PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #34

September 29, 1979

General Instructions

This Comprehensive Examination for Fall 1979 (#34) consists of six problems of equal weight (20 points each). Please check that you have all of them. Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level.

Work carefully, indicate your reasoning briefly and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit--especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, use one bluebook per problem, and be certain that your assigned student letter (but not your name) is on every booklet.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers except pencil (or pen) and bluebook, on the floor. The accompanying booklet contains data and formulas which you may find useful. Please return it at the end of the exam.

1. Consider a system whose eigenstates are $|\alpha\rangle$ and $|\beta\rangle$ such that $\mathcal{H}_0|\alpha\rangle = E_1^0|\alpha\rangle$ and $\mathcal{H}_0|\beta\rangle = E_2^0|\beta\rangle$, where $\mathcal{H}_0$ is a particular Hamiltonian having eigenvalues E_1^0 and E_2^0. Consider an additional interaction, characterized by a Hamiltonian $\mathcal{H}_1$ of magnitude comparable to $\mathcal{H}_0$, to be "switched on" in the system. We also know that $\mathcal{H}_1$ has the properties:

$$\mathcal{H}_1|\alpha\rangle = a_1|\alpha\rangle + b_1|\beta\rangle \tag{1}$$

$$\mathcal{H}_1|\beta\rangle = a_2|\alpha\rangle + b_2|\beta\rangle \tag{2}$$

$$a_1 = -E_1^0\,, \quad b_2 = -E_2^0\,, \quad a_2 = b_1 = \frac{E_1^0 + E_2^0}{2}\,. \tag{3}$$

(10) (a) Determine the energy eigenvalues of the total Hamiltonian $(\mathcal{H}_0 + \mathcal{H}_1)$.

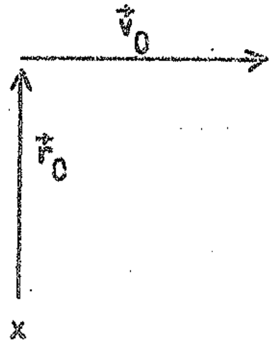
(10) (b) Determine the normalized eigenstates of the total Hamiltonian.

2. There is an attractive central field of force, of constant magnitude mC, with its center at the origin. A particle of mass m is injected into this field with a velocity $\vec{v}_0$ at right angles to the initial position vector $\vec{r}_0$. Let β be the ratio of twice the initial kinetic energy to the initial potential energy (measured with respect to the origin).

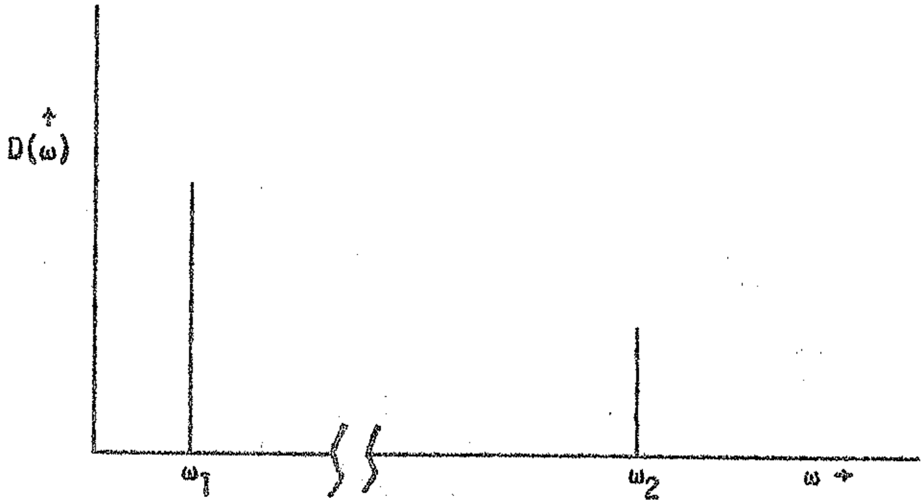
(5) (a) For what value of β will the ensuing motion be circular?

(10) (b) When β is slightly less than the value for circular motion in part (a), what is the half period of radial motion (i.e., the time to go from the largest to the smallest distance from the origin)?

(5) (c) For the time of part (b) (you may call it τ), what is the corresponding angle through which the particle's radius vector has turned?



3. "Soft stuff" is a three-dimensional, bimodal, insulating solid. Specifically, its phonon density of states $D(\omega)$ consists of two spikes; one at frequency ω_1, the other at $\omega_2 \gg \omega_1$ with there being twice as many states at the lower frequency ω_1 as at the higher frequency ω_2 (see figure below).



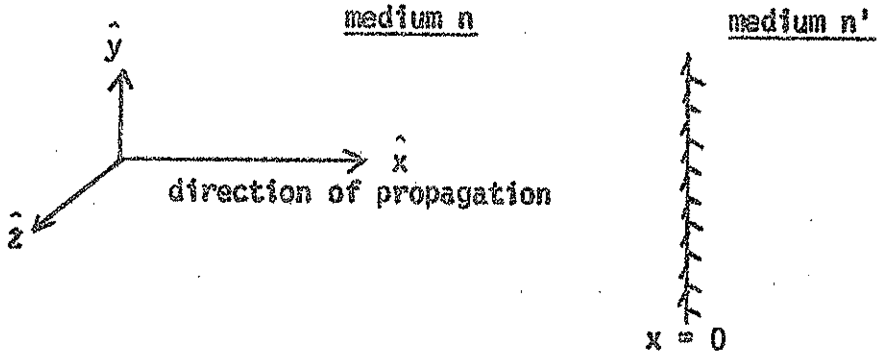
( ) (a) Derive an expression for the specific heat at constant volume (C_v) of this material.

(6) (b) Sketch a rough (approximate) graph of C_v in the following regions:

(i) $T \ll \dfrac{\hbar\omega_1}{k}$, (ii) $\dfrac{\hbar\omega_1}{k} \ll T \ll \dfrac{\hbar\omega_2}{k}$,

(iii) $T \gg \dfrac{\hbar\omega_2}{k}$

4. Consider a plane, linearly polarized, electromagnetic wave normally incident on a dielectric interface. Assume that it propagates from a medium of refractive index n into a medium of index n' as shown:



For simplicity take the direction of polarization to be the y direction.

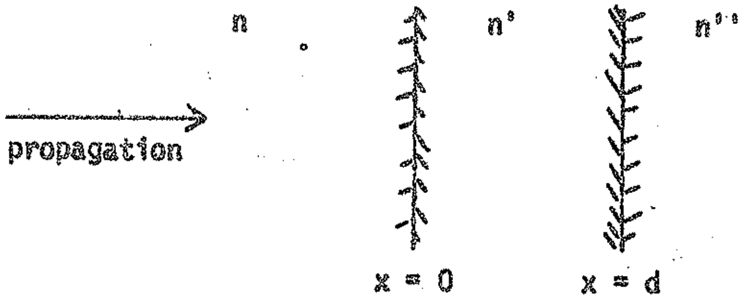
(4) (a) Write down the solutions to Maxwell's equations corresponding to the $\vec{E}$ and $\vec{B}$ fields in the two regions.

) (b) State the boundary conditions satisfied by these fields at the interface.

(4) (c) Derive an expression for the fraction of incident energy reflected by the surface.

(4) (d) Suppose $n < n'$. What is the phase angle between the incident and reflected waves? Answer the same question for the case $n > n'$. (Don't guess the answer; derive it from (a) and (b) above!)

(4) (e) Now assume that there are three regions with indices of refraction $n < n' < n''$ as shown.



Re-derive the expression for the fraction of incident energy that is reflected into the left-hand region. Assume $n < n' < n''$ and neglect multiple internal reflections.

5. Sometimes when light scatters from an atom or molecule there is an internal state change and the light gains or loses energy (Raman effect). Rotational transitions for diatomic molecules are governed by a selection rule $\Delta K = 2$, where $K(K+1)\hbar^2$ is the eigenvalue of the square of the angular momentum operator for the atoms about the center of mass.

(3) (a) Show that the energy shift for the observed lines should be given by

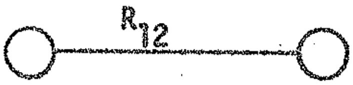
$$E = \pm W(2K+3)$$

where W is a constant and K is the rotational quantum number of the lower state.

(7) (b) If the two nuclei are identical, the requirement of symmetry or antisymmetry of the molecular wavefunction under exchange of the nuclei forbids certain combinations of K and nuclear spin. The rotational Raman Spectrum of the O_2 molecule is observed to fit the formula

$$E = \pm W(4n+5)$$

with W constant and n any non-negative integer. Using the fact that ^{16}O is a spin 0 nucleus and the electronic wavefunction is an odd function of R_{12} (the vector from nucleus 1 to nucleus 2), what conclusions can you draw about the statistics obeyed by the ^{16}O nucleus?



(10) (c) In the H_2 molecule, the nuclei are spin $\frac{1}{2}$ protons obeying Fermi statistics and the electronic wavefunction is even. Show that if a gas of molecules is formed from protons of random spin orientation, there will be an alternation in intensity of the rotational Raman lines of about 3 to 1.

6. Consider a container holding one cubic centimeter of air at standard temperature and pressure.

(15) (a) Approximately how much less probable is it to find 51% of the molecules in one half of the volume than 50% in the same half? (That is, calculate the relative probability of the two configurations.)

(5) (b) What is the difference in entropy between the two configurations?

Problem #1: Solution

$(* \equiv \alpha, \quad \dagger \equiv \beta)$

Matrix of the Hamiltonian:

$$\begin{pmatrix} \langle *|\mathcal{H}|*\rangle & \langle *|\mathcal{H}|\dagger\rangle \\ \langle \dagger|\mathcal{H}|*\rangle & \langle \dagger|\mathcal{H}|\dagger\rangle \end{pmatrix}$$

where
$$\begin{aligned} \langle *|\mathcal{H}_0+\mathcal{H}_1|*\rangle &= E_1^0 + a_1 \\ \langle *|\ \text{''}\ |\dagger\rangle &= a_2 \\ \langle \dagger|\ \text{''}\ |*\rangle &= b_1 \\ \langle \dagger|\ \text{''}\ |\dagger\rangle &= E_2^0 + b_2 \end{aligned}$$

i.e.
$$\begin{pmatrix} E_1^0+a_1 & a_2 \\ b_1 & E_2^0+b_2 \end{pmatrix}$$

We want the representation in which this is diagonal, or

$$\begin{vmatrix} (E_1^0+a_1)-\lambda & a_2 \\ b_1 & (E_2^0+b_2)-\lambda \end{vmatrix} = 0$$

yields
$$\lambda^2 - \left[(E_1^0+a_1)+(E_2^0+b_2)\right]\lambda + (E_1^0+a_1)(E_2^0+b_2) - a_2 b_1 = 0$$

Which, under conditions given becomes

$$\lambda_{1,2} = \pm \left(\frac{E_1^0 + E_2^0}{2} \right).$$ Eigenvalues of the total energy (Hamiltonian).

Eigenstates:

$$(E_1^0 + a_1) c_* + a_2 c_\dagger = \lambda_1 c_*$$

$$c_* = \left(\frac{-a_2}{E_1^0 + a_1 - \lambda_1} \right) c_\dagger = \left(\frac{-a_2}{-\lambda_1} \right) c_\dagger$$

$$\therefore \quad c_* = c_\dagger$$

let $c_* = 1$, then $c_\dagger = 1$ and normalization requires

$$\mu^2 (1^2 + 1^2) = 1 \quad \text{or} \quad \mu = \frac{1}{\sqrt{2}}$$

so $|*\rangle' = \frac{1}{\sqrt{2}} |*\rangle + \frac{1}{\sqrt{2}} |\dagger\rangle$

Similarly,

$$b_1 c_* + (E_2^0 + b_2) c_\dagger = \lambda_2 c_\dagger$$

yields $|\dagger\rangle' = \frac{1}{\sqrt{2}} |*\rangle - \frac{1}{\sqrt{2}} |\dagger\rangle.$

Solution: Dynamics, Problem #2

Given: $\beta = 2KE/PE$, $F = -\frac{dV}{dr} = \text{constant} = -mc$

$\therefore V = mcr + \cancel{\text{Const.}} \rightarrow 0$

$$\beta = \frac{2\,\frac{1}{2} m v_0^2}{m c r_0} = \frac{v_0^2}{c r_0}$$

(A) Circular motion $\Rightarrow$ Force = mass × central acceleration

$$mc = m\frac{v_0^2}{r_0}$$

$$\underline{1 = v_0^2/c r_0 = \beta}$$

(B) Small perturbation on circular orbit : SHM about circle

$a = -\omega^2 x$ fo

1D equivale

$$V_{eff} = mcr + \frac{1}{2}\frac{L^2}{mr^2}$$

where $L = m r_0^2 \dot{\theta}$, $L^2 = m^2 r_0^2 v_0^2$

If $\beta = 1$, $\partial V_{eff}/\partial r = 0$ i.e. circular orbit at r_0

$$\beta = v_0^2/c r_0 = r_0^2 \dot{\theta}^2/c r_0 = L^2/M^2 c r^3$$

$$F = -\frac{\partial V}{\partial r} = \frac{L^2}{mr^3} - mc$$

$$F(r = r_0 + d) = \frac{L^2}{m(r_0+d)^3} - mc \simeq -\Big\{ \underbrace{mc - \frac{L^2}{m r_0^3}}_{= 0 \text{ via equilib}} + \frac{3dL^2}{r_0 m r_0^3} + \cdots$$

$$F = -\frac{3L^2}{m r_0^4}\, d = ma$$

$$-\frac{3L^2}{m r_0^4}\, d = m\ddot{d} \Big\} \text{ this is SHM} \quad \omega^2 = \frac{3L^2}{m^2 r_0^4} , \ \omega = \frac{\sqrt{3}\,L}{m r_0^2} = \frac{\sqrt{3}\,Mr}{mr}$$

$$\Rightarrow \boxed{\frac{T}{2} = \text{half period} = \frac{\pi r_0}{\sqrt{3}\,v_0}}$$

(C) $\underline{L = \text{Constant}}$

$$= m r^2 \dot{\theta} = m r_0 v_0$$

$$\dot{\theta} \simeq \frac{v_0}{r_0} \ (\text{Constant})$$

$$\Delta\theta \simeq \ddot{\theta}\,\Delta t = \frac{v_0}{r_0}\,\frac{r_0}{v_0}\,\frac{\pi}{\sqrt{3}} = \pi/\sqrt{3}$$

Solution, Problem #3

The problem asks for the allowed energies of a set of 3N independent oscillators.

$$E_{\{n_1,n_2,\ldots,n_{3N}\}} = -N\eta + \sum_{i=1}^{3N} n_i\hbar\omega_i$$ where $N\eta$ is a zero point energy, ω_i characterizes an allowed mode frequency and n_i, the number of phonons of frequency ω_i.

Partition function (from which all blessings flow):

$$Z = \sum_{\{n_1,n_2,\ldots,n_{3N}\}} e^{-\beta E_{n_1,n_2,\ldots,n_{3N}}} = e^{\beta N\eta}\sum_{\{n_1,n_2,\ldots,n_{3N}\}} e^{-\beta\sum_i n_i\hbar\omega_i}.$$

Since for phonons (quanta) each n_i can range over all integer values from 0 to ∞ we get

$$Z = e^{\beta N\eta}\left(\sum_{n_1=0}^{\infty} e^{-\beta\hbar\omega_1 n_1}\right)\left(\sum_{n_2=0}^{\infty} e^{-\beta\hbar\omega_2 n_2}\right)\cdots\left(\sum_{n_{3N}=0}^{\infty} e^{-\beta\hbar\omega_{3N} n_{3N}}\right).$$

or $$Z = e^{\beta N\eta}\left(\frac{1}{1-e^{-\beta\hbar\omega_1}}\right)\left(\frac{1}{1-e^{-\beta\hbar\omega_2}}\right)\cdots\left(\frac{1}{1-e^{-\beta\hbar\omega_{3N}}}\right)$$

from which

$$\ln Z = N\eta\beta + \sum_{i=1}^{3N}\ln\left(\frac{1}{1-e^{-\beta\hbar\omega_i}}\right)$$

or $$\ln Z = N\eta\beta - \sum_{i=1}^{3N} \ln(1 - e^{-\beta\hbar\omega_i})$$

Our bimodal phonon spectrum allows us to write

$$\ln Z = N\eta\beta - \left[2N\ln(1-e^{-\beta\hbar\omega_1}) + N\ln(1-e^{-\beta\hbar\omega_2})\right]$$

Now $\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z$

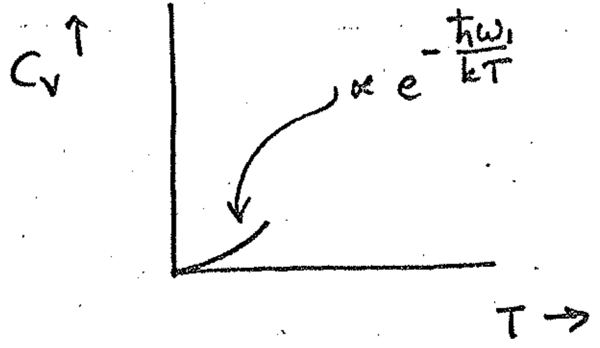
and $C_v = \frac{\partial}{\partial T}\langle E\rangle = -\frac{1}{kT^2}\frac{\partial}{\partial\beta}\langle E\rangle = \frac{1}{kT^2}\frac{\partial^2}{\partial\beta^2}\ln Z$

or

$$C_v = \frac{1}{kT^2}\left\{\frac{2N(\hbar\omega_1)^2 e^{-\beta\hbar\omega_1}}{(1-e^{-\beta\hbar\omega_1})^2} + \frac{N(\hbar\omega_2)^2 e^{-\beta\hbar\omega_2}}{(1-e^{-\beta\hbar\omega_2})^2}\right\}$$

Region (i) $T \ll \frac{\hbar\omega_1}{k}$ means $e^{-\beta\hbar\omega_1}$ and $e^{-\beta\hbar\omega_2} \to 0$

with the first term holding out because $\omega_1 \ll \omega_2$.
Denominators → 1.



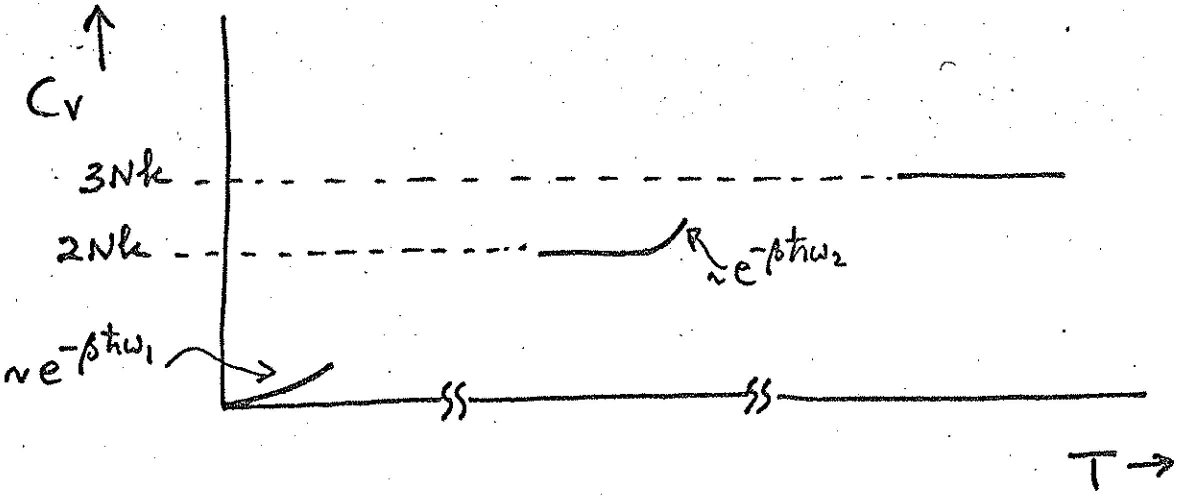
(ii) $\frac{\hbar\omega_1}{k} \ll T \ll \frac{\hbar\omega_2}{k}$

Numerator of the ω_1 term behaves as $e^{-\beta\hbar\omega_1} \to 1$.
Denominator $(1 - e^{-\beta\hbar\omega_1})^2 \to \left(\frac{\hbar\omega_1}{kT}\right)^2$

so ω_1 term contributes $2Nk$ while ω_2 term begins to contribute as $e^{-\beta\hbar\omega_2}$

(iii) Here both ω_1 and ω_2 terms contribute "all they can" as ω_1 term did alone in (ii)

i.e. $2Nk + Nk = 3Nk$



#4 Solutions

Optics - Solutions

Incident wave

$$\vec{E} = E_0 \hat{e}_y e^{ikx-i\omega t}$$

$$\vec{B} = E_0 \sqrt{\mu\epsilon}\, \hat{e}_z e^{ikx-i\omega t}$$

Refracted wave

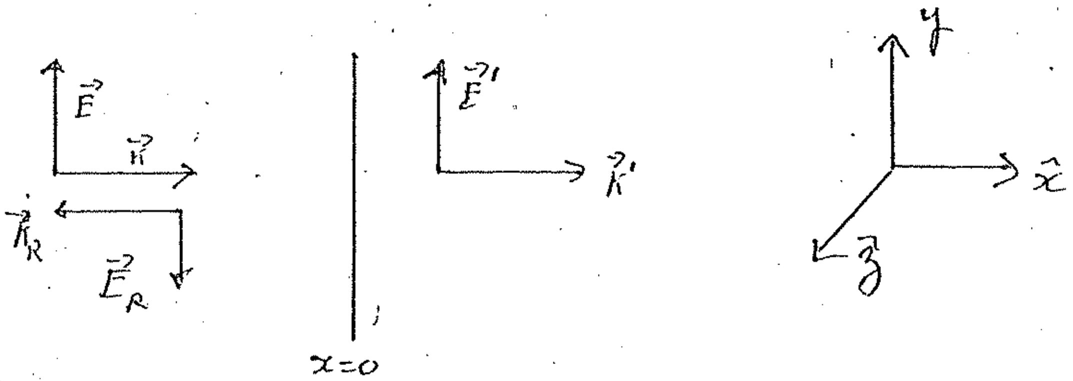
$$\vec{E}' = -E_0' \hat{e}_y e^{ik'x-i\omega t}$$

$$\vec{B}' = E_0' \sqrt{\mu'\epsilon'}\, \hat{e}_z e^{ik'x-i\omega t}$$

Reflected wave

$$\vec{E}_R = E_{0R} \hat{e}_y e^{-ikx-i\omega t}$$

$$\vec{B}_R = E_{0R} \sqrt{\mu\epsilon}\, \hat{e}_z e^{-ikx-i\omega t}$$



(b) The normal components of $\vec{B}$ and $\vec{D}$ as well as the tangential comp. of $\vec{H}$ and $\vec{E}$ are continuous. There are no normal comp. Hence at $x=0$ we have

Continuity in $\vec{E}$ $\Rightarrow$ $E_0 - E_{0R} = E_0'$

continuity in $\vec{H}$ $\Rightarrow$ $\sqrt{\frac{\epsilon}{\mu}} E_0 + \sqrt{\frac{\epsilon'}{\mu'}} E_{0R} = \sqrt{\frac{\epsilon'}{\mu'}} E_0'$

assume $\mu = \mu'$ $\sqrt{\epsilon/\epsilon'} = n/n'$

$$1 + \frac{E_{0R}}{E_0} = \frac{n'}{n}\frac{E_0'}{E_0} = \frac{n'}{n}\left(1 - \frac{E_{0R}}{E_0}\right)$$

$$\frac{E_{0R}}{E_0} = \frac{n'-n}{n'+n}$$

If $n' > n$ then $E_{0R}/E_0 > 0$. This corresponds to phase reversal as shown in the sketch. Now since the energy flow is $\sim |E|^2$ the fraction reflected is

$$\left|\frac{E_{0R}}{E_0}\right|^2 = \left|\frac{n'-n}{n+n}\right|^2$$

From the previous equations

$$\frac{E_0'}{E_0} = 1 - \frac{E_{0R}}{E_0} = \frac{2n}{n'+n}$$

The field reflected from the second surface then is

$$\frac{E_{0R}^{(2)}}{E_0} = \frac{E_{0R}^{(2)}}{E_{0R}^{\prime(2)}} \cdot \frac{E_0^{\prime(2)}}{E_0'} \cdot \frac{E_0'}{E_0}$$

$$= \frac{2n'}{n'+n} \cdot \frac{n''-n'}{n''+n'} \cdot \frac{2n}{n'+n}$$

The phase lag of the wave reflected at the second surface relative to that reflected at the first is $\phi = 2dk$.

$$\vec{E}_R = \vec{E}_R^{(1)} + \vec{E}_R^{(2)}$$

$$= \vec{e}_y \left(E_{0R}^{(1)} + E_{0R}^{(2)} e^{i\phi}\right) e^{-ikx - i\omega t}$$

$$\left|\frac{E_{0R}}{E_0}\right|^2 = \left|\frac{n'-n}{n'+n} + e^{i\phi} \frac{4nn'}{(n+n')^2} \frac{n''-n'}{n''+n'}\right|^2$$

$\boxed{\text{Soltn to \#5 (Q.M)}}$

ⓐ Rotational States with $\Delta K = 2$

$$E_{rot} = \frac{K(K+1)\hbar^2}{2I} \quad , \quad I = \text{moment of inertia}$$

upward transition with $\Delta K = 2$

$$\Delta E = E = \frac{\hbar^2}{2I}\left[(K+2)(K+3) - K(K+1)\right] = \underline{\frac{\hbar^2}{I}(2K+3)}$$

Downward transition

$$E = -\frac{\hbar^2}{I}(2K+3)$$

//

ⓑ O₂ Molecule $\quad E = \pm \overset{w}{\hbar}(4n+5)$

$\Rightarrow 2K+3 = 4n+5$ or $K = 2n+1$ i.e. odd K

The total molecular wavefunction can be written

$$\Psi_{O_2} = \psi_{\substack{\text{electron}\\ \text{space}}}(R_{12})\, \psi_{\substack{\text{nuclear}\\ \text{space}}}\, \chi_{\substack{\text{electron}\\ \text{spin}}}\, \chi_{\substack{\text{nuclear}\\ \text{spin}}}$$

Know ∴ odd (Given) ? ? Even {must be for two Spin 0 particles}

yet since the exchange of the 2 nuclei (Bosons) is equivalent to Parity for identical nuclei, $\psi_{\substack{\text{nuclear}\\ \text{space}}} \longrightarrow (-1)^K \psi_{\substack{\text{nucl.}\\ \text{space}}}$

When we interchange nuclei, $\psi_{\substack{\text{electron}\\ \text{spin}}}$ doesn't change ∴

$$\text{Symmetry} = (-1)(-1)^K = (-1)^{K+1}$$

yet since K is odd via 4n+5, symmetry even

i.e. ^{16}O nuclei are bosons

Ⓒ The variation in intensity of different lines is obviously proportional to the number of molecules with different K values which are formed from a random sample.

Since there is a $\Delta K = 2$ rule present:

$$\frac{\text{Intensity (Even} \to \text{even)}}{\text{Intensity (odd} \to \text{odd)}} = \frac{\#\text{even } K}{\#\text{odd } K}$$

Since protons obey Fermi-Dirac statistics, the overall wavefunction must be odd.

$$\Psi = \underbrace{\psi_{\substack{el\\space}}\;\psi_{\substack{el\\spin}}}_{\text{Even}}\;\underbrace{\psi_{\substack{Nuc\\space}}}\;\psi_{\substack{nuc\\spin}}$$

K = even ⟷ anti sym.

K = odd → symmet.

In General

Number of antisymmetric spin states (of form: $S_{n_1}(1)S_{n_2}(2) - S_{n_1}(2)S_{n_1}(1)$)

$$= \underbrace{(2I+1)}_{\substack{\#\text{ possible}\\ \text{states}}}(2I) \times \tfrac{1}{2} = I(2I+1)$$

for 2nd order

Number of Symmetric ($S_{m_1}(1)S_{m_2}(2)$ form in addition to above)

$$= 2I+1 + I(2I+1) = (2I+1)(I+1)$$

$$\therefore \frac{N_{sym}}{N_{Anti}} = \frac{N_{odd}}{N_{even}} = \frac{\cancel{(2I+1)}(I+1)}{I\cancel{(2I+1)}} = \frac{I+1}{I}$$

SPIN $\frac{1}{2}$

So for $I = 1/2$, $\frac{3/2}{1/2} = 3/1$ as stated.

(for Spin 1/2 know 3 symm. vs 1 antisym)

Solution, #6

The thermodynamic probability for a classical distribution is $W = N! / \prod N_i!$.

Calculate $\ln W/W_0$ where

$$\ln W_0 = N \ln N - \sum N_i \ln N_i = N \ln N - N \ln N/2$$

$$\ln W_0 = N \ln N - (N/2 + \Delta N) \ln (N/2 + \Delta N)$$
$$- (N/2 - \Delta N) \ln (N/2 - \Delta N)$$

$$\ln(N/2 \pm \Delta N) = \ln \frac{N}{2}\left(1 \pm \frac{2\Delta N}{N}\right) \approx \ln \frac{N}{2} \pm \frac{2\Delta N}{N}$$

so $$\ln W/W_0 = -\frac{4}{N}(\Delta N)^2 = -4N\left(\frac{\Delta N}{N}\right)^2$$

1 cc. of air contains

$$N = 10^{-3} \text{ liter} \left(\frac{\text{mole}}{22.4 \text{ liters}}\right)\left(\frac{6 \times 10^{23} \text{ molecules}}{\text{mole}}\right) = 2.7 \times 10^{19} \text{ molecules}$$

$$\frac{W}{W_0} = \exp\left(-4N\left(\frac{\Delta N}{N}\right)^2\right) = \exp\left(-10^{16}\right)$$ (a small number)

the difference in entropy $\Delta S = k \ln W/W_0$

$$= 1.38 \times 10^{-23} \frac{\text{J}}{\text{deg}} \times \left(-10^{16}\right)$$

$$= -1.4 \times 10^{-7} \text{ Joules/deg}.$$