OSU Physics Department
Comprehensive Examination #121

Monday, January 5 and Tuesday, January 6, 2015

Winter 2015 Comprehensive Examination

PARTS 1, 2, 3 and 4

General Instructions

This Winter 2015 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 5, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, January 6, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

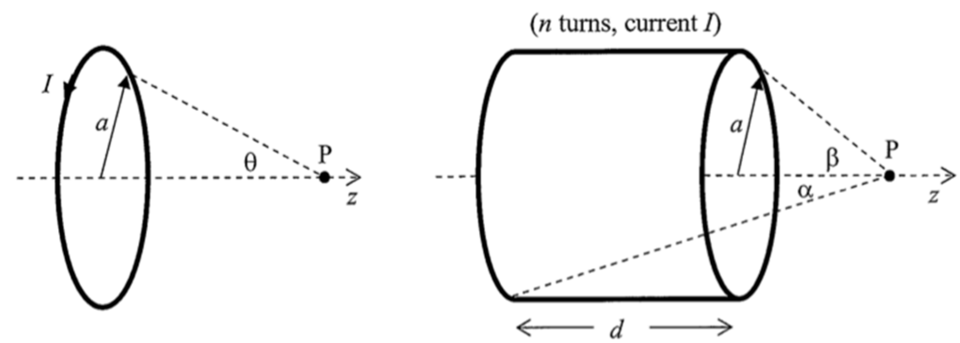


Figure A Figure B

(a) Refer to Figure A in which a single loop of wire is depicted. The loop is a circle with radius a, carries a current I, and μ_o is the permeability of the vacuum. The angle θ is the half-angle of the cone defined by the circular loop as the cone base and the point P as the cone apex, as indicated. Show that the magnetic field at P is

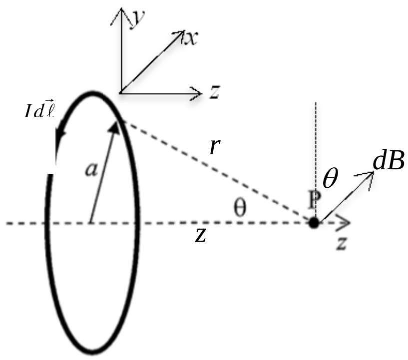
$$\vec{B}(\theta) = \hat{z}\frac{\mu_o I}{2a}\sin^3\theta \tag{1}$$

(b) Refer to Figure B in which an n-turn solenoid of length d, radius a is depicted. The solenoid carries a current I. Use part (a) to calculate the axial magnetic field at point P. Express the answer in terms of α and β, the half-angles of the cones defined by the back and front coils, respectively, as shown.

(c) Show that the result in (b) reduces correctly to the result for an infinitely long solenoid.

(d) Show that for small distances ($z \ll d$) from the center of the narrow ($d \gg a$) but finite solenoid that the axial field dependence is parabolic in z:

$$B_z(z) = B_o\left[1 - \frac{2a}{d^2}\left(1 + \frac{12z^2}{d^2}\right)\right] \tag{2}$$



(a) Let ϕ be the angle that defines the position of the *Idl* element, measured from the *x*-axis. Let *r* be the distance from the *Idl* element to *P* (vector *r* points from *Idl* to *P*).

The Biot-Savart law is:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\ell}\times\vec{r}}{r^3} \qquad (1)$$

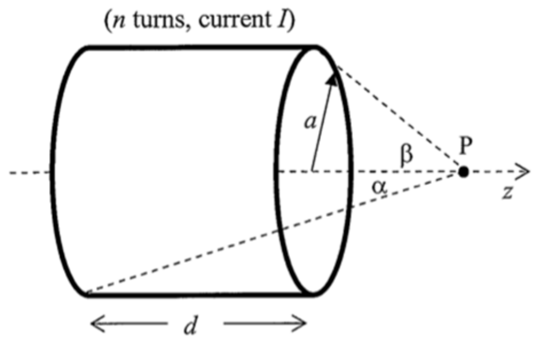
The total field is the integral around the current loop, with *I* and *r* constant.

$$\vec{B} = \frac{\mu_0 I}{4\pi r^3}\oint_{loop} d\vec{\ell}\times\vec{r} \qquad (2)$$

dl and *r* are constant in magnitude around the loop, and they are mutually perpendicular at every point, so *dB* is always constant in magnitude, but traces out a direction on the surface of a cone with apex *P* and at an angle 90˚+θ with the *z*-axis. For this reason, the net **radial field at *P* is zero**, and the field at *P* is axial, and point in the +z direction if positive current flows in the direction indicated in Figure A (right hand rule).

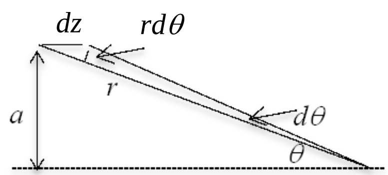
Use $d\ell = ad\phi \qquad \sin\theta = \frac{a}{r} \qquad r^2 = z^2 + a^2$

$$\boxed{\begin{aligned} B_z &= \frac{\mu_0 Ia\sin\theta}{4\pi r^2}\int_{\phi=0}^{2\pi} d\phi = \frac{\mu_0 Ia^2\sin\theta}{2ar^2} = \frac{\mu_0 I\sin^3\theta}{2a} \\ B_r &= 0 \end{aligned}} \quad . \qquad (3)$$



(b) The solenoid can be considered as a stack of current loops, each contributing to the axial B field at *P*. The sum of the axial fields due to all the current loops is an integral of (3) with θ varying from α to β. Each current loop contributes:

$$dB_z = \frac{\mu_0 \sin^3\theta}{2a} dI_{loop} = \frac{n\mu_0 I \sin^3\theta}{2ad} dz \quad (4)$$ where $dI_{loop} = I\frac{n}{d}dz$ and we must relate dz to $d\theta$.



$$\sin\theta = \frac{a}{r} = \frac{rd\theta}{dz} \qquad (5)$$

so that

$$dB_z = \frac{n\mu_0 I \sin^2\theta}{2ad} rd\theta = \frac{n\mu_0 I \sin\theta}{2d} d\theta$$

Perform the integration:

$$B_z = \frac{n\mu_0 I}{2d}\int_\alpha^\beta \sin\theta\, d\theta = \frac{n\mu_0 I}{2d}(-1)\cos\theta\Big|_\alpha^\beta$$

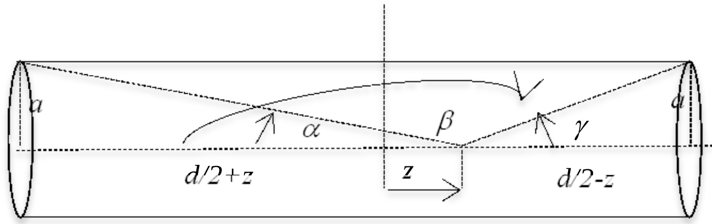
$$\boxed{B_z = \frac{n\mu_0 I}{2d}(\cos\alpha - \cos\beta)} \qquad (6)$$

(c) Infinitely long solenoid has $\alpha = 0;\ \beta = \pi$.

$$\boxed{B_z = \frac{n\mu_0 I}{2d}(\cos 0 - \cos\pi) = \frac{n\mu_0 I}{2d}(1-(-1)) = \mu_0 I\left(\frac{n}{d}\right)} \qquad (7)$$

as required.

(d) At the center of a long and narrow solenoid, we have the following. α and γ are small, so a cosine expansion to parabolic order suffices.



Angles:

$$\alpha \approx \tan\alpha = \frac{a}{\frac{d}{2}+z}; \qquad \gamma = \pi - \beta \approx \tan\gamma = \frac{-a}{\frac{d}{2}-z}$$

$$\cos\alpha \approx 1-\frac{\alpha^2}{2} = 1-\frac{a^2}{2\left(\frac{d}{2}+z\right)^2} \qquad (8)$$

$$\cos\beta = -\cos\gamma \approx -\left(1-\frac{\gamma^2}{2}\right) = -\left(1-\frac{a^2}{2\left(\frac{d}{2}-z\right)^2}\right)$$

Putting together:

$$B_z = \frac{n\mu_0 I}{2d}\left(\cos\alpha - \cos\beta\right)$$

$$B_z \approx \frac{n\mu_0 I}{2d}\left(1-\frac{a^2}{2\left(\frac{d}{2}+z\right)^2} + \left[1-\frac{a^2}{2\left(\frac{d}{2}-z\right)^2}\right]\right)$$

$$= \frac{n\mu_0 I}{2d}\left(2-\frac{a^2}{\left(\frac{d}{2}+z\right)^2} - \frac{a^2}{\left(\frac{d}{2}-z\right)^2}\right)$$

$$= \frac{n\mu_0 I}{2d}\left(2-\frac{a^2}{2}\left[\frac{4}{d^2\left(1+2z/d\right)^2} + \frac{4}{d^2\left(1-2z/d\right)^2}\right]\right)$$

$$= \frac{n\mu_0 I}{d}\left(1-\frac{a^2}{d^2}\left[1-\cancel{4\frac{z}{d}} + \frac{24z^2}{2d^2} + 1 + \cancel{4\frac{z}{d}} + \frac{24z^2}{2d^2}\right]\right)$$

And the required result:

$$\boxed{B_z = \frac{n\mu_0 I}{d}\left(1-\frac{2a^2}{d^2}\left[1+\frac{12z^2}{d^2}\right]\right)} \qquad (9)$$

A bead of mass m is constrained to slide under the influence of gravity on a smooth(frictonless) parabolic wire with equation $z = \frac{1}{2}\alpha^2 x^2$, where the z-axis is directed vertically upwards. The wire rotates about the z-axis with constant angular velocity ω.

(a) Show that the Hamiltonian is

$$H(p, x) = \frac{p^2}{2m(1 + \alpha^4 x^2)} + \frac{mx^2}{2}(g\alpha^2 - \omega^2) \tag{5}$$

where $p = p(x, \dot{x})$ is the generalized momentum of the bead confined to the parabolic wire.

(b) *Roughly* sketch the phase curves (i.e. lines of constant H) for the two cases (i) $g\alpha^2 > \omega^2$ and (ii) $g\alpha^2 < \omega^2$. Briefly comment qualitatively on what your plots tells you about the bead's motion in each case. For simplicity you may assume $\alpha^4 x^2 \ll 1$.

(c) Show that if $g\alpha^2 > \omega^2$, the bead always oscillates about the bottom of the parabola with period,

$$T = \frac{2\sqrt{2m}}{\beta} \int_0^{\pi/2} \left(1 + \frac{\alpha^4 E}{\beta^2} \sin^2\theta\right)^{1/2} d\theta \tag{6}$$

where E is the total system energy and $\beta^2 = \frac{m}{2}(g\alpha^2 - \omega^2)$.

(d) Explain qualitatively and justify mathematically what happens in the special case $g\alpha^2 = \omega^2$.

A bead of mass m is constrained to slide under the influence of gravity on a smooth(frictonless) parabolic wire with equation $z = \frac{1}{2}\alpha^2 x^2$, where the z-axis is directed vertically upwards. The wire rotates about the z-axis with constant angular velocity ω.

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Solution:
(8 points) Given that $z = \frac{1}{2}\alpha^2 x^2$ we get $\dot{z} = \alpha^2 x\dot{x}$. This is a 2D problem $(x, z(x))$, but the bead is effected by the external angular momentum introduced. Cartesian coordinate make sense for this problem b/c the bead is NOT free to orbit the angular coordinate.

Step 1: Write the Lagrangian, $L = T - U$. Note, it is necessary to find L, as the definition of the classical generalized momentum is $p = \frac{\partial L}{\partial \dot{q}}$.
The only potential is gravitational, so $U = mgz = \frac{1}{2}mg\alpha^2 x^2$. Similarly the kinetic can be expressed as,

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{z}^2 + x^2\omega^2) && (8)\\
&= \frac{1}{2}m(\dot{x}^2 + \alpha^4 x^2\dot{x}^2 + x^2\omega^2) && (9)\\
&= \frac{1}{2}m(\dot{x}^2(1 + \alpha^4 x^2) + x^2\omega^2) && (10)
\end{aligned}$$

The system Lagrangian is: $L = \frac{1}{2}m(\dot{x}^2(1 + \alpha^4 x^2) + x^2\omega^2) - \frac{1}{2}mg\alpha^2 x^2$.

Step 2: Find the generalized x-momentum of the bead, $p(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}} = m\dot{x}(1 + \alpha^4 x^2)$. This also implies that $\dot{x} = \frac{p}{m(1+\alpha^4 x^2)}$.

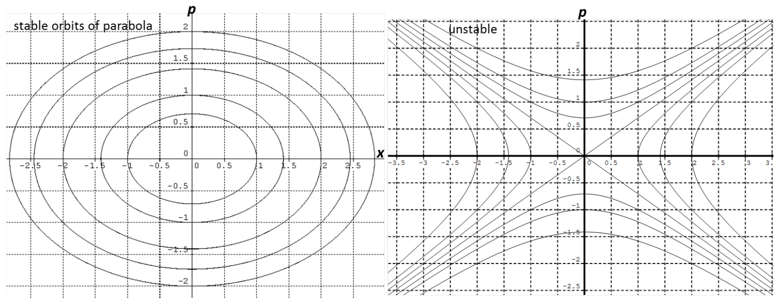
Step 3: Find the Hamiltonian, $H(p, x)$. A Legendre transform is used to introduce our generalized momentum to the Hamiltonian.

$$\begin{aligned}
H(p, x)) &= p\dot{x} - L && (11)\\
&= \frac{p^2}{m(1 + \alpha^4 x^2)} - \left[\frac{1}{2}m(\dot{x}^2(1 + \alpha^4 x^2) + x^2\omega^2) - \frac{1}{2}mg\alpha^2 x^2\right] && (12)\\
&= \frac{p^2}{m(1 + \alpha^4 x^2)} - \left[\frac{1}{2}m\left(\left(\frac{p}{m(1 + \alpha^4 x^2)}\right)^2 (1 + \alpha^4 x^2) + x^2\omega^2\right) - \frac{1}{2}mg\alpha^2 x^2\right] && (13)\\
&= \frac{p^2}{2m(1 + \alpha^4 x^2)} + \frac{mx^2}{2}(g\alpha^2 - \omega^2) && (14)
\end{aligned}$$

(the desired result)

(b) *Roughly* sketch the phase curves (i.e. lines of constant H) for the two cases (i) $g\alpha^2 > \omega^2$ and (ii) $g\alpha^2 < \omega^2$. Briefly comment qualitatively on what your plots tells you about the bead's motion in each case. For simplicity you may assume $\alpha^4 x^2 \ll 1$.

Solution:
(4 points) In the limit, $\alpha^4 x^2 \ll 1$, our Hamiltonian is $H = \frac{p^2}{2m} + \frac{mx^2}{2}(g\alpha^2 - \omega^2)$. In a phase curve, we draw lines of constant energy $H = E$. The resulting orbits are either elliptical (left)for case i, $g\alpha^2 > \omega^2$ and mean that the bead oscillates in the parabola with complete periods. In case ii (right), $g\alpha^2 < \omega^2$ and the phase curves are hyperbolic and the system in unstable, means the bead prefers to climb the parabola rather than oscillate about it.



(c) Show that if $g\alpha^2 > \omega^2$, the bead always oscillates about the bottom of the parabola with period,

$$T = \frac{2\sqrt{2m}}{\beta} \int_0^{\pi/2} \left(1 + \frac{\alpha^4 E}{\beta^2} \sin^2\theta\right)^{1/2} d\theta \tag{15}$$

where E is the total system energy and $\beta^2 = \frac{m}{2}(g\alpha^2 - \omega^2)$.

Solution:
(6 points) The conventional approach to finding the system period is to apply energy conservation ($H = E$) and separation of variables.
Step one is to express the velocity $\dot{x}$ in term of x and t only.

$$\frac{dx}{dt} = \frac{p}{m(1 + \alpha^4 x^2)} \tag{16}$$

$$= \frac{\sqrt{2m(1 + \alpha^4 x^2)\left[E - \frac{m}{2}(g\alpha^2 - \omega^2)x^2\right]}}{m(1 + \alpha^4 x^2)} \tag{17}$$

Step 2 is to separate variables and integrate both side over a full period oscillation. Since $g\alpha^2 > \omega^2$, the oscillations are stable and it will oscillate between endpoints $\pm x_o$. Endpoint x_o is obtained from the Hamiltonian when $p = 0$ (no kinetic energy, all potential). Accordingly,

$$\pm x_o = \pm \frac{2E}{m(g\alpha^2 - \omega^2)} \tag{18}$$

Separation of variables and integration give, the period (T),

$$T = 2\int_{-x_o}^{x_o} \frac{dx}{\dot{x}} \tag{19}$$

$$= 4\int_0^{x_o} \frac{m(1+\alpha^4 x^2)}{\sqrt{2m(1+\alpha^4x^2)\left[E - \frac{m}{2}(g\alpha^2-\omega^2)x^2\right]}}dx \tag{20}$$

$$= 4\int_0^{x_o} \sqrt{\frac{\frac{m}{2}(1+\alpha^4x^2)}{E-\frac{m}{2}(g\alpha^2-\omega^2)x^2}}dx \tag{21}$$

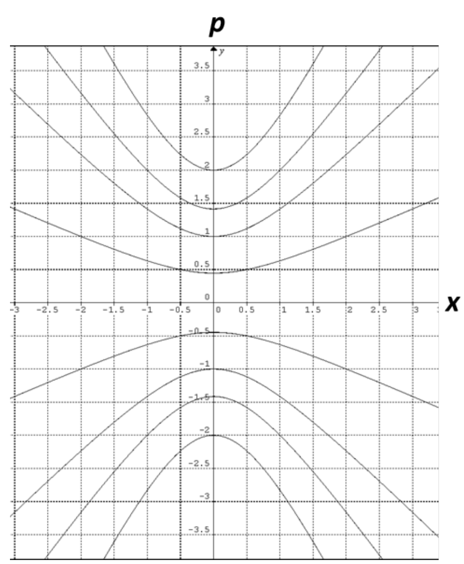
Lastly to simplify, we substitute $x = x_o \sin\theta = \frac{\sqrt{E}}{\beta}\sin\theta$ to get rid of the ugly denominator above, where β (given) is $\beta^2 = \frac{m}{2}(g\alpha^2 - \omega^2)$. Plugging in, we obtain the desired (readily integrable) result for the bead's period,

$$T = \frac{2\sqrt{2m}}{\beta}\int_0^{\pi/2}\left(1+\frac{\alpha^4 E}{\beta^2}\sin^2\theta\right)^{1/2} d\theta \tag{22}$$

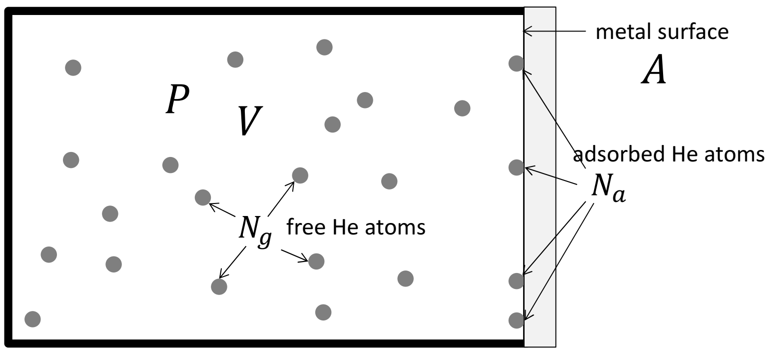
$$\tag{23}$$

(d) Explain qualitatively and justify mathematically what happens in the special case $g\alpha^2 = \omega^2$.

Solution:
(2 points) When $g\alpha^2 = \omega^2$ there is no potential energy, just and x-dependent inertia on the bead. So for any x, the bead will rest stay at rest if $p = 0$ or the bead will move with remaining p, otherwise. In both cases the rotational and gravitational effects cancel. The phase curve plots are shown below.



Adsorption equilibrium A helium gas (atomic mass, m) at a pressure P is confined in a box (volume, V) at temperature T. One side of the box is a metal plate (area, A). Some atoms are adsorbed on the metal surface with the adsorption energy U, an amount of work being necessary to remove a helium atom from the metal surface to infinity. Assume that the adsorbed atoms do not interact with each other and are free to move on the metal surface, i.e., the atoms on the metal surface form a two-dimensional ideal gas. The number of the adsorbed atoms is $N_a(\gg 1)$, and the number of gas-phase helium atoms is $N_g(\gg 1)$.



(a) Find the partition functions of the gas-phase atoms Z_g and the adsorbed atoms Z_a.

(b) Find the total partition function Z and the free energy F of the helium system including both the gas-phase and the adsorbed atoms.

(c) Calculate the chemical potential of a gas-phase atom μ_g and of an adsorbed atom μ_a.

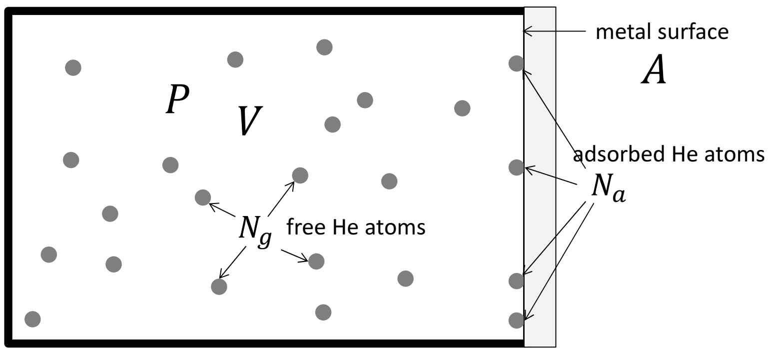
(d) What is the number of atoms adsorbed per unit area at equilibrium? Show that it depends on P, T, and U.

Useful formula:

$$\ln n! \cong n \ln n - n$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Adsorption equilibrium A helium gas (atomic mass, m) at a pressure P is confined in a box (volume, V) at temperature T. One side of the box is a metal plate (area, A). Some atoms are adsorbed on the metal surface with the adsorption energy U, an amount of work being necessary to remove a helium atom from the metal surface to infinity. Assume that the adsorbed atoms do not interact with each other and are free to move on the metal surface, i.e., the atoms on the metal surface form a two-dimensional ideal gas. The number of the adsorbed atoms is $N_a (\gg 1)$, and the number of gas-phase helium atoms is $N_g (\gg 1)$.



(a) Find the partition functions of the gas-phase atoms Z_g and the adsorbed atoms Z_a.

Solution:
Since the helium atoms are indistinguishable while not mutually interacting, the partition function of the helium gas can be written as

$$Z_g = \frac{1}{N_g!}[Z_{g1}]^{N_g}, \tag{24}$$

where the single-atom partition function,

$$\begin{aligned} Z_{g1} &= \int\int \frac{d^3\mathbf{p}d^3\mathbf{r}}{h^3} e^{-\frac{p^2}{2mk_BT}} \\ &= \frac{V}{h^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2mk_BT}(p_x^2+p_y^2+p_z^2)} dp_x dp_y dp_z \\ &= \frac{V}{h^3}\left(\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mk_BT}} dp_x\right)^3 \\ &= \frac{V}{h^3}(2\pi m k_B T)^{3/2}. \end{aligned} \tag{25}$$

Similarly, the partition function of the adsorbed atoms is expressed as

$$Z_a = \frac{1}{N_a!}[Z_{a1}]^{N_a}, \tag{26}$$

where

$$
\begin{aligned}
Z_{a1} &= \int\int \frac{d^2\mathbf{p}d^2\mathbf{r}}{h^2} e^{-\frac{p^2}{2mk_BT}+\frac{U}{k_BT}} \\
&= \frac{A}{h^2} e^{\frac{U}{k_BT}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2mk_BT}(p_x^2+p_y^2)} dp_x dp_y \\
&= \frac{A}{h^2} e^{\frac{U}{k_BT}} \left(\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mk_BT}} dp_x\right)^2 \\
&= \frac{A}{h^2}(2\pi m k_B T) e^{\frac{U}{k_BT}}. \qquad (27)
\end{aligned}
$$

(b) Find the total partition function Z and the free energy F of the helium system including both the gas-phase and the adsorbed atoms.

Solution:

The total partition function is written as

$$
\begin{aligned}
Z &= Z_g Z_a \\
&= \frac{1}{N_g! N_a!}\left[\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right]^{N_g}\left[\frac{A}{h^2}(2\pi m k_B T) e^{\frac{U}{k_BT}}\right]^{N_a}. \qquad (28)
\end{aligned}
$$

The free energy is

$$
\begin{aligned}
F &= -k_BT \ln Z \\
&= -k_BT\left[N_g \ln\left\{\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right\} + N_a \ln\left\{\frac{A}{h^2}(2\pi m k_B T) e^{\frac{U}{k_BT}}\right\}\right. \\
&\quad \left. - \ln(N_g!) - \ln(N_a!)\right] \qquad (29)
\end{aligned}
$$

Since $N_g, N_a \gg 1$,

$$
\begin{aligned}
F &\cong -k_BT\left[N_g \ln\left\{\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right\} + N_a \ln\left\{\frac{A}{h^2}(2\pi m k_B T) e^{\frac{U}{k_BT}}\right\}\right. \\
&\quad \left. -N_g \ln(N_g) + N_g - N_a \ln(N_a) + N_a\right] \qquad (30)
\end{aligned}
$$

(c) Calculate the chemical potential of a gas-phase atom μ_g and of an adsorbed atom μ_a.

Solution:

$$
\begin{aligned}
\mu_g &= \left(\frac{\partial F}{\partial N_g}\right) \\
&= -k_BT\left[\ln\left\{\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right\} - \ln N_g\right] \qquad (31)
\end{aligned}
$$

and

$$\begin{aligned}\mu_a &= \left(\frac{\partial F}{\partial N_a}\right) \\ &= -k_BT\left[\ln\left\{\frac{A}{h^2}(2\pi m k_B T)e^{\frac{U}{k_BT}}\right\} - \ln N_a\right] \qquad (32)\end{aligned}$$

(d) What is the number of atoms adsorbed per unit area at equilibrium? Show that it depends on P, T, and U.

Solution:
Setting on $\mu_g = \mu_a$ at equilibrium, we obtain

$$\frac{N_a}{N_g} = \frac{Ah}{V}\frac{1}{\sqrt{2\pi m k_B T}}e^{\frac{U}{k_BT}}. \qquad (33)$$

The number of atoms adsorbed per unit area is therefore

$$\begin{aligned}n_a &= \frac{N_a}{A} = h\left(\frac{N_g}{V}\right)\frac{1}{\sqrt{2\pi m k_B T}}e^{\frac{U}{k_BT}} \\ &= \frac{hP}{k_BT}\frac{1}{\sqrt{2\pi m k_B T}}e^{\frac{U}{k_BT}} \\ &= \frac{h}{\sqrt{2\pi m k_B^3}}\frac{P}{T^{3/2}}e^{\frac{U}{k_BT}} \qquad (34)\end{aligned}$$

This shows that n_a is a function of P, T, and U.

Useful formula:

$$\ln n! \cong n\ln n - n$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

In a hydrogen-like atom the $2s$ state has slightly higher energy than the $2p$ states. The energy difference is Δ. The effect causing Δ has a negligible influence on the wave functions of the states. The atom is placed in a constant uniform electric field of strength $\mathcal{E}$.

When answering the following questions, neglect the effects of electronic and nuclear spin.

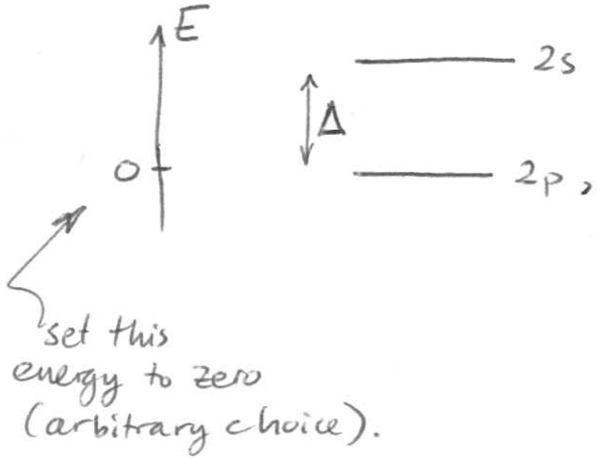
(a) Sketch an energy level diagram of the $n = 2$ states (the $2s$ and $2p$ states) for the case $\mathcal{E} = 0$. Indicate any degeneracy of the levels.

(b) Neglecting the influence of more distant levels (such as $n = 1$ and $n = 3$ states) obtain general expressions for the energy shift of **all** $n = 2$ levels as a function of $\mathcal{E}$.
Note: You do not have to evaluate explicitly any non-zero integrals. Assign an algebraic symbol to represent the value of any non-zero integral.

(c) Plot the energy of the levels as a function of the electric field strength $\mathcal{E}$. Label all the curves.

#4

a)



2p, 3-fold degeneracy $l=1, m_l=-1,0,+1$

i.e. $2P_{-1}$, $2P_0$, $2P_1$

b) Express the Hamiltonian using matrix representation.

The basis states are $2S$, $2P_0$, $2P_{-1}$, $2P_{+1}$

$$H = \begin{bmatrix} \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{when } \mathcal{E}=0.$$

When $\mathcal{E}>0$ there is a perturbation to Hamiltonian.

$$H = H_0 + H' = H_0 + \mathcal{E}z$$

Calculate the change in each matrix element

$$\langle 2S|\mathcal{E}z|2S\rangle = \int |\psi_{2s}|^2 \mathcal{E}z \, d^3\vec{r} = 0$$

since $|\psi_{2s}|^2$ is symm function w.r.t $\pm z$.

Similarly
$$\langle 2P_0|\mathcal{E}z|2P_0\rangle = 0$$
$$\langle 2P_1|\mathcal{E}z|2P_1\rangle = 0$$
$$\langle 2P_{-1}|\mathcal{E}z|2P_{-1}\rangle = 0$$

$$\langle 2S | \mathcal{E} z | 2P_0 \rangle = \mathcal{E} \underbrace{\int \psi_{2S}^* \, z \, \psi_{2P_0} \, d^3\vec{r}}_{} = \mathcal{E} a$$

Non-zero integral because ψ_{2P_0} is an odd function w.r.t. z

All other matrix elements are zero.

When $\mathcal{E} > 0$

$$H = \begin{bmatrix} \Delta & a\mathcal{E} & 0 & 0 \\ a\mathcal{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Diagonalize the 2×2 matrix (top left corner)

$$\det \begin{bmatrix} \Delta - \lambda & a\mathcal{E} \\ a\mathcal{E} & -\lambda \end{bmatrix} = 0$$

$$(\Delta - \lambda)(-\lambda) - a^2 \mathcal{E}^2 = 0$$

$$\lambda^2 - \lambda \Delta = a^2 \mathcal{E}^2$$

$$\left(\lambda - \frac{\Delta}{2}\right)^2 = a^2 \mathcal{E}^2 + \left(\frac{\Delta}{2}\right)^2$$

$$\lambda = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + a^2 \mathcal{E}^2}$$

The 2S state has energy shift

$$\frac{\Delta}{2} + \sqrt{\left(\frac{\Delta}{2}\right)^2 + a^2 \mathcal{E}^2} - \Delta = \boxed{-\frac{\Delta}{2} + \sqrt{\left(\frac{\Delta}{2}\right)^2 + a^2 \mathcal{E}^2}}$$

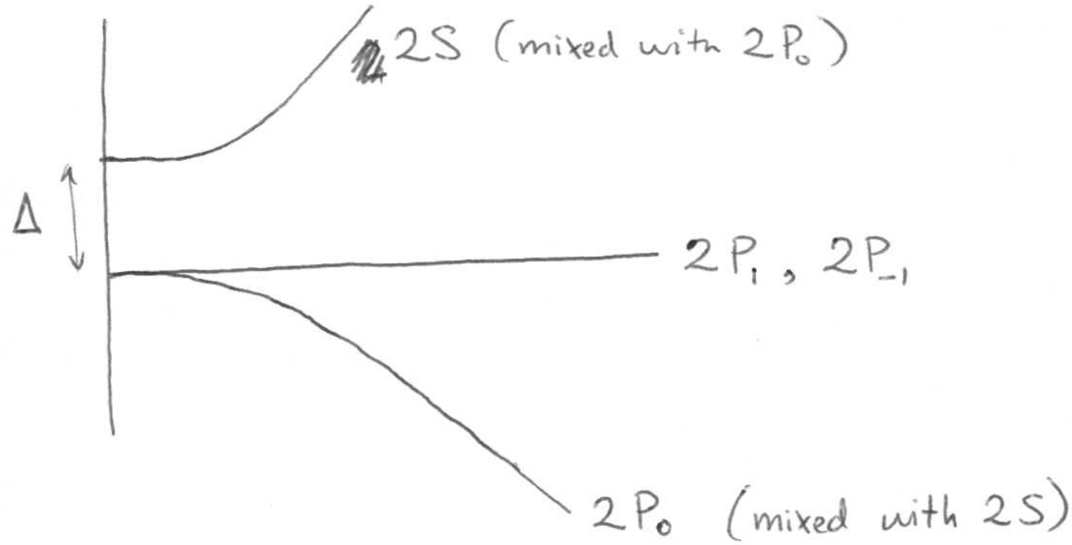
The $2P_0$ state has energy shift

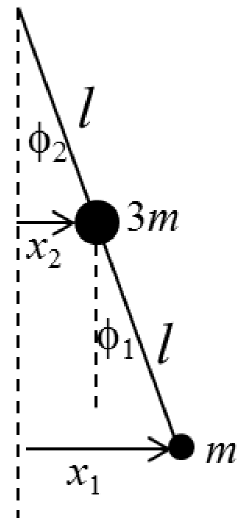
$$\frac{\Delta}{2} - \sqrt{\left(\frac{\Delta}{2}\right)^2 + a^2 \mathcal{E}^2}$$

Other states ($2P_{-1}$, $2P_1$) have no shift.

---

c)





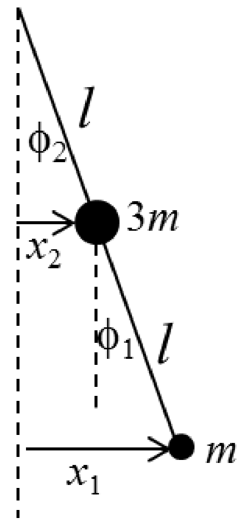
A pendulum is composed of two point masses, $3m$ and m, and two massless strings of equal length l as shown. At $t = 0$, the system is released from rest with upper (heavier) mass not displaced and the lower mass displaced slightly to the right: i.e. $x_2(0) = 0$ and $x_1(0) = a$.

(a) In the limit that $a \ll l$, explicitly evaluate an analytic equation for the subsequent motion of the lower mass, $x_1(t)$.

(b) Suppose instead that we displaced x_1 by a much larger value, such that the angular motion of the masses cannot be be neglected. Show that the Lagrangian of this system is given by:

$$L = 2ml^2\dot{\phi}_2^2 + \frac{1}{2}ml^2\dot{\phi}_1^2 + ml^2\dot{\phi}_2\dot{\phi}_1\cos(\phi_2 - \phi_1) + 4mgl\cos\phi_2 + mgl\cos\phi_1 \qquad (35)$$

(c) Again consider the limit that $a \ll l$, show that the above Lagrangian simplifies to a Lagrangian describing coupled simple harmonic motion.



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(a) In the limit that $a \ll l$, explicitly evaluate an analytic equation for the subsequent motion of the lower mass, $x_1(t)$.

Solution: (Revised, 8 pts)
(10 points, *Question credit: Prof. Tom Greytek, MIT Part 1 PhD exam, Fall 2000*)
In the limit that $a \ll l$, one must immediately recognize that is the standard condition when a pendulum (angular harmonic) become a 1D simple harmonic motion (analogous to a spring) where gravity is the restoring force − this true b/c when $a \ll l$, $\sin\theta \cong \theta \cong x$. The same argument holds for a double pendulum.

As such, all angular motion (i.e. the vertical motion component) can be ignored, and (to first order) the tension of each string is just mg and $4mg$.

Using Newtonian mechanics (i.e. $F = m\ddot{x}$) we can readily write a coupled differential equation systems (this problem is now precisely analogous to the coupled mass-on-a spring problem, see the 120^{th} comp exam for details). By inspection that for the lower mass we get:

$$m\ddot{x}_1 = -\left(\frac{x_1 - x_2}{l}\right) mg \quad (36)$$

$$0 = \ddot{x}_1 + \frac{g}{l}x_1 - \frac{g}{l}x_2 \quad (37)$$

$$(38)$$

For the upper mass we get:

$$3m\ddot{x}_2 = -\frac{x_2}{l}4mg + \frac{x_1 - x_2}{l}mg \tag{39}$$
$$0 = 3\ddot{x}_2 - \frac{g}{l}x_1 + 5\frac{g}{l}x_2 \tag{40}$$
$$\tag{41}$$

Let's define $\omega_o^2 = g/l$. Assume the solution may be obtained with an exponential of form $x_1(t) = A_1 \exp i\omega t$ and $x_2(t) = A_2 \exp i\omega t$, and we may solve the coupled ODEs by method of determinants where the matrix equation is $\ddot{x} = Cx$ or more explicitly
$C = \begin{pmatrix} \omega_o^2 & -\omega_o^2 \\ -\omega_o^2 & 5\omega_o^2 \end{pmatrix}$
We can now solve for the eigenvalues (or characteristic frequencies) of matrix C by,

$$\det\left(C - \omega^2 I\right) = \left(\omega_o^2 - \omega^2\right)\left(5\omega_o^2 - 3\omega^2\right) - \omega_o^4 = 0 \tag{42}$$
$$3\omega^4 - 8\omega^2\omega_o^2 + 4\omega_o^4 = 0 \tag{43}$$
$$(\omega/\omega_o)^2 = \frac{8 \pm \sqrt{64 - 48}}{6} \tag{44}$$

Hence characteristic frequencies are $\omega_1 = \sqrt{2}\omega_o$ and $\omega_2 = \sqrt{2/3}\omega_o$

Now solving for the normal modes (eigenvectors), we obtain
$\det\left(C - \omega^2 I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$
Solving for the eigenvectors, we find that $x_2 = -x_1$ and $x_2 = \frac{1}{3}x_1$. The normal modes Q_1 and Q_2 are $Q_1 = x_1 + x_2$ and $Q_2 = x_2 - \frac{1}{3}x_1$. This enables us to write the solution as superposition of normal modes:

$$x_1 = A\cos(\omega_1 t) + B\cos(\omega_2 t) \tag{45}$$
$$x_2 = -A\cos(\omega_1 t) + \frac{B}{3}\cos(\omega_2 t) \tag{46}$$
$$\tag{47}$$

At $t = 0$, $x_1 = a$ and $x_2 = 0a$. Applying these initial conditions to the above system, we get:

$$a = A + B \tag{48}$$
$$0 = -A + \frac{B}{3} \tag{49}$$
$$\tag{50}$$

OR that $A = \frac{a}{4}$ and $B = \frac{3a}{4}$.

Therefore the explicit motion of the lower mass in the $a \ll l$ limit is:

$$x_1(t) = \frac{a}{4}\cos(\omega_1 t) + \frac{3a}{4}\cos(\omega_2 t) \tag{51}$$

(b) Suppose instead that we displaced x_1 by a much larger value, such that the angular motion of the masses cannot be be neglected. Show that the Lagrangian of this system is given by:

$$L = 2ml^2\dot{\phi}_2^2 + \frac{1}{2}ml^2\dot{\phi}_1^2 + ml^2\dot{\phi}_2\dot{\phi}_1\cos(\phi_2 - \phi_1) + 4mgl\cos\phi_2 + mgl\cos\phi_1 \tag{52}$$

Solution:
(6 points) Working in Cartesian coordinates we have:

$$x_2 = l\sin\phi_2 \tag{53}$$
$$y_2 = -l\cos\phi_2 \tag{54}$$
$$x_1 = l\sin\phi_1 + l\sin\phi_2 \tag{55}$$
$$y_1 = -l\cos\phi_1 - l\cos\phi_2 \tag{56}$$

where we observe that $l^2 = x_1^2 + y_1^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. The Lagrangian is $L = T - U = T_1 + T_2 - U_1 - U_2$. The Kinetic energy of the upper mass is:

$$T_2 = \frac{3m}{2}(\dot{x}_2^2 + \dot{y}_2^2) \tag{57}$$
$$= \frac{3ml^2}{2}\dot{\phi}_2^2 \tag{58}$$

Likewise for the lower mass (applying an double angle identity), we get:

$$T_1 = \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) \tag{59}$$
$$= \frac{ml^2}{2}\left[(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2)^2 + (\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2)^2\right] \tag{60}$$
$$= \frac{ml^2}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)) \tag{61}$$
$$= \frac{ml^2}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\cos(\phi_2 - \phi_1)) \tag{62}$$

For the gravitational potential energies we obtain,

$$U_2 = 3mgy_2 = -3mgl\cos\phi_1 \tag{63}$$
$$U_1 = mgy_1 = -mgl(\cos\phi_1 + \cos\phi_2) \tag{64}$$

Finally putting it all together we get,

$$L = T_1 + T_2 - U_1 - U_2 \tag{65}$$
$$= \frac{ml^2}{2}(\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_2 - \phi_1)) + \frac{3ml^2}{2}\dot\phi_2^2 + mgl(\cos\phi_1 + \cos\phi_2) + 3mgl\cos\phi_1 \tag{66}$$
$$= 2ml^2\dot\phi_2^2 + \frac{1}{2}ml^2\dot\phi_1^2 + ml^2\dot\phi_2\dot\phi_1\cos(\phi_2 - \phi_1) + 4mgl\cos\phi_2 + mgl\cos\phi_1 \tag{67}$$

which is the desired result.

(c) Again consider the limit that $a \ll l$, show that the above Lagrangian simplifies to a Lagrangian describing coupled simple harmonic motion.

Solution:
(4 points) In limit that $a \ll l$, $\sin\phi \cong \phi$ and $\dot\phi \cong \dot x$, $\cos(\phi_2 - \phi_1) \cong 1$. For the potential energy terms, we must include the second order terms (otherwise the potential energy would be constant), i.e $\cos\phi_2 \cong 1 - x_2^2/2$ and $\cos\phi_1 \cong 1 - (x_1 - x_2)^2/2$. Using these approximations we get:

$$L = 2ml^2\dot\phi_2^2 + \frac{1}{2}ml^2\dot\phi_1^2 + ml^2\dot\phi_2\dot\phi_1\cos(\phi_2 - \phi_1) + 4mgl\cos\phi_2 + mgl\cos\phi_1 \tag{68}$$
$$\cong 2ml^2\dot x_2^2 + \frac{1}{2}ml^2\dot x_1^2 + ml^2\dot x_2\dot x_1 + 5mgl - mgl(x_2^2/2 + 2(x_1 - x_2)^2) \tag{69}$$
$$\tag{70}$$

We can obtain similar equations of motion as found in part a, by applying the respective Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot x_i} = \frac{\partial L}{\partial x_i}$ to the above approximation to the Lagrangian, i.e. we obtain

$$4ml^2\ddot x_2 + ml^2\ddot x_1 = -mgl(5x_2 - 4x_1) \tag{71}$$
$$ml^2\ddot x_1 + ml^2\ddot x_2 = -4mgl(x_1 - x_2) \tag{72}$$

These equation describes simple harmonic coupled motion, as they are a superposition of the equations of the similar form to those derived in part a. i.e. subtracting these two equations, one obtains $3ml^2\ddot x_2 = mgl(8x_1 - 6x_2)$ which is an ODE describing a simple coupled oscillator.

Note: there are various levels of approximation possible to make this problem work, any reasonable set of approximations that result in a coupled simple harmonic differential system are acceptable.

(a) A free particle of mass M is confined to move around the perimeter of a circle. At time $t = 0$ its wave function is given by

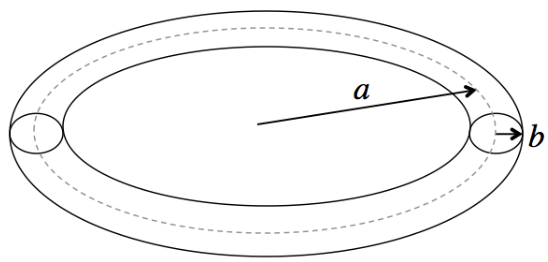
$$\Psi(\phi, t = 0) = A(\exp(2i\phi) + 2\cos(\phi)) \tag{73}$$

where ϕ gives the position of the particle on the circle.

i. If the angular momentum of the particle is measured at time t_0, what are the possible results of the measurement and the corresponding probabilities for each possible outcome?

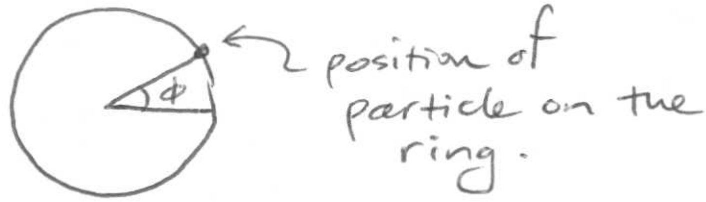
ii. If the energy of the particle is measured at time t_0 (instead of measuring the angular momentum), what are the possible results of the measurement and the corresponding probabilities for each possible outcome?

(b) A free particle of mass M is confined to the surface of a torus of radii a and b as shown below. You may assume $a \gg b$.



Find the eigenenergies of this system.

#6 a) i)



Ring → Periodic boundary condition

$$\Psi(\phi) = \Psi(\phi + 2\pi)$$

Free to move → Set $V = 0$, potential is independent of ϕ.

(unnormalized) Eigenstates are $\ldots e^{-2i\phi}, e^{-i\phi}, 1, e^{i\phi}, e^{2i\phi}, \ldots$

These are eigenstates of both energy and angular momentum.

We are told $\Psi(\phi) = A\left(e^{2i\phi} + 2\cos\phi\right)$

$$= A\left(\underbrace{e^{2i\phi} + e^{i\phi} + e^{-i\phi}}_{\text{superposition of 3 eigenstates.}}\right)$$

The angular momentum operator is

$$L_z = i\hbar \frac{\partial}{\partial\phi} \qquad (L_x \,\&\, L_y = 0 \text{ for this problem})$$

Therefore, possible outcomes are

$$\boxed{L = 2\hbar,\ \hbar \text{ or } -\hbar}$$

Probability $1/3$ $\quad 1/3 \quad 1/3$

ii) The energy is related to angular momentum

$$E = \frac{1}{2}\frac{L^2}{I}$$

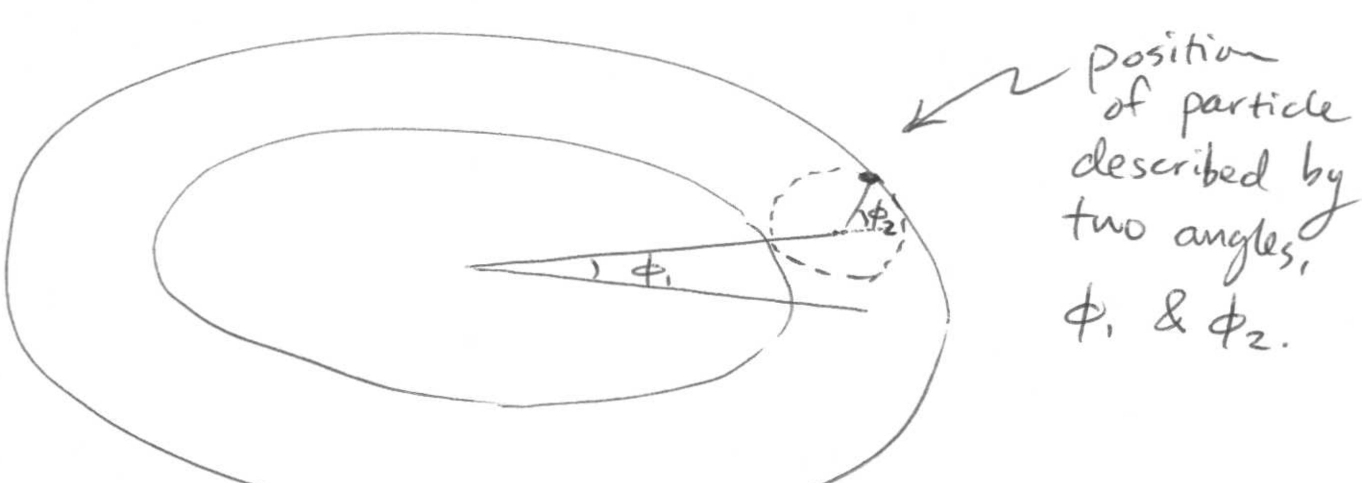
where $I = MR^2$ (R is the radius of the circle).

$$\Rightarrow \quad E = \frac{1}{2}\frac{4\hbar^2}{MR^2} \quad \text{or} \quad \frac{1}{2}\frac{\hbar^2}{MR^2}$$

$$\boxed{E = \frac{2\hbar^2}{MR^2} \quad \text{or} \quad \frac{\hbar^2}{2MR^2}}$$

Probability: $1/3$ (for $\frac{2\hbar^2}{MR^2}$), $2/3$ (for $\frac{\hbar^2}{2MR^2}$)

b)



Periodic boundary conditions on ϕ_1 and ϕ_2.

Free particle.

When $a \gg b$, the ϕ coordinates are independent.

Therefore $\Psi = Ae^{in\phi_1}e^{im\phi_2}$ describes eigenstates

where $n = \ldots -2, -1, 0, 1, 2 \ldots$

$m = \ldots -2, -1, 0, 1, 2 \ldots$

$$E = \frac{n^2\hbar^2}{2Ma^2} + \frac{m^2\hbar^2}{2Mb^2}$$

(a) Derive the wave equation for electromagnetic fields $\vec{E}$ and $\vec{B}$ in a linear, non-dispersive, homogeneous dielectric with dielectric constant ϵ and magnetic permeability μ (no free charges or currents). This derivation should be performed in a general vector space that does not rely on a particular choice of coordinate system.

(b) To discuss the solutions, use Cartesian coordinates to simplify the algebra, and assume solutions of the form

$$\vec{E}(\vec{r},t) = \vec{E}_o e^{i(kx-\omega t)} \tag{75}$$

$$\vec{B}(\vec{r},t) = \vec{B}_o e^{i(kx-\omega t)} \tag{76}$$

where $\vec{E}_o$, $\vec{B}_o$ are constant vectors, and ω and k are constant scalars.

(i) Show that these fields satisfy the wave equation, and that they propagate in the x direction. Hence define the wave velocity.

(ii) Show that $\vec{E}_o$, $\vec{B}_o$ are vectors in the yz plane.

(iii) Show $|E_o| = v|B_o|$, where v is the velocity defined in (i).

(iv) Show that $\vec{E}_o \perp \vec{B}_o$

(c) The medium is now conductive, *i.e.* a free current $\vec{J}_f = \sigma\vec{E}$ is allowed, where σ is the conductivity. Show that the waves attenuate by a factor of $1/e$ in a length $\sqrt{\frac{2}{\omega\mu\sigma}}$ in the high conductivity limit.

Comprehensive Exam, Winter 2015 E&M Graduate (Solution)

(a) The Maxwell equations are (equation sheet):

$$\nabla \bullet \vec{D} = \frac{\rho_f}{\varepsilon} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + J_f \qquad (1)$$

where, in a **linear** medium,

$$\vec{D} = \varepsilon \vec{E}; \quad \vec{H} = \frac{\vec{B}}{\mu} \; .$$

In the **absence of free charge and current**

$\rho_f = 0; \quad J_f = 0 \quad , \quad (2)$

and in a **homogeneous medium** where ε and μ do not depend on position,

$$\nabla \bullet \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \bullet \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \; . \qquad (3)$$

Take the curl of the curl equations:

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial \left(\nabla \times \vec{B}\right)}{\partial t}$$
$$\nabla \times \nabla \times \vec{B} = \mu\varepsilon \frac{\partial \left(\nabla \times \vec{E}\right)}{\partial t} \qquad (4)$$

Use standard vector identities (equation sheet) on the LHS, and use the relation of the curl of one field to the time derivative of the other (Eqs 3) on the RHS:

$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\nabla\left(\nabla \bullet \vec{B}\right) - \nabla^2 \vec{B} = -\mu\varepsilon \frac{\partial^2 \vec{B}}{\partial^2 t} \qquad (5)$$

Use the zero divergence of both fields (Eqs 3), and rearrange:

$$\boxed{\begin{aligned} \nabla^2 \vec{E} &= \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{B} &= \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} \end{aligned}} \qquad (6)$$

Eqs. (6) are the decoupled wave equations for the E and B fields, and it is conventional to define $v \equiv \frac{1}{\sqrt{\mu\varepsilon}}$, which has dimensions of [length/time], for later identification with the speed of wave propagation.

(b) To simplify the algebra, assume solutions of the form

$$\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(kx-\omega t)}$$
$$\vec{B}(\vec{r},t)=\vec{B}_0 e^{i(kx-\omega t)} \qquad (7)$$

where $\vec{E}_0, \vec{B}_0$ are constant vectors, and ω and *k* are constant scalars.

(i) Show that these fields satisfy the wave equation, and that they propagate in the *x* direction. Hence define the wave velocity.

Any feature of the waveform can be identified by a particular value of the phase $\phi=\omega t-kx$. A point of constant phase is characterized by $d\phi=\omega dt-kdx=0$, so that $\omega dt=kdx$, and

$$\frac{\omega}{k}=\frac{dx}{dt}\equiv v_x \qquad (8)$$

which identifies ω/*k* as the velocity of the waveform **in the *x*-direction**.

If $\vec{E}_0, \vec{B}_0$ are constant vectors, independent of space and time, then the assumed form (7) for E substituted into the wave equations (6) gives:

$$\vec{E}_0 \frac{\partial^2 e^{i(\omega t-kx)}}{\partial x^2}=\mu\varepsilon\vec{E}_0 \frac{\partial^2 e^{i(\omega t-kx)}}{\partial t^2}$$
$$-\vec{E}_0 k^2 e^{i(\omega t-kx)}=-\omega^2\mu\varepsilon\vec{E}_0 e^{i(\omega t-kx)}$$
$$\Rightarrow \frac{\omega^2}{k^2}=\frac{1}{\mu\varepsilon}$$

That is, the given forms satisfy the wave equations provided $\boxed{v_x=\frac{\omega}{k}=\frac{1}{\sqrt{\mu\varepsilon}}}$. (9)

An **identical result holds for B** because the wave equation and solution have the same form as for E.

(ii) Show that $\vec{E}_0, \vec{B}_0$ are vectors in the *yz* plane.

There is no dependence on *y* and *z* and the *x* dependence is all in the exponent.

To satisfy the divergence equations (3),

$$\nabla\bullet\vec{E}=\frac{\partial E_x}{\partial x}+\underbrace{\frac{\partial E_y}{\partial y}}_{0}+\underbrace{\frac{\partial E_z}{\partial z}}_{0}=0$$
$$\Rightarrow \cancel{\frac{\partial E_{0x}}{\partial x}} e^{i(kx-\omega t)}-ikE_{0x}e^{i(kx-\omega t)}=0 \qquad (10)$$

which means that E_{0x} = 0, so that the $\vec{E}_0$ **has no *x* component, and must point in the *yz* plane**, perpendicular to the direction of propagation.
The **same argument is true for** $\vec{B}_0$ because its divergence is zero also and it has the same form.

(iii) & (iv) Use one of the curl equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = i\omega\vec{B} \quad (11)$$

The x components are zero as shown in (ii) and there is no y or z dependence of any kind.

$$\hat{x}\left(\cancel{\frac{\partial E_z}{\partial y}} - \cancel{\frac{\partial E_y}{\partial z}}\right) - \hat{y}\left(\cancel{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}}\right) = i\omega \cancel{B_{0x}} i\omega B_{0y}\hat{y} + i\omega B_{0z}\hat{z}$$

Equate components to see that

$$\hat{y}\left(\frac{\partial E_z}{\partial x}\right) = i\omega B_{0y}\hat{y} \Rightarrow ikE_{0z}e^{i(kx-\omega t)} = i\omega B_{0y}e^{i(kx-\omega t)} \Rightarrow E_{0z} = \frac{\omega}{k}B_{0y}$$
$$\hat{z}\left(\frac{\partial E_y}{\partial x}\right) = i\omega B_{0z}\hat{z} \Rightarrow ikE_{0y}e^{i(kx-\omega t)} = i\omega B_{0z}e^{i(kx-\omega t)} \Rightarrow E_{0y} = \frac{\omega}{k}B_{0z} \quad (12)$$

(iv) Thus $\vec{E}_0, \vec{B}_0$ are mutually perpendicular because a $y(z)$ component of $\vec{E}_0 \Rightarrow$ a $z(y)$ component of $\vec{B}_0$.

(iii) Furthermore, the amplitudes are related:

$$E_{0z}{}^2 + E_{0y}{}^2 = \left(\frac{\omega}{k}\right)^2\left(B_{0y}{}^2 + B_{0z}{}^2\right) \Rightarrow \boxed{|E_0| = \frac{\omega}{k}|B_0|} \quad (13)$$

The x direction is not special and in general (with n direction perpendicular to k direction)

$$\vec{E} = E_0 e^{i(\vec{k}\bullet\vec{r}-\omega t)}\hat{n}; \quad \vec{B} = \frac{1}{v}\vec{k}\times\vec{E}$$

(d) The medium is now conductive, *i.e*. a free current $\vec{J}_f = \sigma\vec{E}$ is allowed, where σ is the conductivity. Show that the waves attenuate by a factor of $1/e$ in a length $\sqrt{\frac{2}{\omega\mu\sigma}}$ in the high conductivity limit.

$$\nabla \bullet \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \bullet \vec{B} = 0 \quad \nabla \times B = \mu\varepsilon\frac{\partial \vec{E}}{\partial t} + \mu\sigma\vec{E} \quad (14)$$

Same principle as in (a), but now

$$\nabla\left(\nabla \bullet \vec{E}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla\left(\nabla \bullet \vec{B}\right) - \nabla^2 \vec{B} = -\mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} - \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

and with zero divergence:

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \qquad (15)$$

$$\nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

Again plane waves:

$$\vec{E} = \vec{E}_0 e^{i(kx-\omega t)} \qquad (16)$$

$$\vec{B} = \vec{B}_0 e^{i(kx-\omega t)}$$

(E-field only because B-field has identical mathematics)

$$\vec{E}_0 \frac{\partial^2 e^{i(kx-\omega t)}}{\partial x^2} = \mu\varepsilon \vec{E}_0 \frac{\partial^2 e^{i(kx-\omega t)}}{\partial t^2} + \mu\sigma \vec{E}_0 \frac{\partial e^{i(kx-\omega t)}}{\partial t}$$

$$-\vec{E}_0 k^2 e^{i(kx-\omega t)} = -\omega^2 \mu\varepsilon \vec{E}_0 e^{i(kx-\omega t)} - i\omega\mu\sigma \vec{E}_0 e^{i(kx-\omega t)} \qquad (17)$$

$$k^2 = \omega^2 \mu\varepsilon + i\omega\mu\sigma$$

So now k is complex, so set $k = k_1 + ik_2$ giving

$$\vec{E} = \vec{E}_0 e^{-k_2 x} e^{i(k_1 x - \omega t)}$$

with k_1 in the role previously played by k (determines velocity and wavelength) and k_2 representing an inverse attenuation length because the field attenuates by e^{-1} in a length $1/k_2$.

$$k = k_1 + ik_2 \Rightarrow k^2 = k_1^2 - k_2^2 + 2ik_1k_2 \qquad (18)$$

Equate real and imaginary parts of (17) and (18).

$$k_1^2 - k_2^2 = \mu\varepsilon\omega^2 \qquad (19)$$

$$2k_1k_2 = \mu\omega\sigma$$

Solve two equation simultaneously. Substitute 2nd of (19) into first of (19) to get k_2

$$\left(\frac{\mu\omega\sigma}{2k_2}\right)^2 - k_2^2 = \mu\varepsilon\omega^2$$

$$k_2^4 + \mu\varepsilon\omega^2 k_2^2 - \left(\frac{\mu\omega\sigma}{2}\right)^2 = 0$$

$$k_2^2 = -\frac{\mu\varepsilon\omega^2}{2} \pm \sqrt{\left(\frac{\mu\varepsilon\omega^2}{2}\right)^2 + \left(\frac{\mu\omega\sigma}{2}\right)^2} = \omega^2\frac{\mu\varepsilon}{2}\left[-1 \pm \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right] \qquad (20)$$

$$k_2 = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[-1 \pm \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}$$

Choose positive sign to recover correct k_2 ->0 as σ ->0 result.

In the high conductivity limit, $\frac{\sigma}{\varepsilon\omega} >> 1$

$$k_2 \approx \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\frac{\sigma}{\varepsilon\omega}\right]^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad (21)$$

1/*e* attenuation length

$$k_2^{-1} \approx \sqrt{\frac{2}{\omega\mu\sigma}} \qquad (22)$$

Einstein's theory of heat capacity Einstein treated the atoms in a crystal as simple harmonic oscillators, all having the same vibrational frequency ω. The energy levels of the harmonic oscillators are given as

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

Consider a crystal consisting of N atoms.

(a) Find the partition function Z_1 for a single oscillator and the mean energy per oscillator u at temperature T. What is the total energy of the N-atom crystal?

(b) What is the heat capacity of the crystal at constant volume C_V? Show that $C_V \cong 3Nk_B$ at the high temperature limit ($k_BT \gg \hbar\omega$) and $C_V \to 0$ at the low temperature limit ($k_BT \ll \hbar\omega$).

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(a) Find the partition function Z_1 for a single oscillator and the mean energy per oscillator u at temperature T. What is the total energy of the N-atom crystal?

Solution:
Assuming the oscillators are in thermal equilibrium at T, the partition function for a single oscillator is

$$\begin{aligned} Z_1 &= \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}, \text{ where } \beta \equiv \frac{1}{k_B T} \\ &= e^{-x/2}\sum_{n=0}^{\infty} e^{-nx} = \frac{e^{-x/2}}{1-e^{-x}}, \text{ where } x \equiv \beta\hbar\omega \\ &= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}. \end{aligned} \tag{79}$$

The mean energy per oscillator is then

$$\begin{aligned} u &= -\frac{d}{d\beta}\ln Z_1 \\ &= \frac{d}{d\beta}\left\{\frac{1}{2}\beta\hbar\omega + \ln\left(1-e^{-\beta\hbar\omega}\right)\right\} \\ &= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}. \end{aligned} \tag{80}$$

The N-atom crystal is composed of $3N$ identical oscillators of the resonance frequency ω. Using the fact that energy is an extensive property, the energy of the $3N$ oscillators in the N atom crystal is

$$U = 3Nu = 3N\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}\right). \tag{81}$$

(b) What is the heat capacity of the crystal at constant volume C_V? Show that $C_V \cong 3Nk_B$ at the high temperature limit ($k_BT \gg \hbar\omega$) and $C_V \to 0$ at the low temperature limit ($k_BT \ll \hbar\omega$).

Solution:
The heat capacity is

$$\begin{aligned} C_V &= \left(\frac{\partial U}{\partial T}\right)_V = 3N\left(\frac{\partial U}{\partial \beta}\right)_V \frac{\partial \beta}{\partial T} \\ &= 3Nk_B \frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2} && (82) \\ &= 3Nk_B \frac{x^2 e^x}{\left(e^x-1\right)^2}, \text{ where } x = \beta\hbar\omega. && (83) \end{aligned}$$

When $k_BT \gg \hbar\omega$ (i.e., $x \ll 1$), $e^x - 1 \cong x$ and $e^x \cong 1$, therefore

$$C_V \cong 3Nk_B \frac{x^2 \cdot 1}{x^2} = 3Nk_B.$$

When $k_BT \ll \hbar\omega$ (i.e., $x \gg 1$), $e^x - 1 \cong e^x$, therefore $C_V \cong 3Nk_B x^2 e^{-x} \to 0$.