PHYSICS DEPARTMENT COMPREHENSIVE EXAMINATION #22

October 11, 1975

General Instructions

This Comprehensive Examination for Fall 1975 (#22) consists of six problems of equal weight (20 points each). Half of the problems are judged to be at intermediate undergraduate level, the other half at graduate level. Work carefully and show all your steps so that partial credit can be given liberally in case you do not complete a problem. Use no scratch paper; do all work in the bluebook, using one bluebook per problem.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers, except pencil and bluebook, on the floor.

Some information you may find useful:

$$\log N! = N \log N - N \qquad N \gg 1$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

1. The entropy of a system in thermodynamic equilibrium may be defined by $S = -k_B \sum_i p_i \log p_i$ where p_i is the probability that the system will be found in the i^{th} state, k_B is Boltzmann's constant, and the sum is taken over all states of the system.

The quantities p_i are found by the condition that they maximize the entropy function subject to the normalization side condition plus any other side conditions that specify knowledge about the system. In the important case that the mean value of the energy of the system constitutes a constraint the p_i take the values

$$p_i = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$$

where E_i is the energy of the i^{th} state of the system, $\beta = \frac{1}{k_B T}$ with T the temperature. When the p_i have this form they are called the state probabilities of a canonical ensemble.

(10) a) Starting from the definition of entropy given above and defining the partition function, Helmholtz free energy and the mean value of the energy, show that for a canonical ensemble

$$F = E - TS$$

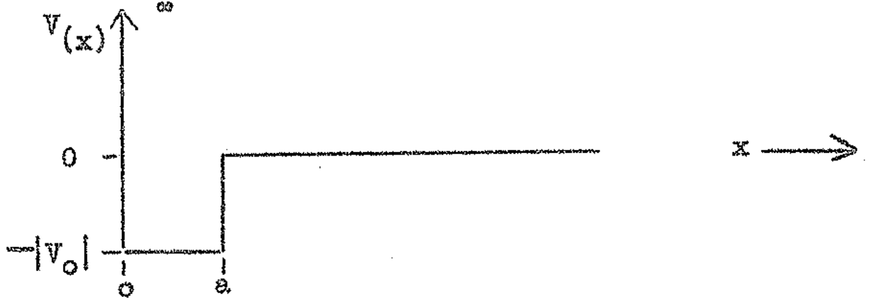
where F is the Helmholtz free energy, and
E is the mean value of the energy.

(10) b) Starting from the definition of entropy given above and by defining pressure, show that for a canonical ensemble

$$TdS = dE + PdV$$

where P is the pressure and V is the volume of the system.

2. Consider a particle of mass m moving in the following one-dimensional potential:

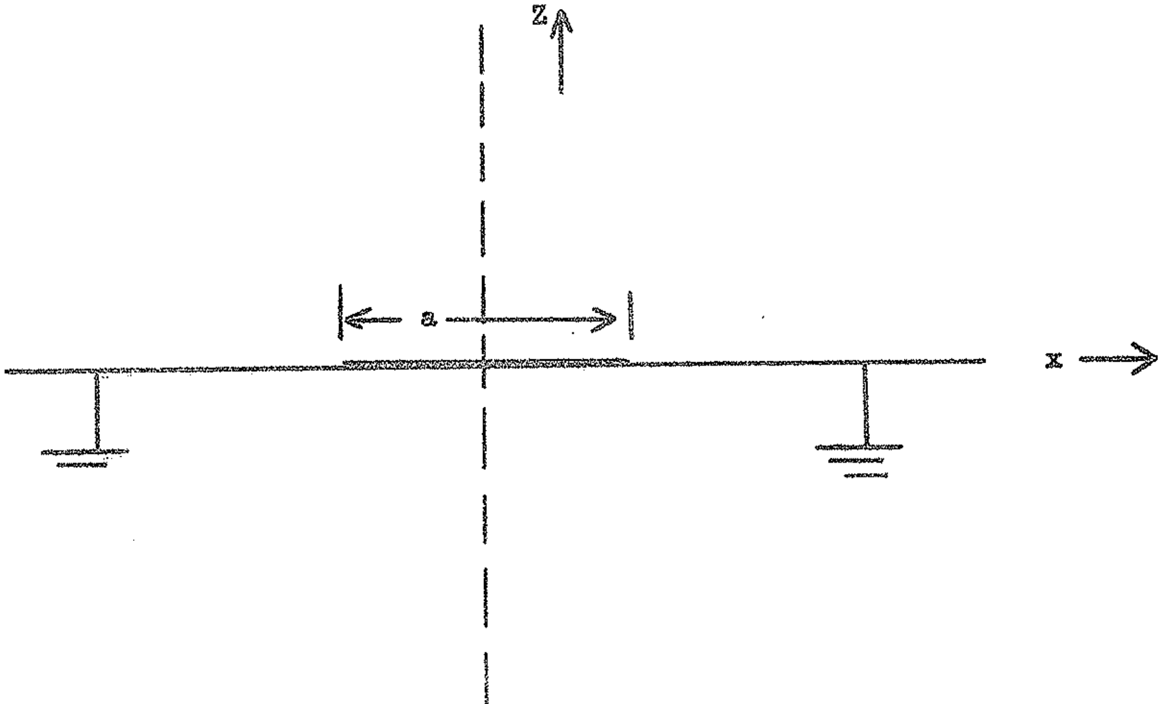


(10) a) Find an equation from which the allowed <u>bound</u> energy eigenvalues of the particle may be determined.

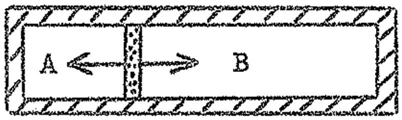
(10) b) Find the condition under which exactly n bound states exist. (HINT: Use a graphical method.)

(5) 3. a) An infinite line charge with linear density λ is located a distance z_o above an infinite grounded conducting plane. Find the electric potential everywhere above the plane.

(15) b) An infinitely long strip of width a is charged to a potential V_o and rests upon (but is insulated from, of course) an infinite grounded conducting surface. (See diagram.) Find the electric potential everywhere in the region $z>0$.
(HINT: Use the result from part <u>a</u>.)



4. Consider a gas tight but frictionless piston of small but non-negligible thermal conductivity which is free to slide in a thermally insulated cylinder.



Compartments A and B each contain a mole of an ideal monatomic gas. Initially the temperature of the gas in compartment A is T_o and that of the gas in compartment B is $3T_o$. Moreover, the system is initially in mechanical equilibrium.

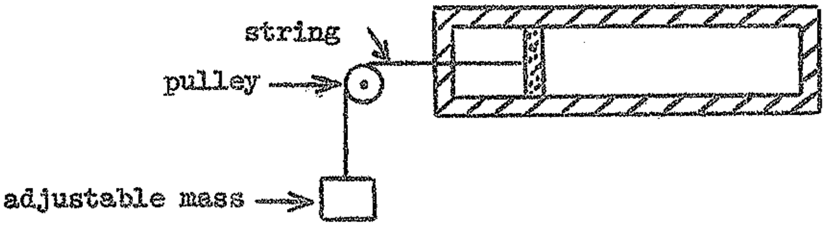
After a long time the system returns to mechanical equilibrium and reaches thermal equilibrium as well.

(2) a) Compute the ratio of volumes $\frac{V_A}{V_B}$ in the initial and final states.

(3) b) What is the final temperature of the gas in each compartment?

(4) c) Compute the entropy difference between the final and initial states.

Suppose we redesign the apparatus so that the heat transferred across the piston takes place reversibly with the cylinder still thermally insulated from its surroundings. Such an apparatus might look like



(8) d) Under these reversible conditions what would be the final temperature of the gas in each compartment?

(3) e) How much work would be done in this reversible process?

5. A system has the Hamiltonian

$$H = H_o + V(\vec{x}) \cos \omega t e^{-\alpha|t|} \qquad (\alpha > 0)$$

where α is small. H_o is time independent with energy levels E_n and corresponding wavefunctions $\Psi_n(\vec{x})$ $(n = 0, 1, 2, \ldots)$ and at time $t = -\infty$ the system is in the state $\Psi_o(\vec{x})$.

(15) a) Find, to lowest order in $V(\vec{x})$, the probability that at time $t = \infty$ the system is in the state $\Psi_m(\vec{x})$ $(m \neq 0)$. Show that it is proportional to a function $f(\omega,\alpha)$ which has the properties

$$\lim_{\alpha \to 0} f(\omega,\alpha) = 0 \quad \text{for} \quad \hbar\omega \neq E_m - E_o \quad \text{and} \quad \int_{-\infty}^{\infty} f(\omega,\alpha) d\omega = 1$$

(5) b) How would you find the probability of the system's remaining in the state $\Psi_o(\vec{x})$ at $t = \infty$?

6. At temperatures below 1000° the hydrogen molecule (H_2) can be regarded as a rigid rotator. The total wavefunction for such a molecule can be written

$$\Psi(\vec{R}_A,\vec{R}_B) = \Psi^{J,M}_{rot}(\theta,\phi)\,\Psi_N(\vec{R}_A,\vec{R}_B)$$

where $\Psi^{J,M}_{rot}$ is a rotational wavefunction with eigenvalues $\frac{\hbar^2}{2\mathcal{I}}J(J+1)$ and Ψ_N is the nuclear spin wavefunction. Here, $\vec{R}_A$ and $\vec{R}_B$ refer to the coordinates of the two nuclei, θ,ϕ are the polar and azimuthal angles of $\vec{R}_A - \vec{R}_B$, J, M are the good quantum numbers of the rotational states, and $\mathcal{I}$ is the moment of inertia of the molecule.

(2) a) What is the effect of the operator P_{AB} (the operator which permutes the nuclear coordinates) on

i) the total wavefunction Ψ

ii) the rotational wavefunction $\Psi^{J,M}_{rot}$?

(2) b) If α and β are the eigenfunctions of the spin operator S_z,

i.e. $$S_z\alpha = \frac{\hbar}{2}\alpha$$
$$S_z\beta = \frac{\hbar}{2}\beta$$

list the possible nuclear spin wavefunctions in terms of $\alpha(R_A),\beta(R_A)$ $\alpha(R_B),\beta(R_B)$. Label each nuclear spin wavefunction by its total angular momentum I and its projection I_z .

(4) c) What combinations of $\Psi^{J,M}_{rot}$ and Ψ_N will satisfy the overall symmetry requirement for Ψ ? What restrictions does this place on the allowed energy levels for the molecule. Sketch a few of the lowest lying levels indicating the quantum numbers J,I associated with them.

(NOTE: The lowest lying level is associated with parahydrogen and the next higher with orthohydrogen.)

(3) d) Write down (but do not attempt to sum) the rotational partition function for

i) orthohydrogen

ii) parahydrogen.

Why does one distinguish between the two kinds of hydrogen in writing a partition function instead of merely writing one partition function for "hydrogen"?

(6) e) Write the partition function for a gas of N_o molecules of orthohydrogen and N_p molecules of parahydrogen. From the partition function of the ortho-para mixture derive an expression which gives the equilibrium ratio of ortho hydrogen to parahydrogen at any temperature T .

(3) f) Describe briefly how this analysis would have to be modified for a discussion of D_2 . HD .

1\. (a) $S = -k_B \sum_i \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} \left[(-\beta E_i) - \log \sum_i e^{-\beta E_i} \right]$

$= -k_B \left\{ (-\beta) \langle E \rangle - \log \sum_i e^{-\beta E_i} \right\}$

Define the partition function

$$Z = \sum_i e^{-\beta E_i}$$

and Helmholtz Free Energy

$$F = -\frac{1}{\beta} \log Z$$

$$S = \frac{\langle E \rangle}{T} - \frac{F}{T}$$

&

$$F = \langle E \rangle - TS$$

(b) $dS = \left(\frac{\partial S}{\partial E}\right)_{\beta,V} dE + \left(\frac{\partial S}{\partial \beta}\right)_{E,V} d\beta + \left(\frac{\partial S}{\partial V}\right)_{E,\beta} dV$

$\left(\frac{\partial S}{\partial E}\right)_{\beta,V} = \frac{1}{T}$ $\qquad \left(\frac{\partial S}{\partial \beta}\right)_{E,V} = 0$ $\qquad \left(\frac{\partial S}{\partial V}\right)_{E,\beta} = \frac{(k_\beta)(-\beta) \sum_i \frac{\partial E_i}{\partial V} e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$

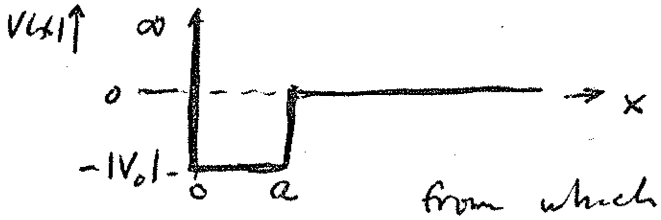
$P = \cancel{\sum} \frac{\partial E}{\partial V} = \frac{\sum \frac{\partial E_i}{\partial V} e^{-\beta E_i}}{\sum Z}$ Define $\frac{P}{T} = -\frac{1}{T} \frac{\sum_i \left(\frac{\partial E_i}{\partial V}\right) e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$

So

$$dS = \frac{1}{T} dE + \frac{P}{T} dV$$

Q.M. Undergrad

2\. Consider a particle/of mass m in the following one dimensional potential:



(10) (a) Find an ~~expression~~ equation ~~which is satisfied by~~ from which ~~the~~ the/bound allowed ~~states~~/energy eigenvalues of the particle may be ~~calcul~~ determined.

b) Find the condition for which ~~there~~ permits exactly n bound states.

(10) Solution: As with discontinuous constant potentials we solve the problem in the various regions of constant potential

HINT: TRY A GRAPHICAL METHOD

In region I

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V_0|\right]\phi_I(x) = E\,\phi_I(x)$$

In region II

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\phi_{II}(x) = E\,\phi_{II}(x)$$

For Bound states $(E<0)$ $\qquad k_1 = \sqrt{\frac{2m}{\hbar^2}(|V_0|-|E|)}$

$$\phi_1(x) = A\,e^{iK_1x} + B\,e^{-iK_1x}$$

~~[illegible]~~

$$\phi_2(x) = C\,e^{-K_2x} \qquad k_2 = \sqrt{\frac{2m}{\hbar^2}|E|}$$

Since $\phi_1(0) = 0$ $\qquad A + B = 0$

$$\phi_1 = A\sin K_1x$$

Matching function and slope at $x = a$

$$A \sin k_1 a = C e^{-k_2 a}$$

$$k_1 A \sin k_1 a = -C k_2 e^{-k_2 a}$$

The eigen value expression is

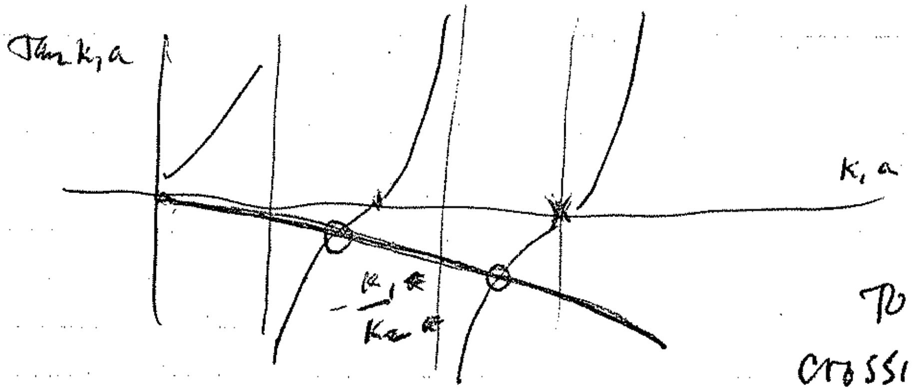
(a) $$\boxed{\frac{1}{k_1} \tan k_1 a = -\frac{1}{k_2}}$$

To solve part (b) a graphical method is called for. A not too elegant but straightforward method suggests writing

$$\tan k_1 a = -\frac{k_1}{k_2}$$

$$\tan k_1 a = -\sqrt{\left|\frac{V_0}{E}\right| - 1}$$

Roughly the two curves look like



The circles mark the allowed eigenvalues.

To just catch n crossings min $E \to 0$ the last one $(k_1 a)\big|_{E=0} = (2n-1)\frac{\pi}{2}$.

Thus we have

$$a\sqrt{\frac{2m}{\hbar^2}|V_0|} = \left(n-\tfrac{1}{2}\right)\pi$$

as the condition for exactly n bound state

3\. (a) Initially

$P_A = P_0$ $\qquad$ $P_B = P_0$

$T_A = T_0$ $\qquad$ $T_B = 3T_0$

$P_A V_A = RT_A$

$P_0 V_A = RT_0$ $\qquad$ $\frac{V_A}{V_B} = \frac{1}{3}$

$P_0 V_B = R(3T_0)$

Finally

$P_A = P_f$ $\quad$ $P_B = P_f$

$T_A = T_f$ $\quad$ $T_B = T_f$

$P_f V_A = RT_f$ $\qquad$ $\frac{V_A}{V_B} = 1$

$P_f V_B = RT_f$

(b) From 1st Law of Thermodynamics

$$\Delta E = Q - W,$$

since $Q = 0$

and $W = 0$

$$\Delta E = \Delta E_A + \Delta E_B = 0$$

$$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV$$

For an ideal gas

$$\left(\frac{\partial E}{\partial V}\right)_T = 0 \quad \text{so}$$

$$dE = C_v \, dT$$

Thus

$$\Delta E_A = C_v (T_f - T_0)$$

$$\Delta E_B = C_v (T_f - 3T_0)$$

So $T_f = 2T_0$

(c) Since the gas is ideal, from the above consideration we see that $\Delta P_A = \Delta P_B = 0$

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT$$

$$= \frac{C_P}{T} dT =$$

$$\Delta S_A = C_p \ln \frac{T_f}{T_0} = C_p \ln \frac{2T_0}{T_0} = C_p \ln 2$$

$$\Delta S_B = C_p \ln \frac{T_f}{3T_0} = C_p \ln \frac{2T_0}{3T_0} = C_p \ln \frac{2}{3}$$

$$\Delta S = \Delta S_A + \Delta S_B = C_p \ln \frac{4}{3}$$

Or, without realizing that $\Delta P = 0$ we could do the more straightforward

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

$$= \frac{C_v}{T} dT + \left(\frac{\partial P}{\partial T}\right)_V dV$$

$$= \frac{C_v}{T} dT + \frac{R}{V} dV$$ etc. . .

(d) This time work is being done so

$$\Delta E \neq 0,$$

but now $\Delta S = 0$

$$\Delta S_A + \Delta S_B = 0$$

So write

$$\int_i^f [dS_A + dS_B] = \int_i^f \left[\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \right]_A$$

$$+ \left[\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV \right]_B = 0$$

$$0 = \int_i^f \left(\frac{C_v}{T} dT + \frac{R}{V} dV \right)_A + \left(\frac{C_v}{T} dT + \frac{R}{V} dV \right)_B$$

$$= C_v \ln \frac{T_f}{T_{0A}} + R \log \frac{V_f}{V_{0A}} + C_v \ln \frac{T_f}{T_{0B}} + R \log \frac{V_f}{V_{0B}}$$

$$= 2 C_v \ln \frac{T_f}{T_0} + R \ln \frac{4}{3} - C_v \ln 3$$

So $$T_f = T_0 \left[\frac{3}{\left(\frac{4}{3}\right)^{\frac{R}{C_v}}} \right]^{1/2}$$

(e) Thus, since

$$\Delta E = -W$$

so $$C_v \left[\left(T_0 \sqrt{\frac{3}{\left(\frac{4}{3}\right)^{\frac{R}{C_v}}}} - T_0 \right) + T_0 \sqrt{\frac{3}{\left(\frac{4}{3}\right)^{\frac{R}{C_v}}}} - 3T_0 \right] = -W$$

4. <u>Solution</u>:

The Green's function method is the best way to approach this problem as per Jackson Ch I.

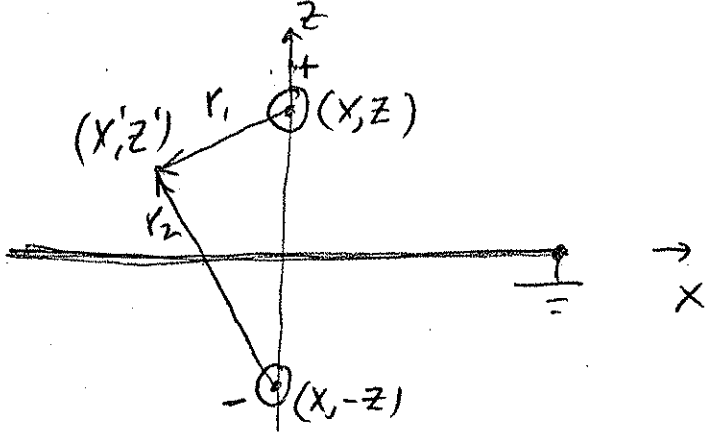
We can write immediately

$$\phi(\vec{x}) = \Big[\int d\vec{x}'\,\rho(\vec{x}')\,G(\vec{x},\vec{x}') \;\blacksquare\; \frac{1}{4\pi}\int_Z \phi_Z(\vec{x}')\,\hat{n}'\cdot\nabla' G(\vec{x},\vec{x}')\,d$$

where the second integral is all that matters since $\rho(x) = 0$. It is an integral only over the surface of the region in question and thus contains the required B.C. information.

The only hard part is to find $G(\vec{x},\vec{x}')$ and this is quickly done by the method of images.

Two unit line charges ^of opposite sign have an infinite grounded plane as an equipotential surface



Thus the G.F., which is a solution of

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(x-x'),$$

is also the solution of the problem of a line charge, unit strength, sitting above an infinite grounded conducting plane, and can be ~~solved~~ replaced by the "image" problem stated above.

So

$$G(x,x') = +2 \times \frac{1}{2} \log \frac{r_1}{r_2}$$

Next we need to evaluate

$$\hat{n}\cdot\vec{\nabla}G(x,x') = \frac{\partial}{\partial z} G(x,x')$$

$$= -2\left[\frac{(z'-z)}{(x'-x)^2+(z'-z)^2} + \frac{(z'+z)}{(x'-x)^2+(z'+z)^2}\right]$$

and perform the integral over the surface $z=0$

$$+\frac{2\times 2V_0}{4\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{z'}{(x'-x)^2+z'^2}\,dx \quad + \int_{-\infty}^{-\frac{a}{2}} (0)G\,dx + \int_{\frac{a}{2}}^{\infty} (0)G\,dx$$

Using one of the integrals on the cover sheet

$$\Phi(x',z') = -\frac{4V_0}{4\pi}\left[\operatorname{Tan}^{-1}\left(\frac{x'-\frac{a}{2}}{z'}\right) - \operatorname{Tan}^{-1}\left(\frac{x'+\frac{a}{2}}{z'}\right)\right]$$

QM II (Grad)

5. A system has a Hamiltonian

$$H = H_0 + V(\vec{x}) \cos \omega t \, e^{-\alpha |t|} \qquad (\alpha > 0)$$

where α is small. H_0 is time independent with energy levels E_n and corresponding eigenfunctions $\Phi_n(\vec{x})$ $(n = 0, 1, 2 \cdots)$. At time $t = -\infty$ the system is in the state Φ_0. Find, to lowest order in V, the probability that the system is in the state Φ_m $(m \neq 0)$

(15) at $t = +\infty$, and show that it is proportional to the square of a function $f(\omega, \alpha)$ which has the properties

$$\lim_{\alpha \to 0} f(\omega, \alpha) = 0 \quad \text{for} \quad \hbar\omega \neq E_m - E_0$$

and $$\int_{-\infty}^{\infty} f(\omega, \alpha)\, d\omega = 1$$

How would you find the probability of

(5) the system's remaining in the state Φ_0 at $t = +\infty$?

Sol QM I ①

(1) $\Psi = \sum_n a_n(t)\, \varphi_n e^{-iE_n/\hbar\, t}$ where

(2) $\mathcal{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$ and $\mathcal{H}_0 \varphi_n = E_n \varphi_n$

$$\int \varphi_m^* \varphi_n \, d\tau = \delta_{mn}$$

Let $\omega_n = E_n/\hbar$

Put (1) in (2)

$$\cancel{\sum_n a_n(t) E_n \varphi_n e^{-i\omega_n t}} + \sum_n a_n(t) \mathcal{H}' \varphi_n e^{-i\omega_n t} = i\hbar \sum_n \dot{a}_n \varphi_n e^{-i\omega_n t} + \cancel{\sum \hbar\omega_n a_n \varphi_n e^{-i\omega_n t}}$$

$\times\, \varphi_m^*$ on left & $\int \quad d\tau$

Let $\int \varphi_m^* \mathcal{H}' \varphi_n \, d\tau \equiv \langle m|\mathcal{H}'|n\rangle$

(3) $\sum a_n \langle m|\mathcal{H}'|n\rangle e^{-i\omega_n t} = i\hbar\, \dot{a}_m e^{-i\omega_m t}$

$$\dot{a}_m = \frac{1}{i\hbar} \sum_n a_n \langle m|\mathcal{H}'|n\rangle e^{-i(\omega_n - \omega_m)t}$$

Set $\mathcal{H}' \to \lambda \mathcal{H}'$ and $a_m = a_m^{(0)} + \lambda a_m^{(1)} + \lambda^2 a_m^{(2)} + \dots$ etc

& put in (3) $\omega_{nm} = \omega_n - \omega_m = -\omega_{mn}$

$$\dot{a}_m^{(0)} + \lambda \dot{a}_m^{(1)} + \lambda^2 a_m^{(2)} + \dots = \frac{1}{i\hbar} \sum_n \lambda a_n^{(0)} \langle m|\mathcal{H}'|n\rangle e^{-i\omega_{nm} t} + \lambda^2 a_n^{(1)} \langle m|\mathcal{H}'|n\rangle e^{-i\omega_{nm} t} + \dots$$

Set powers of λ = & Let $\lambda \to 1$

$\dot{a}_m^{(0)} = 0$ So all $a_m^{(0)} = \text{const}$

4) $\dot{a}_m^{(1)} = \frac{1}{i\hbar} \sum a_n^{(0)} \langle m|\mathcal{H}'|n\rangle e^{i\omega_{mn}t}$

at $t = -\infty$ $\Big\{$ choose $a_0^{(0)} = 1$ all $a_m^{(0)}(m \neq 0) = 0$

$\therefore \; a_n^{(0)} = \delta_{n0}$

1st order

$\dot{a}_n^{(1)} = \frac{1}{i\hbar} \langle m|\mathcal{H}'|0\rangle e^{i\omega_{m0}t}$

5) $a_m^{(1)}(t) = \frac{1}{i\hbar} \int_{-\infty}^{t} \langle m|\mathcal{H}'(t')|0\rangle e^{i\omega_{m0}t'} dt'$

for $t = +\infty$ $a_m^{(1)} \propto$ Fourier Xform of $\langle m|\mathcal{H}'(t)|0\rangle$

$P_m^{(1)} = |\langle\Psi(t)|\phi_m\rangle|^2 = |a_m^{(1)}|^2$ ≡ Probability to 1st order of finding the system in the state m

For higher orders $\dot{a}_m^{(2)} = \frac{1}{i\hbar} \sum_n a_n^{(1)} \langle m|\mathcal{H}'|n\rangle e^{i\omega_{mn}t}$ can be iterated but will involve higher than first order in $V(x)$ where $\mathcal{H}' = V(x)\, f(t)$

$$P_m^{(1)}(\infty) = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} \langle m|\mathcal{H}|0\rangle e^{i\omega_{m0}t} dt \right|^2$$

$$= \frac{1}{\hbar^2} |\langle m|V(x)|0\rangle|^2 \left| \int_{-\infty}^{\infty} \cos\omega t \, e^{-\nu|t|} e^{i\omega_{m0}t} dt \right|^2$$

Let $\int_{-\infty}^{\infty} \cos\omega t \, e^{-\nu|t|} e^{i\omega_{m0}t} dt = I + iI'$

In $I = \int_{-\infty}^{\infty} \cos\omega t \cos\omega_{m0} t \, e^{-\nu|t|} dt = 2\int_0^{\infty} \cos\omega t \cos\omega_{m0} t \, e^{-\nu t} dt$

$I' = \int_{-\infty}^{\infty} \cos\omega t \sin\omega_{m0} t \, e^{-\nu|t|} dt = 0$ since odd integrand

$$I = \int_0^{\infty} \cos(\omega-\omega_{m0})t \; e^{-\nu t} dt + \int_0^{\infty} \cos(\omega+\omega_{m0})t \; e^{-\nu t} dt$$

$$= \frac{\nu}{(\omega-\omega_{m0})^2+\nu^2} + \frac{\nu}{(\omega+\omega_{m0})^2+\nu^2}$$

Let $f(\omega\nu) = \frac{1}{2\pi}\left\{\frac{\nu}{(\omega-\omega_{m0})^2+\nu^2} + \frac{\nu}{(\omega+\omega_{m0})^2+\nu^2}\right\}$

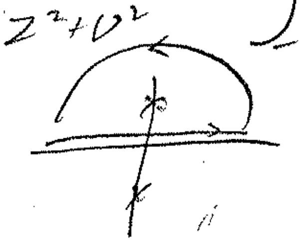
{for $\omega \gg \nu$ and $\omega \approx \omega_{m0}$ the 2nd term may be neglected but this is not necessary}

$$P_m^{(1)} = \frac{1}{\hbar^2} |\langle m|V|0\rangle|^2 \left[2\pi f(\omega,\nu)\right]^2$$

note that for $\omega \neq \omega_{m0} = (E_m - E_0)/\hbar$, $f(\omega,0) = 0$

and $\int_{-\infty}^{\infty} \frac{\nu}{(\omega\pm\omega_{m0})^2+\nu^2} d\omega = \nu\int_{-\infty}^{\infty} \frac{dz}{z^2+\nu^2}$ let $z = \omega \pm \omega_{m0}$

and $\int_{-\infty}^{\infty} \frac{dz}{z^2+\nu^2} = \int_{-\infty}^{\infty} \frac{dz}{(z+i\nu)(z-i\nu)} = 2\pi i \, \mathrm{Res}(i\nu) = \frac{2\pi i}{2i\nu} = \frac{\pi}{\nu}$



$\therefore \int_{-\infty}^{\infty} f(\omega,\nu) = \frac{1}{2\pi}\left\{\nu\frac{\pi}{\nu} + \nu\frac{\pi}{\nu}\right\} = 1$

$$P_0^{(1)} = \frac{1}{\hbar^2} |\langle 0|V(x)|0\rangle|^2 \left|\frac{\nu}{\omega^2+\nu^2}\right|^2 = 1 - \sum_{n\neq 0} P_m^{(1)}$$

#⑥ Solution

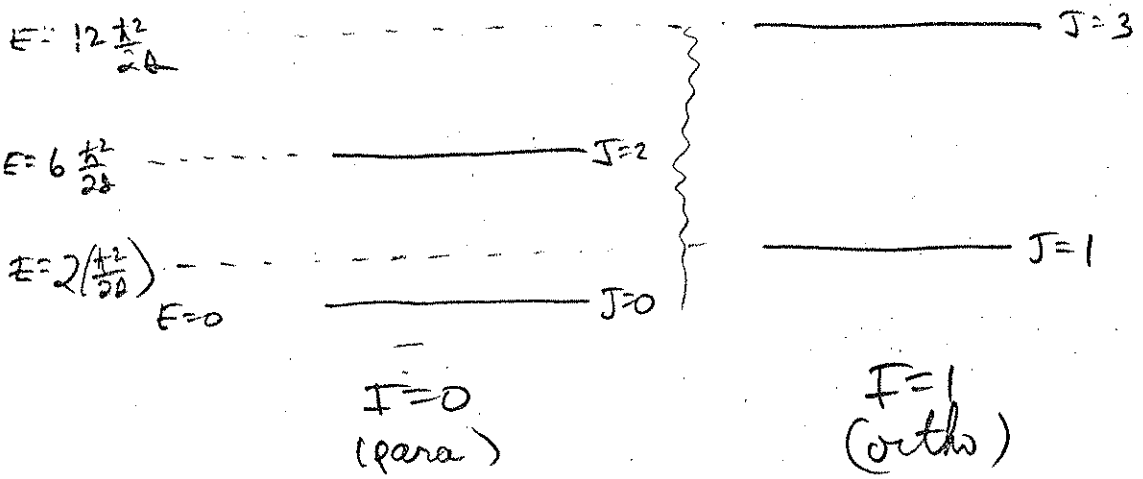
a) $P_{AB}\psi = -\psi$

$P_{AB}\psi_{JM} = (-1)^J \psi_{JM}$

b)

	I	I_z
$\alpha_A \alpha_B$	1	1
$\frac{1}{\sqrt{2}}(\alpha_A\beta_B + \alpha_B\beta_A)$	1	0
$\beta_A\beta_B$	1	-1
$\frac{1}{\sqrt{2}}(\alpha_A\beta_B - \alpha_B\beta_A)$	0	0

c) Since ψ must be antisymmetric, we must have ψ_{JM} symmetric (J = even) for ψ_N antisymmetric ($I=0$) and ψ_{JM} antisymmetric (J = odd) for ψ_N symmetric ($I=1$)



$$E = \frac{\hbar^2}{2I} J(J+1)$$

d) $I=1$ levels are 3-fold degenerate ($I_z = 0, \pm 1$)

all levels are $(2J+1)$-fold degenerate

$$\therefore\ Z_{para} = \sum_{J=\text{even}} (2J+1)\, e^{-\frac{\hbar^2}{2\theta} J(J+1)/kT}$$

$$Z_{ortho} = \sum_{J=\text{odd}} 3(2J+1)\, e^{-\frac{\hbar^2}{2\theta} J(J+1)/kT}$$

e)

$$Z = \frac{(Z_{para})^{N_p}}{N_p!}\ \frac{(Z_{ortho})^{N_o}}{N_o!}$$

$$F = -kT \ln Z \qquad , \mu = \frac{\partial F}{\partial N}$$

$$F = -kT \ln\left[\frac{Z_p^{N_p}}{N_p!}\ \frac{Z_o^{N_o}}{N_o!}\right] = -kT\left\{\begin{matrix} N_p \ln Z_p - N_p \ln N_p + N_p \\ + N_o \ln Z_o - N_o \ln N_o + N_o \end{matrix}\right\}$$

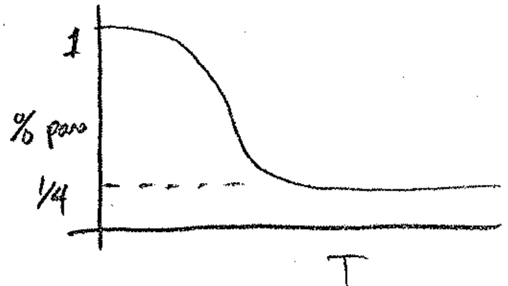
$$\left.\begin{matrix} \mu_p = \frac{\partial F}{\partial N_p} = -kT \ln \frac{Z_p}{N_p} \\ \mu_o = \frac{\partial F}{\partial N_o} = -kT \ln \frac{Z_o}{N_o} \end{matrix}\right\} \Rightarrow \begin{matrix} \mu_p = \mu_o \\ \therefore \boxed{\frac{N_o}{N_p} = \frac{Z_o}{Z_p}} \end{matrix}$$

for $T \to \infty$ $\quad \sum_{\text{all } J} \to \int \quad \therefore \sum_{\substack{J\,\text{even} \\ \text{or } J\,\text{odd}}} \to \frac{1}{2}\int$

$$Z_p = \frac{1}{2}\int_0^\infty (2J+1)\, e^{-xJ(J+1)}\, dJ = \frac{1}{2x} \qquad \left(x = \frac{\hbar^2}{2\theta}\frac{1}{kT}\right)$$

$$Z_o = \frac{3}{2x}$$

$T \to 0$ goes into lowest energy state



f). for D_2 the 2 nuclear particle have spin 1 + are thus bosons. Thus the total wave function must be symmetric, and the lowest energy state becomes $I=1$.
For HD, the fermion solution is again obtained.