt{2}\Delta l_0}{l_0 + \frac{\sqrt{2}\Delta l_0}{2}}\right) = -\frac{mg}{\Delta l_0}\left[\frac{1 + 2\left(\frac{\sqrt{2}\Delta l_0}{2l_0}\right)}{1 + \left(\frac{\sqrt{2}\Delta l_0}{2l_0}\right)}\right]
\end{aligned}$$

The expression $\sqrt{2}\Delta l_0/2l_0$ that appears at the end is ıthe relative string elongation in the two-string configuration. Let us call it λ; using this symbol, one can write the above equation in elegant form:

$$\left(\frac{dF}{dz}\right)_{z=0} = -\frac{mg}{\Delta l_0}\left(\frac{1+2\lambda}{1+\lambda}\right) \tag{9}$$

Inserting this expression into Eq. (5) yields the final form of the equation of motion:

$$\frac{d^2z}{dt^2} = -\Delta z\frac{g}{\Delta l_0}\left(\frac{1+2\lambda}{1+\lambda}\right), \tag{10}$$

which is the equation of a harmonic oscillator with oscillation frequency:

$$\omega' = \sqrt{\frac{g}{\Delta l_0}\left(\frac{1+2\lambda}{1+\lambda}\right)} = \omega_0\sqrt{\frac{1+2\lambda}{1+\lambda}}, \tag{11}$$

As follows from the above, the oscillation frequency changes by a factor $\sqrt{(1+2\lambda)/(1+\lambda)}$ in comparison th the frequency of the single-string oscillator. If $\Delta l_0/l_0 \ll 1$, i.e., $\lambda \ll 1$, then than in the first approximation $\omega' \cong \omega_0$; and if the string elongation is very large, then $\omega' \to \sqrt{2}\omega_0$.

Comment: A common mistake that people make when solving this problem is that they assume that the change in the φ angle can be neglected, because the oscillations are "small". This leads to the result $\omega' = \omega_0$ in all cases — however, as we have shown, this is a true result only in the case of $\Delta l_0/l_0 \to 0$.

(a) What, in the context of electrostatics, is a Green function? Explain carefully how a Green function is defined, what conditions it must satisfy, and how it is used. What is the difference between a Dirichlet and a Neumann Green function?

(b) Find the Dirichlet Green function for the inside of an infinitely long cylinder of radius a. Assume that the potential does not depend on z, the coordinate measured along the length of the cylinder.

#2

1 a) A Dirichlet Green function is the potential produced by a unit point charge in the presence of a grounded conductor, i.e. a surface S on which $\phi=0$. It is used to find the potential due to an arbitrary charge distribution $\rho(\vec{r})$ in the presence of an arbitrary but known potential on S.

$$\phi(\vec{r}) = \int_V \rho(\vec{r}')\, G_D(\vec{r},\vec{r}')\, d\vec{r}' - \frac{1}{4\pi}\int_S \phi(\vec{r}')\, \frac{\partial G_D}{\partial n'}\, da'$$

The Neumann function is used when ϕ is not known on S but $\frac{\partial \phi}{\partial n}$ is. Then

$$\phi(\vec{r}) = \int_V \rho(\vec{r}')\, G_N(\vec{r},\vec{r}')\, d\vec{r}' + \frac{1}{4\pi}\int_S \frac{\partial \phi}{\partial n'}\, G_N\, da' + \langle\phi\rangle_S$$

Here $\langle\phi\rangle_S$ is the average potential over the surface and $\frac{\partial G_N}{\partial n'} = -\frac{4\pi}{S}$ for $\vec{r}'$ on S.

b) This is really a 2-d problem, and it is easiest to use images. The analog to the unit point charge is a line charge with unit charge per length. The potential due to the line charge in empty space can be found from Gauss's law

$$\int \vec{E} \cdot \hat{n}\, da = 4\pi Q$$

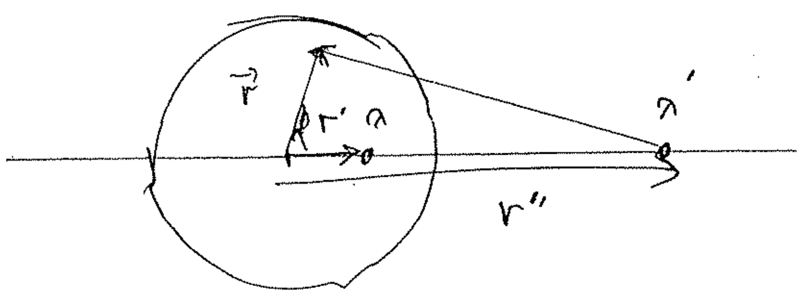
$$E\, 2\pi r \not{L} = 4\pi \lambda \not{L}$$

or $$E = 2\lambda / r$$

where λ can be taken to be 1 i.e. unit charge per unit length

then $$\Phi = -2\lambda \ln(r)$$

plus a constant (which is actually infinite)



at an arbitrary point r.

$$\phi(r) = -2\lambda \ln\left(\sqrt{r^2 + r'^2 - 2rr'\cos\phi}\right)$$
$$- 2\lambda' \ln\left(\sqrt{r^2 + r''^2 - 2rr''\cos\phi}\right)$$

If this is to vanish when $r = a$

$$0 \overset{?}{=} -\lambda \ln\left(a^2 + r'^2 - 2ar'\cos\phi\right) - \lambda' \ln\left(a^2 + r''^2 - 2ar''\cos\phi\right)$$

let $r'' \to a^2/r'$ $\quad (a^2 + r''^2 - 2ar'' \cos\phi) \Rightarrow \frac{a^2}{r'^2}(r'^2 + a^2 - 2ar' \cos\phi)$

Let $\lambda' = -\lambda = -1$ then

$$\phi(r=a) = 2\ln(a/r') = 2\ln a - 2\ln r'$$

The constant is inconsequential ~~(there~~ is an ∞ additive const. anyway) The remaining term can be canceled with another image charge at the origin. This will produce a potential

$$\phi''(r) = -2\lambda'' \ln r$$

We must have $\quad \cancel{2}\ln a - 2\ln r' - \cancel{2}\lambda'' \ln a = 0$

$$\lambda'' = 1 - \frac{\ln r'}{\ln a}$$

so $\quad \phi''(r) = -2\left(1 - \frac{\ln r'}{\ln a}\right)\ln r$

Finally

$$G(r,r') = -\ln\left[\frac{r^2 + r'^2 - 2rr'\cos\phi}{\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\phi}\right]$$

$$+ 2\ln a + 2\ln r' - 2\ln r + \frac{2\ln r \ln r'}{\ln a}$$

Consider a paramagnetic sample for which the total average magnetic moment can be written as $\overline{M} = V\chi H$, where V is the volume of the sample, and $\chi = \frac{C^*}{T}$, where C^* is a constant, and T is the absolute temperature. The heat capacity at constant magnetic field is given by $C_H = C_H(T,H)$, where $C_H(T,0) = \frac{aV}{T^2}$, over the temperature range of interest, a being a constant.

Show that if the sample undergoes an adiabatic demagnetization, its final temperature, T_f, is given by

$T_f = \left\{\frac{a + C^* H_f^2}{a + C^* H_i^2}\right\}^{1/2} T_i$, where H_f is the final magnetic field, and where H_i and T_i are the initial magnetic field and initial temperature, respectively.

Hint: You might want to consider $\left(\frac{\partial C_H}{\partial H}\right)_T$, and make use of the following Maxwell relation $\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H$.

The differential expression for entropy, TdS, can be written as

$$TdS = T\left(\frac{\partial S}{\partial T}\right)_H dT + T\left(\frac{\partial S}{\partial H}\right)_T dH = C_H(T,H)dT + T\left(\frac{\partial M}{\partial T}\right)_H dH$$

Now,

$$\left(\frac{\partial C_H}{\partial H}\right)_T = \left(\frac{\partial}{\partial H}\right)_T\left[T\left(\frac{\partial S}{\partial T}\right)_H\right] = T\frac{\partial^2 S}{\partial H\partial T} = T\frac{\partial^2 S}{\partial T\partial H} = T\left(\frac{\partial}{\partial T}\right)_H\left(\frac{\partial S}{\partial H}\right)_T$$

$$\left(\frac{\partial C_H}{\partial H}\right)_T = T\left(\frac{\partial}{\partial T}\right)_H\left(\frac{\partial M}{\partial T}\right)_H = T\left(\frac{\partial^2 M}{\partial T^2}\right)_H = VTH\left(\frac{\partial^2 \chi}{\partial T^2}\right)_H$$

Therefore,

$$\int_0^H\left(\frac{\partial C_H}{\partial H'}\right)dH' = VT\int_0^H H'\left(\frac{\partial^2 \chi}{\partial T^2}\right)_{H'} dH'$$

and so,

$$C_H(T,H) - C_H(T,0) = \frac{C^* V H^2}{T^2}$$

which means that

$$C_H(T,H) = (aV + C^* V H^2)\frac{1}{T^2}.$$

Now since the process occurs adiabatically, $dS = 0$, and so we get $0 = (a + C^* H^2)dT - HTC^* dH$, or

$\frac{dT}{T} = \frac{C^* H dH}{a + C^* H^2}$, integration of which yields

$$T_f = \left\{\frac{a + C^* H_f^2}{a + C^* H_i^2}\right\}^{1/2} T_i.$$

Consider an electron of a linear triatomic molecule formed by three equidistant atoms, as shown in the diagram at the bottom. We use $|\Phi_A\rangle$, $|\Phi_B\rangle$, $|\Phi_C\rangle$ to denote three orthonormal states of this electron, corresponding to localization about the three atoms A, B, and C. When we neglect the possibility of the electron jumping from one atom to another, then its energy is described by the Hamiltonian, H_0, whose eigenstates are $|\Phi_A\rangle$, $|\Phi_B\rangle$, $|\Phi_C\rangle$, with the same eigenvalue, E_0. The coupling between the states $|\Phi_A\rangle$, $|\Phi_B\rangle$, $|\Phi_C\rangle$ is described by an additional Hamiltonian, W, defined by

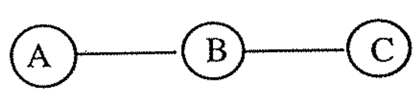
$$\begin{aligned} W\,|\Phi_A\rangle &= -a\,|\Phi_B\rangle \\ W\,|\Phi_B\rangle &= -a\,|\Phi_A\rangle - 2a\,|\Phi_C\rangle \\ W\,|\Phi_C\rangle &= -2a\,|\Phi_B\rangle \end{aligned}$$

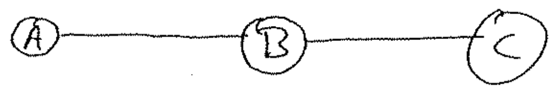
where a is a positive constant.

(a) Write down the matrix representation of the full Hamiltonian $H = H_0 + W$ in the basis defined by the states $|\Phi_A\rangle$, $|\Phi_B\rangle$, $|\Phi_C\rangle$.

(b) Calculate the energies and the eigenstates of the full Hamiltonian $H = H_0 + W$.

(c) At time t = 0, the electron is in the state $|\Phi_A\rangle$. Find the wave function at subsequent times t (expressed in terms of the basis defined by the states $|\Phi_A\rangle$, $|\Phi_B\rangle$, $|\Phi_C\rangle$). Are there any times, t, for which the electron is perfectly localized about atom A, B, or C?





$|\phi_A\rangle$ $|\phi_B\rangle$ $|\phi_C\rangle$ 3 orthonormal states

label matrices as $\begin{pmatrix} |\phi_A\rangle & |\phi_B\rangle & |\phi_C\rangle \\ & & \\ & & \end{pmatrix}$

Since $|\phi_A\rangle, |\phi_B\rangle, |\phi_C\rangle$ are energy eigenstates of H_0, then H_0 must be diagonal in this basis:

$$H_0 = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix} = E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hamiltonian W represents coupling between states:

$$W|\phi_A\rangle = -a|\phi_B\rangle$$

$$W|\phi_B\rangle = -a|\phi_A\rangle - 2a|\phi_C\rangle$$

$$W|\phi_C\rangle = -2a|\phi_B\rangle$$

$\Rightarrow \langle\phi_A|W|\phi_A\rangle = 0$ since $\langle\phi_A|\phi_B\rangle = 0$
$\langle\phi_B|W|\phi_A\rangle = -a$ etc

$$\Rightarrow W = \begin{pmatrix} 0 & -a & 0 \\ -a & 0 & -2a \\ 0 & -2a & 0 \end{pmatrix}$$

Note that it is Hermitian.

(a) Full Hamiltonian is $H = H_0 + W$

$$H = \begin{pmatrix} E_0 & -a & 0 \\ -a & E_0 & -2a \\ 0 & -2a & E_0 \end{pmatrix}$$

(b) Find eigenvalues : $\det |H - \lambda I| = 0$

$$\begin{vmatrix} E_0-\lambda & -a & 0 \\ -a & E_0-\lambda & -2a \\ 0 & -2a & E_0-\lambda \end{vmatrix} = 0$$

$$(E_0-\lambda)\left[(E_0-\lambda)^2 - 4a^2\right] + a\left[-a(E_0-\lambda)\right] = 0$$

$$(E_0-\lambda)\left[(E_0-\lambda)^2 - 5a^2\right] = 0$$

solns: $\lambda = E_0$; $E_0 - \lambda = \pm\sqrt{5}\,a$

$$\Rightarrow \boxed{\lambda = E_0,\ E_0 \pm \sqrt{5}\,a}$$

Find eigenstates, (label them $|\psi_i\rangle$, $i = 1,2,3$)

$\underline{E_0}$: $H|\psi_1\rangle = E_0|\psi_1\rangle$ let $|\psi_1\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$$\Rightarrow \begin{aligned} E_0\alpha - a\beta &= E_0\alpha \\ -a\alpha + \beta E_0 - 2a\gamma &= E_0\beta \\ -2a\beta + E_0\gamma &= E_0\gamma \end{aligned}$$

$\Rightarrow \beta = 0$, $\alpha = -2\gamma$ normalize: $\alpha^2 + \gamma^2 = 1$

$\Rightarrow 5\gamma^2 = 1$

$\gamma =$ [illegible]

$$\Rightarrow \quad |\psi_1\rangle = \frac{2}{\sqrt{5}}|\phi_A\rangle - \frac{1}{\sqrt{5}}|\phi_C\rangle$$

<u>$E_0 + \sqrt{5}a$:</u>

eqns:

$$E_0\alpha - a\beta = (E_0+\sqrt{5}a)\alpha$$
$$-a\alpha + \beta E_0 - 2a\gamma = (E_0+\sqrt{5}a)\beta$$
$$-2a\beta + E_0\gamma = (E_0+\sqrt{5}a)\gamma$$

$$\beta = -\sqrt{5}\alpha = -\frac{\sqrt{5}}{2}\gamma$$

$$\alpha^2+\beta^2+\gamma^2 = 1$$
$$\frac{\beta^2}{5} + \beta^2 + \frac{4}{5}\beta^2 = 1 \Rightarrow \beta^2 = \frac{1}{2}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{10}}|\phi_A\rangle - \frac{1}{\sqrt{2}}|\phi_B\rangle + \frac{2}{\sqrt{10}}|\phi_C\rangle$$

<u>$E_0 - \sqrt{5}a$:</u>

eqns:

$$E_0\alpha - a\beta = (E_0-\sqrt{5}a)\alpha$$
$$-a\alpha + \beta E_0 - 2a\gamma = (E_0-\sqrt{5}a)\beta$$
$$-2a\beta + E_0\gamma = (E_0-\sqrt{5}a)\gamma$$

$$\beta = \sqrt{5}\alpha = \frac{\sqrt{5}}{2}\gamma$$

$$|\psi_3\rangle = \frac{1}{\sqrt{10}}|\phi_A\rangle + \frac{1}{\sqrt{2}}|\phi_B\rangle + \frac{2}{\sqrt{10}}|\phi_C\rangle$$

note that $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ as required

E_i	$\|\psi_i\rangle$
E_0	$\frac{2}{\sqrt{5}}\|\phi_A\rangle - \frac{1}{\sqrt{5}}\|\phi_C\rangle$
$E_0 \pm \sqrt{5}a$	$\frac{1}{\sqrt{10}}\|\phi_A\rangle \mp \frac{1}{\sqrt{2}}\|\phi_B\rangle + \frac{2}{\sqrt{10}}\|\phi_C\rangle$

(c) at $t=0$ electron in state $|\phi_A\rangle$

$$\Rightarrow |\psi(0)\rangle = |\phi_A\rangle$$

rewrite in energy basis $|\psi_i\rangle$

$$|\phi_A\rangle = \sum_n c_n |\psi_n\rangle$$

$$c_n = \langle \psi_n | \phi_A \rangle$$

$$\Rightarrow c_1 = \frac{2}{\sqrt{5}}, \quad c_2 = \frac{1}{\sqrt{10}}, \quad c_3 = \frac{1}{\sqrt{10}}$$

$$\Rightarrow |\phi_A\rangle = \frac{2}{\sqrt{5}}|\psi_1\rangle + \frac{1}{\sqrt{10}}|\psi_2\rangle + \frac{1}{\sqrt{10}}|\psi_3\rangle$$

Since those are energy eigenstates, it is trivial to write down time evolution

$$|\psi(t)\rangle = \frac{2}{\sqrt{5}} e^{-iE_0 t/\hbar}|\psi_1\rangle + \frac{1}{\sqrt{10}} e^{-i(E_0+\sqrt{5}a)t/\hbar}|\psi_2\rangle + \frac{1}{\sqrt{10}} e^{-i(E_0-\sqrt{5}a)t/\hbar}|\psi_3\rangle$$

Now write this in terms of localized states:

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar}\left\{\frac{4}{5}|\phi_A\rangle - \frac{2}{5}|\phi_C\rangle + \frac{1}{10}e^{-i\sqrt{5}at/\hbar}\left[|\phi_A\rangle - \sqrt{5}|\phi_B\rangle + 2|\phi_C\rangle\right] + \frac{1}{10}e^{i\sqrt{5}at/\hbar}\left[|\phi_A\rangle + \sqrt{5}|\phi_B\rangle + 2|\phi_C\rangle\right]\right\}$$

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar}\left\{\frac{1}{5}\left(4 + \cos\frac{\sqrt{5}at}{\hbar}\right)|\phi_A\rangle + \frac{i}{\sqrt{5}}\sin\frac{\sqrt{5}at}{\hbar}|\phi_B\rangle - \frac{2}{5}\left(1 - \cos\frac{\sqrt{5}at}{\hbar}\right)|\phi_C\rangle\right\}$$

when $\cos \frac{\sqrt{5}\,a t}{\hbar} = 1$, then $|\psi(t)\rangle = e^{-iE_0 t/\hbar} |\phi_A\rangle$

so electron is perfectly localized about atom A.

i.e. $P_A = |\langle \phi_A | \psi(t)\rangle|^2 = 1$

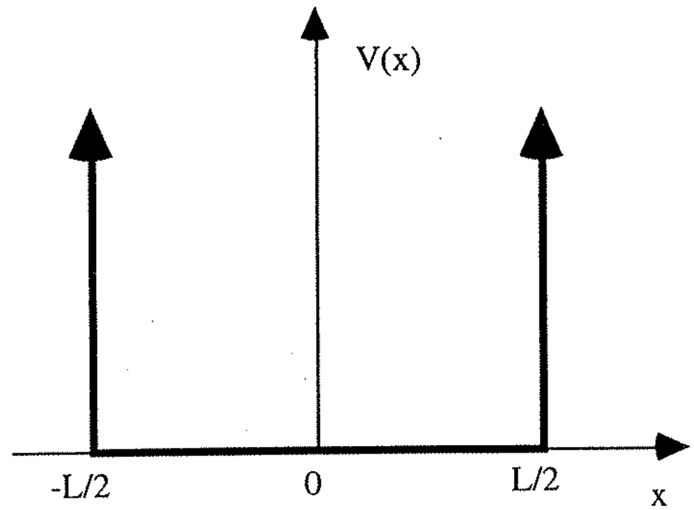
note that at these times $P_B = P_C = 0$

$$\Rightarrow \frac{\sqrt{5}\,a t}{\hbar} = 2\pi n \qquad n = 0, 1, 2, \ldots$$

$$\boxed{t = \frac{2\pi \hbar}{\sqrt{5}\,a}\, n} \qquad n = 0, 1, 2 \ldots$$

There are no times at which P_B or $P_C = 1$.

Consider an infinite square well potential as shown below.

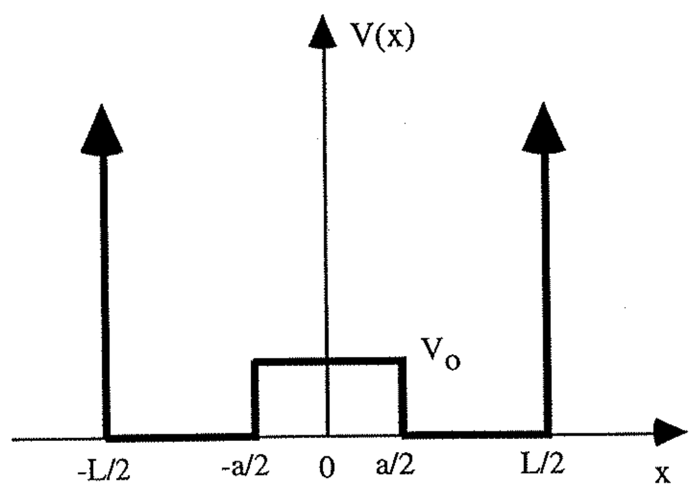


a) What is the ground state wave function for two identical bosons of mass m confined to the box?

b) What is the ground state wave function for two identical fermions of mass m confined to the box?

c) What are the ground state energies for cases (a) and (b)?

d) Consider a small perturbation to this potential, as shown below:



Find the first order corrections to the ground state energies for both cases (a) and (b). You need not do any of the integrations (leave your results in integral form).

#5.

The infinite square well has sin + cos solns

$$\phi_n = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 3, 5, 7, \cdots 2j+1$$

$$\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad n = 2, 4, 6, \cdots 2j$$

$$H = \frac{P^2}{2m} \qquad (\text{since } V = 0 \text{ in well})$$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$H\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}\sqrt{\frac{2}{L}}\left(\frac{n\pi}{L}\right)^2(-)\cos\left(\frac{n\pi x}{L}\right) = E\sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

Ground state is $\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right)$, $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$

a) If we put 2 bosons in well, then total wave function must be symmetric with respect to exchange of 2 particles

$$\Rightarrow |\psi_{Ground, boson}\rangle = |\psi_1(1)\rangle\,|\psi_1(2)\rangle$$

↑ particle 1 ↑ particle 2

or $$\boxed{\phi_b(x_1, x_2) = \frac{2}{L}\cos\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)}$$

b) If we put 2 fermions in well, then state must be antisymmetric. If you try to put both particles in state $|\psi_1\rangle$, then antisymmetrization gives a null wave-function. This is just the Pauli exclusion principle. Thus must put one particle in ϕ_1, one in ϕ_2 and antisymmetrize:

$$|\psi_{\text{Ground, fermion}}\rangle = \frac{1}{\sqrt{2}}\left[|\psi_1(1)\rangle|\psi_2(2)\rangle - |\psi_2(1)\rangle|\psi_1(2)\rangle\right]$$

or

$$\boxed{\phi_f(x_1, x_2) = \frac{\sqrt{2}}{L}\left[\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{2\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)\right]}$$

c) Total Hamiltonian $H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m}$

$$= \frac{-\hbar^2}{2m}\left(\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2}\right)$$

$$H\psi = E\psi$$

$$\Rightarrow H\phi_b = \frac{-\hbar^2}{2m}\left(-\frac{\pi^2}{L^2} - \frac{\pi^2}{L^2}\right)\phi_b$$

$$\Rightarrow \boxed{E_b = \frac{\hbar^2\pi^2}{mL^2} = 2E_1}$$

$$H\phi_f = -\frac{\hbar^2}{2m}\left(\frac{\sqrt{2}}{L}\right)\left\{-\frac{\pi^2}{L^2}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \frac{4\pi^2}{L^2}\sin\left(\frac{2\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)\right.$$

$$\left. - \frac{4\pi^2}{L^2}\cos\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \frac{\pi^2}{L^2}\sin\left(\frac{2\pi x_1}{L}\right)\cos\left(\frac{\pi x_2}{L}\right)\right\}$$

$$= \frac{\hbar^2}{2m}\left[\frac{\pi^2}{L^2} + \frac{4\pi^2}{L^2}\right]\phi_f$$

$$\Rightarrow \boxed{E_f = \frac{5\hbar^2\pi^2}{2mL^2} = E_1 + E_2}$$

d) Perturbation $H' = V_0 \quad ; \; -\frac{a}{2} \le x \le \frac{a}{2}$

$= 0 \qquad$ elsewhere

First order perturbation theory gives

$$E_n^{(1)} = H'_{nn} = \langle \psi_n | H' | \psi_n \rangle$$

which for our case would give:

$$E_b^{(1)} = \langle \phi_b | H' | \phi_b \rangle \qquad \underline{\text{for bosons}}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_2 \;\; \phi_b^*(x_1, x_2) \, V_0 \, \phi_b(x_1, x_2)$$

$$= V_0 \cdot \frac{4}{L^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x_1}{L}\right) dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x_2}{L}\right) dx_2$$

$$\boxed{E_b^{(1)} = V_0 \frac{4}{L^2} \left[\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x}{L}\right) dx \right]^2}$$

$$E_f^{(1)} = \langle \phi_f | H' | \phi_f \rangle \qquad \text{for fermions}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_2 \;\; \phi_f^*(x_1, x_2) \, V_0 \, \phi_f(x_1, x_2)$$

$$= V_0 \cdot \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_2 \left(\phi_1^*(x_1)\,\phi_2^*(x_2) - \phi_2^*(x_1)\,\phi_1^*(x_2) \right) \cdot \left(\phi_1(x_1)\,\phi_2(x_2) \right.$$

$$\left. \phi_?(x_?)\,\phi_?(x_?) \right)$$

$$E_f^{(1)} = \frac{V_0}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} dx_2 \Big\{ |\phi_1(x_1)|^2 |\phi_2(x_2)|^2 + |\phi_2(x_1)|^2 |\phi_1(x_2)|^2 -$$

$$\phi_2^*(x_1)\,\phi_1(x_1)\,\phi_1^*(x_2)\,\phi_2(x_2) - \phi_1^*(x_1)\,\phi_2(x_1)\,\phi_2^*(x_2)\,\phi_1(x_2) \Big\}$$

Since $\phi_1(x)$ is even and $\phi_2(x)$ is odd, the last 2 terms give zero, leaving:

$$E_f^{(1)} = \frac{V_0}{2} \cdot \frac{4}{L^2} \cdot 2 \left[\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x}{L}\right) dx \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin^2\left(\frac{2\pi x}{L}\right) dx \right]$$

$$\boxed{E_f^{(1)} = V_0 \frac{4}{L^2} \left[\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{\pi x}{L}\right) dx \int_{-\frac{a}{2}}^{\frac{a}{2}} \sin^2\left(\frac{2\pi x}{L}\right) dx \right]}$$

Consider a system of N independent, distinguishable, spin 1/2 particles at temperature, T, in the presence of a uniform external magnetic field, H.

Derive an expression for the entropy, S, of the system in terms of the system's partition function, Z, and show that S goes to zero, as T goes to zero.

As usual, the partition function, Z, can be written

$Z = \sum e^{-\beta E} \approx \int e^{-\beta E}\Omega(E)dE \approx e^{-\beta \overline{E}}\Omega(\overline{E})\delta E$, where $\Omega(E)dE$ is the number of states with energy between E and E+dE. The sharpness of the distribution for any reasonable number of particles allows us to approximate the logarithm of the partition function by

$$\ln Z \approx -\beta\overline{E} + \ln\Omega(\overline{E}) + \ln(\delta E),$$

where $\overline{E}$ is the average energy.

The definition of entropy, S, is given by $S = k\ln\Omega(E)$, and so we can rewrite the above expression as

$$S = \frac{\overline{E}}{T} + k\ln Z$$

Now, the average energy, $\overline{E}$, can be written in terms of the partition function, Z, as

$$\overline{E} = -\frac{\partial}{\partial\beta}\ln Z$$

Thus, the entropy of the system can be written totally in terms of the partition function, Z, as

$$S = k\ln Z + kT\frac{\partial}{\partial T}\ln Z$$

For the case of N non-interacting spin 1/2 particles, the partition function is particularly simple. It's just

$Z = (e^{\alpha H/T} + e^{-\alpha H/T})^N$, and so in the limit that T goes to zero,

we have

$$S_{T\to 0} = Nk\left\{\frac{\alpha H}{T} - \frac{\alpha H}{T}\tanh\left(\frac{\alpha H}{T}\right)\right\} \to 0$$

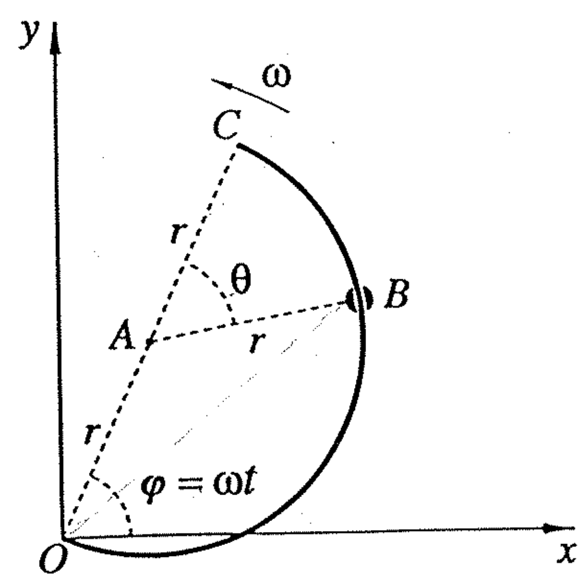
OSU Physics Dept. Comp. Exam #78 Sept 30 - Oct 1, 1996 **PROB 7**

A stiff wire formed into a semi-circle having radius $r = AO = AB = AC$, as shown in the figure at the bottom, rotates counter-clockwise with angular velocity ω about a vertical axis that passes through one of the wire's ends (O), and is perpendicular to the plane of the semi-circle. Consider a bead (B) that can slide with no friction along the wire:

(a) Determine the bead's equation of motion using an appropriate formalism;

(b) Suppose that the bead is initially placed at exactly the rotation center (point O, $\theta = 180°$), which is the point of unstable equilibrium - and then given a slight push toward the other end of the wire. What is the speed of the bead in the laboratory frame at the moment it slides off the wire?

Hints: You will probably not be able to find the exact solution of the equation you should get in part (a) (it's not an elementary function). But after finding the equation, you should notice that it is very similar to the equation of a physical system you certainly know very well..... After realizing that, you should be able to find the answer to part (b).



#7

CLASSICAL MECHANICS, PROBLEM II — SOLUTION

The bead in this problem has only one degree of freedom — its motion is fully determined by the $\theta(t)$ dependence. So, the θ angle is a certainly a "good" generalized coordinate for analysis in terms of Lagrange's Equation.

Since there are no applied forces, and the potential energy is constant, the Lagrange's Equation in the present problem takes a simple form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = 0, \tag{1}$$

where T is the kinetic energy. In order to find T and the $\partial T/\partial\theta$ and $\partial T/\partial\dot{\theta}$ derivatives, it's convenient to use the Cartesian coordinates xy. From the figure it can be readily seen that the angle beetwen the radius vector connecting the bead with the semicircle center (the AB line) and the x axis is $\varphi - \theta = \omega t - \theta$. Using simple trigonometry, one can write:

$$\begin{aligned} x &= r\cos\omega t + r\cos(\omega t - \theta) \\ y &= r\sin\omega t + r\sin(\omega t - \theta), \end{aligned}$$

from which it follows that:

$$\begin{aligned} \dot{x} &= -r\omega\sin\omega t - r(\omega - \dot{\theta})\sin(\omega t - \theta) \\ \dot{y} &= \omega r\cos\omega t + r(\omega - \dot{\theta})\cos(\omega t - \theta). \end{aligned}$$

Furthermore:

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{mr^2}{2}\left\{\omega^2 + (\omega - \dot{\theta})^2 + 2\omega(\omega - \dot{\theta})\left[\cos\omega t\cos(\omega t - \theta) + \sin\omega t\sin(\omega t - \theta)\right]\right\} \\ &= \frac{mr^2}{2}\left[\omega^2 + (\omega - \dot{\theta})^2 + 2\omega(\omega - \dot{\theta})\cos(\omega t - \omega t + \theta)\right] \\ &= \frac{mr^2}{2}\left[\omega^2 + (\omega - \dot{\theta})^2 + 2\omega(\omega - \dot{\theta})\cos\theta\right], \end{aligned}$$

so that:

$$\frac{\partial T}{\partial \theta} = -mr^2\omega(\omega - \dot{\theta})\sin\theta \tag{2}$$

and:

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2(\dot{\theta} - \omega - \omega\cos\theta).$$

From the latter result one obtains:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = mr^2(\ddot{\theta} + \omega\dot{\theta}\sin\theta). \tag{3}$$

By inserting Eqs. (2) and (3) into Eq. (1), one gets the Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = mr^2\ddot{\theta} + mr^2\omega^2\sin\theta = 0,$$

which can be rewritten as:

$$\frac{d^2\theta}{dt^2} = -\omega^2 \sin\theta. \tag{4}$$

The well-known equation of motion of a simple pendulum with the length l has the form:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta \tag{5}$$

So, from Eq. (4) it follows that the $\theta(t)$ dependence for the bead, before it slides off the wire, is the same as in the case of a single pendulum of length $l = g/\omega^2$ (if the wire had the shape of a full circle instead of a semicircle, the bead would oscillate about the B point).

Unfortunately, the simple pendulum equation cannot be integrated in terms of elementary functions, so that determining the exact $\theta(t)$and $\dot{\theta}(t)$ dependence for an arbitrary set of initial values, $\theta(t = t_0) = \theta_0$, and $\dot{\theta}(t = t_0) = \Omega_0$, is a complicated problem.

But if the bead's motion starts at $\theta = 180°$, and we only need to find the value of $\dot{\theta}$ for $\theta = 0$, then the recipe is not difficult — one can take advantage of the the analogy with the simple pendulum. If a pendulum of length l is held at upright position ($\theta = 180°$), its potential energy (relative to its lowermost position, i.e., $\theta = 0$) is $2mgl =$. When the pendulum reaches the $\theta = 0$ position, all this energy is converted into kinetic energy, so that the *linear* velocity V is such that:

$$\frac{mV^2}{2} = 2mgl \quad \Longrightarrow \quad V = 2\sqrt{gl},$$

and the angular velocity at this instant is:

$$\dot{\theta} = \frac{V}{l} = 2\sqrt{\frac{g}{l}}.$$

Well, if this is the *theta* solution of Eq. (6) for the special case we discuss, then the solution of Eq. (5) for the same initial conditions can be obtained simply by replacing the g/l coefficient with ω^2. In other words, the solution for the bead is:

$$\dot{\theta}(\theta = 0) = 2\omega.$$

At this very instant, the bead's linear velocity in the rotating wire frame is:

$$\mathcal{U} = \dot{\theta}r = 2\omega r.$$

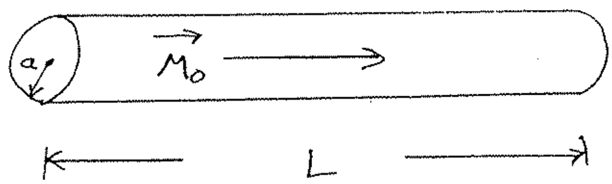
In the laboratory frame, this velocity ads up to the linear velocity of the wire's free end, $\mathcal{U}'$, which also is:

$$\mathcal{U}' = 2\omega r.$$

Since these two velocity vectors are parallel, the answer to our question is: after sliding off the wire, the bead flies away with the speed $4\omega r$, twice the linear speed of the rotating wire's free end.

A cylinder of length L and radius a is uniformly magnetized along its z axis. The magnetization is $\vec{M}_0$, a known quantity. Calculate $\vec{B}$ and $\vec{H}$ along the z axis, both inside and outside the cylinder. Plot your results as functions of z.

#8



The magnetization $\vec{M}$ produces surface "charges" on the ends of the cylinder $\sigma_M = \hat{n} \cdot \vec{M}$.

$$\Phi_M = \int_S \frac{\sigma_M(\vec{r}\,')\, da'}{|\vec{r} - \vec{r}\,'|}$$

We take the origin, $z=0$, at the left end of the cylinder. Then

$$\Phi_M = \int_0^{2\pi} d\varphi' \int_0^a \frac{(-M_0)\,\rho' d\rho'}{(\rho'^2 + z^2)^{1/2}} + \int_0^{2\pi} d\varphi' \int_0^a \frac{(+M_0)\,\rho' d\rho'}{(\rho'^2 + (L-z)^2)^{1/2}}$$

$$= -2\pi M_0 \left[\sqrt{a^2 + z^2} - |z| \right] + 2\pi M_0 \left[\sqrt{a^2 + (L-z)^2} - |L-z| \right]$$

$$= 2\pi M_0 \left[\sqrt{a^2 + (L-z)^2} - \sqrt{a^2 + z^2} + |z| - |L-z| \right]$$

Since there are no true currents $\vec{H} = -\vec{\nabla}\Phi_M$ gives

$\vec{H}$ everywhere. On the z-axis there is only one component

$$H_z = -\frac{\partial \Phi}{\partial z} = 2\pi M_0 \left[\frac{(L-z)}{\sqrt{a^2+(L-z)^2}} + \frac{z}{\sqrt{a^2+z^2}} \right]$$

$$-4\pi M_0 \quad \text{if} \quad 0 \le z \le L$$

Inside $\quad B_z = H_z + 4\pi M_0$

$$= 2\pi M_0 \left[\frac{(L-z)}{\sqrt{a^2+(L-z)^2}} + \frac{z}{\sqrt{a^2+z^2}} \right]$$

Outside $\quad B_z = H_z = \quad$ same

so $$\frac{B}{4\pi M_0} = \frac{1}{2}\left\{ \frac{1-z/L}{\sqrt{a^2/L^2+(1-z/L)^2}} + \frac{z/L}{\sqrt{a^2/L^2+(z/L)^2}} \right\}$$

Inside the bar $$\frac{H}{4\pi M_0} = \frac{B}{4\pi M_0} - 1$$