OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #114

Monday, September 24 and Tuesday, September 25, 2012

Fall 2012 Comprehensive Examination

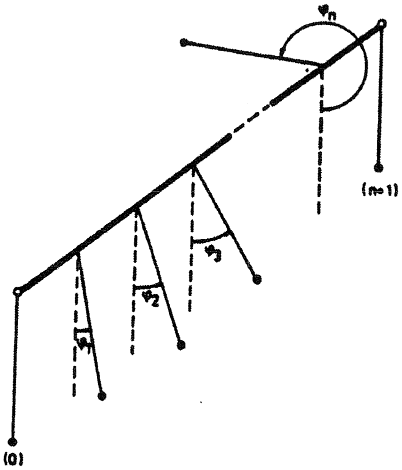
PART 1, Monday, September 24, 9:00am

General Instructions

This Fall 2012 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 24, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 25, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

A system consists of n identical simple pendulums of length l and mass m which are suspended along a straight line and which swing in planes perpendicular to that line. The deviations from the vertical are measured by angles ϕ_i. The pendulums are coupled by harmonic forces in such a way that the torque acting between the i-th and $(i+1)$-th pendulum is given by $-k(\phi_{i+1}-\phi_i)$. The line of suspension may be thought of as realized by a torsion bar. The chain is fixed at both ends and we formally add two more motionless pendulums at either end of the bar, which we label with numbers 0 and $(n+1)$. This means that the angles ϕ_0 and ϕ_{n+1} are taken to be zero at all times. In this problem the deviations (angles) ϕ_i are not small!



a) Construct the Lagrangian of the system and derive the equations of motion. Analyze the equations of motions in two cases: (i) when gravity is ignored, $g=0$; and (ii) when $k=0$.

b) No longer neglecting g or k, solve the equations of motion in the case of small deviations from the vertical. Find the eigenfrequencies and normal modes. You may first solve these equations for two or three moving pendulums or directly for n. Describe the slowest mode.

c) Let the horizontal separation of the pendulums be d, so that the length of the chain is $L=(n+1)d$. Consider the transition to the continuous system by taking the limit $n\to\infty$ and $d\to 0$. Keep $v^2=\dfrac{kd^2}{ml^2}$ finite in the same limit. Take the continuum limit of the equations, where the countable variables $\phi_1(t),\ldots,\phi_n(t)$ are replaced with the continuous variable $\phi(x,t)$ and x is taking over the role of the counting index, which runs from 1 to n. Use the fact that

$$(\phi_{i+1}-\phi_i)\to d\frac{\partial\phi}{\partial x}\bigg|_{x=id+d/2} .$$

a) $$T = \frac{1}{2} m \ell^2 \sum_{j=0}^{n+1} \dot{\phi}_j^2$$

$$U = mg\ell \sum_{j=0}^{n+1} (1-\cos\phi_j) + \frac{k}{2} \sum_{j=0}^{n} (\phi_{j+1} - \phi_j)^2$$

$$L = T - U = \frac{1}{2} m\ell^2 \sum_{j=0}^{n+1} \dot{\phi}_j^2 - mg\ell \sum_{j=0}^{n+1} (1-\cos\phi_j) - \frac{k}{2} \sum_{j=0}^{n} (\phi_{j+1} - \phi_j)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow m\ell^2 \ddot{\phi}_j + mg\ell \sin\phi_j - k(\phi_{j+1} - \phi_j) + k(\phi_j - \phi_{j-1}) = 0$$

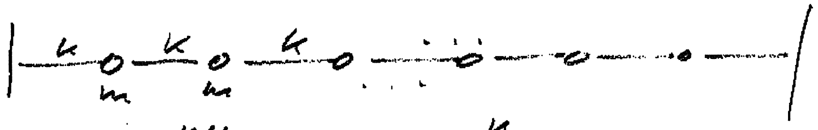
$$\Leftrightarrow \ddot{\phi}_j + \omega_g^2 \sin\phi_j - \omega_0^2 \left[(\phi_{j+1} - \phi_j) - (\phi_j - \phi_{j-1})\right] = 0 \qquad j = 1, \dots, n$$

with $\omega_g^2 = \frac{g}{\ell}$ $\omega_0^2 = \frac{k}{m\ell^2}$ $j = 1, \dots, n$.

(i) $g = 0 \rightarrow \omega_g = 0$

$$\ddot{\phi}_j - \omega_0^2 \left[(\phi_{j+1} - \phi_j) - (\phi_j - \phi_{j-1})\right] = 0 \qquad j = 1, \dots, n$$

equivalent to point masses on a string moving in one dimension (transverse motion)



$$L = \frac{m}{2} \sum_{j=0}^{n+1} \dot{x}_j^2 - \frac{k}{2} \sum_{j=0}^{n} (x_{j+1} - x_j)^2$$

(ii) $k=0$ → n independent pendulums.

b) small angles $\sin\phi_j \sim \phi_j$

$$\Rightarrow \ddot{\phi}_j + (\omega_g^2 + 2\omega_0^2)\phi_j - \omega_0^2(\phi_{j+1} + \phi_{j-1}) = 0$$

$j = 1, \dots, n$

ansatz: $\phi_j = a_j e^{i\Omega t}$

leads to $\sum_{j=1}^{n} (-\Omega^2 t_{ij} + b_{ij}) a_j = 0$

for $n=2$ $\underline{\underline{t}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

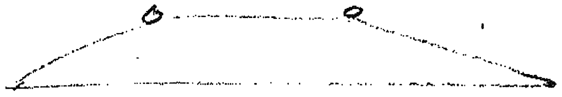
$$\underline{\underline{b}} = \begin{pmatrix} \omega_g^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_g^2 + 2\omega_0^2 \end{pmatrix}$$

solutions for $\det(b_{ij} - \Omega^2 t_{ij}) \overset{!}{=} 0$

$$(\omega_g^2 + 2\omega_0^2 - \Omega^2)^2 - \omega_0^4 = 0$$

$$\Rightarrow \Omega_1^2 = \omega_g^2 + \omega_0^2$$
$$\Omega_2^2 = \omega_g^2 + 3\omega_0^2$$

lowest mode:
(in case of a string)



general solution:

$$\phi_j^{(p)}(t) = A^{(p)} \sin\left(\frac{jp\pi}{n+1}\right) \sin(\Omega_p t)$$

with $\Omega_p^2 = \omega_g^2 + 2\omega_0^2\left(1 - \cos\left(\frac{p\pi}{n+1}\right)\right)$

$$= \omega_g^2 + 4\omega_0^2 \sin^2\left(\frac{p\pi}{2(n+1)}\right)$$

$p = 1, \dots, n$

c) continuum limit:

pendulums at positions $x = jd = j\frac{L}{n+1}$

$$\phi_j(t) \to \phi(x,t)$$

use $$\lim_{d\to 0} \frac{\phi((j+1)d,t)-\phi(jd,t)}{d} = \left.\frac{\partial \phi}{\partial x}\right|_{x=jd+\frac{d}{2}}$$

$$\Rightarrow \phi_{j+1}-\phi_j \to d\left.\frac{\partial \phi}{\partial x}\right|_{x=jd+\frac{d}{2}}$$

$$\phi_j-\phi_{j-1} \to d\left.\frac{\partial \phi}{\partial x}\right|_{x=jd-\frac{d}{2}}$$

$$\Rightarrow (\phi_{j+1}-\phi_j)-(\phi_j-\phi_{j-1}) \longrightarrow d^2\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x=jd}$$

in the limit $n\to\infty$, $d\to 0$, $v^2 = \frac{kd^2}{ml^2}$ finite

$$\Rightarrow \frac{\partial^2\phi(x,t)}{\partial t^2} + \omega_g^2 \sin\phi(x,t) - \underbrace{\omega_0^2 d^2}_{\frac{kd^2}{ml^2} \overset{(*)}{=} v^2 \text{ finite}} \frac{\partial^2\phi(x,t)}{\partial x^2} = 0$$

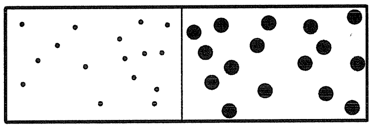
$$\Rightarrow \frac{\partial^2\phi(x,t)}{\partial t^2} - v^2\frac{\partial^2\phi(x,t)}{\partial x^2} + \omega_g^2 \sin\phi(x,t) = 0$$

"Sine Gordon equation"

Remark: continuum limit: linear mass density $\rho = \frac{m}{d}$, $k = \frac{\eta}{d}$ where $\eta \sim$ to torsion of the bar.

when $d\to 0 \Rightarrow k\to\infty$ such that $\eta = kd$ finite: $\frac{kd^2}{ml^2} = \frac{\eta d}{\rho d l^2} = \frac{\eta}{\rho l^2} = v^2$

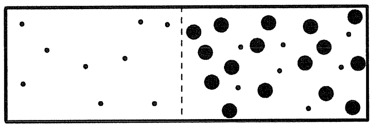
Consider a thermally insulated box containing two monatomic ideal gasses, separated by an impermeable barrier.



On the left of the partition are N_0 atoms of a small atom type S, stored in volume V_0 at temperature T_0. On the right side of the partition are N_0 atoms of a large atom type L, which occupy volume V_0 and are in thermal equilibrium with the smaller atoms on the left side of the barrier.

(a) What is the pressure on each side of the partition?

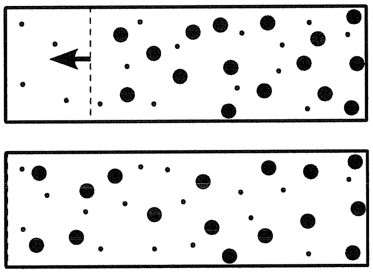
(b) The barrier between the two sides of the box is now made permeable to the small atoms only, while remaining impermeable to the large atoms on the right side.



After the box has reached equilibrium, what will be the pressure on each side of the box? What is the ratio between this pressure and the initial pressure?

(c) What is the change in entropy of the system (defined as everything in the box) between its initial state and its equilibrium state with the permeable membrane?

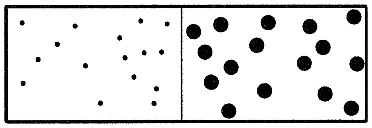
(d) Now we will slowly move the very thin, permeable partition to the left side of the box, until it reaches the left-hand wall, at which point there will be only one enclosure with volume equal to $2V_0$.



At this stage, what is the pressure in the box?

(e) What is the change in entropy of the system (enclosed by the box) as a result of moving the permeable membrane to its edge?

Consider a thermally insulated box containing two monatomic ideal gasses, separated by an impermeable barrier.



On the left of the partition are N_0 atoms of a small atom type S, stored in volume V_0 at temperature T_0. On the right side of the partition are N_0 atoms of a large atom type L, which occupy volume V_0 and are in thermal equilibrium with the smaller atoms on the left side of the barrier.

(a) What is the pressure on each side of the partition?

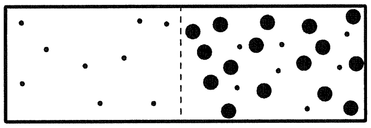
Solution:
We can find the pressure on the left side using the ideal gas law:

$$p_{left} = \frac{N_0 k_B T_0}{V_0} \tag{1}$$

The temperature is equal on each side, since the two sides are in thermal equilibrium. Thus we find that the pressure on the right hand side is the same (since the volume and number are also the same):

$$p_{right} = \frac{N_0 k_B T_0}{V_0} \tag{2}$$

(b) The barrier between the two sides of the box is now made permeable to the small atoms only, while remaining impermeable to the large atoms on the right side.



After the box has reached equilibrium, what will be the pressure on each side of the box? What is the ratio between this pressure and the initial pressure?

Solution:
We will again use the ideal gas law, but this time we will need to work out the number of atoms on each side of the box, as well as the temperature.

The temperature is pretty easy, since no work is being done, and the box is insulated. This means that by the First Law, the internal energy of the system cannot change. Since the internal energy of an ideal gas is only dependent on temperature, we can conclude that the temperature has not changed.

The number of atoms on each side is also relatively straightforward. The number of large atoms is unchanged, since they cannot move. The small atoms will arrange themselves so as to maximize entropy, which will result in an equal number of small atoms being on each side of the partition, this being the "least ordered" state, which means there will be $N_0/2$ small atoms on each side of the partition. Thus we find that:

$$p_{left} = \frac{\frac{N_0}{2} k_B T_0}{V_0} \tag{3}$$

$$= \frac{1}{2} p_0 \tag{4}$$

The right side has both big and small atoms, so it gets a pressure of:

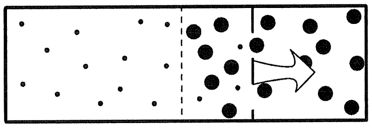
$$p_{right} = \frac{\left(N_0 + \frac{N_0}{2}\right) k_B T_0}{V_0} \tag{5}$$

$$= \frac{3}{2} p_0 \tag{6}$$

(c) What is the change in entropy of the system (defined as everything in the box) between its initial state and its equilibrium state with the permeable membrane?

Solution:

Finding ΔS requires that we find a reversible path from the starting point to the ending point. One way of doing this uses two separators, one of which is permeable to small atoms but blocks big ones, while the other is permeable to big atoms but blocks small ones (don't ask me how!).



We can now use the one separator to keep the big atoms on the right, while the other one slowly moves to the right, allowing the small atoms to fill the entire container. We want to do this isothermally, since the initial and final states have the same temperature.

We now want to solve for ΔS using

$$\Delta S = \int \frac{dQ}{T} \tag{7}$$

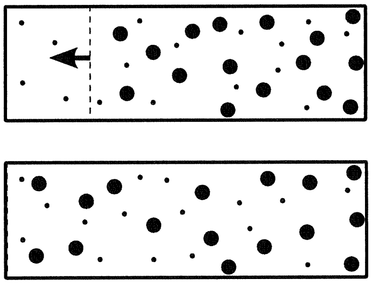
$$= \frac{Q}{T} \tag{8}$$

where the latter equality came about because the temperature is fixed. We can solve for Q using the First Law (and the fact that for an ideal gas the internal energy only depends on temperature):

$$\begin{aligned}
\Delta U &= Q + W && (9)\\
0 &= Q - \int p dV && (10)\\
Q &= \int p dV && (11)\\
&= \int \frac{Nk_B T}{V} dV && (12)\\
&= Nk_B T \int \frac{dV}{V} && (13)\\
&= Nk_B T \ln \frac{2V_0}{V_0} && (14)\\
&= Nk_B T \ln 2 && (15)\\
\Delta S &= Nk_B \ln 2 && (16)
\end{aligned}$$

We could alternatively have gotten the same answer from a statistical approach, but that seems harder to me.

(d) Now we will slowly move the very thin, permeable partition to the left side of the box, until it reaches the left-hand wall, at which point there will be only one enclosure with volume equal to $2V_0$.



At this stage, what is the pressure in the box?

Solution:

Since work is being done on the partition in this case (and the system is thermally insulated), the temperature is going to change. Once we find the change in temperature, we will know what the final pressure is. To find the final temperature, we just need to find the amount of work done.

$$\begin{aligned} U &= \frac{3}{2}N_{\text{small}}k_BT + \frac{3}{2}N_{\text{large}}k_BT && (17)\\ &= 3Nk_BT && (18)\end{aligned}$$

So let's look at what the work is (keeping in mind $dS = 0$ for this adiabatic process).

$$\begin{aligned} dU &= TdS - pdV && (19)\\ &= -pdV && (20)\\ &= -\frac{Nk_BT}{V}dV && (21)\\ &= 3Nk_BdT && (22)\\ \frac{dV}{V} &= -3\frac{dT}{T} && (23)\\ \int\frac{dV}{V} &= -\int 3\frac{dT}{T} && (24)\\ \ln\frac{V_f}{V_i} &= -3\ln\frac{T_f}{T_i} && (25)\\ \ln\frac{V_f}{V_i} &= \ln\frac{T_i^3}{T_f^3} && (26)\\ \frac{V_f}{V_i} &= \frac{T_i^3}{T_f^3} && (27)\\ T_f^3 &= T_i^3\frac{V_i}{V_f} && (28)\\ &= \frac{T_o^3}{2} && (29)\\ T_f &= \frac{T_o}{\sqrt[3]{2}} && (30)\end{aligned}$$

Now that we have the final temperature, we can use the known volume $2V_0$ and number of atoms $2N$ with the ideal gas law to find the pressure.

$$\begin{aligned} p_f &= \frac{2Nk_BT_f}{2V_0} && (31)\\ &= \frac{1}{\sqrt[3]{2}}\frac{Nk_BT_0}{V_0} && (32)\\ &= \frac{p_0}{\sqrt[3]{2}} && (33)\end{aligned}$$

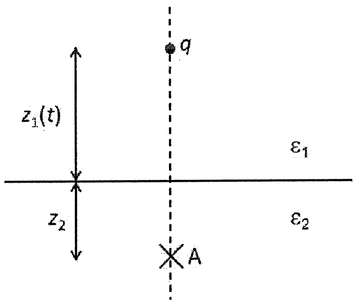
So the final pressure is $\sqrt[3]{2}$ smaller than the initial pressure.

(e) What is the change in entropy of the system (enclosed by the box) as a result of moving the permeable membrane to its edge?

Solution:

Since this is a reversibale, adiabatic process, $\Delta S = 0$.

Consider an experiment in which a moving charge q is monitored using a detector at point A as illustrated below.



The moving charge has fixed value q and moves through a liquid with relative dielectric constant ε_1. The detector is inside a semi-infinite dielectric slab with relative dielectric constant ε_2. The charge q is constrained to move along an imaginary line (dashed line in the Figure). This imaginary line is perpendicular to the dielectric interface and passes through point A. The detector at point A is sensitive to the local electrostatic potential. Assume a "perfect detector" that does not modify the electrostatic potential that is being sampled.

a) What boundary conditions does the electric field satisfy at the dielectric interface?

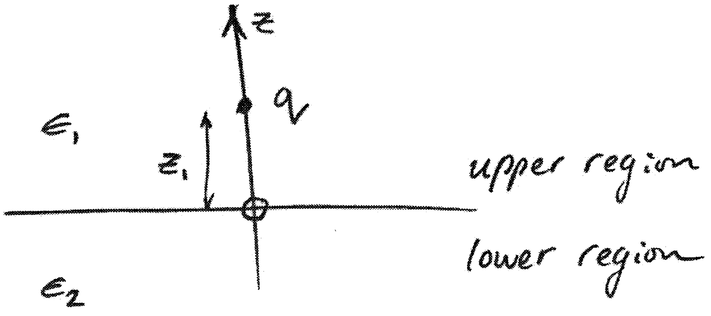
b) Solve for the electrostatic potential $\phi(r)$ on either side of the dielectric interface.

c) Find an expression for $\Delta\phi$, the change in electrostatic potential at point A, when the charge moves from z_1 to $z_1 + \delta$.

①

Solutions to problem 3

a)

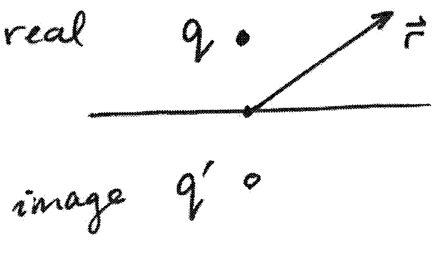


$$\epsilon_1 E_z^{upper}(x,y,0) = \epsilon_2 E_z^{lower}(x,y,0)$$

$$E_y^{upper}(x,y,0) = E_y^{lower}(x,y,0)$$

$$E_x^{upper}(x,y,0) = E_x^{lower}(x,y,0)$$

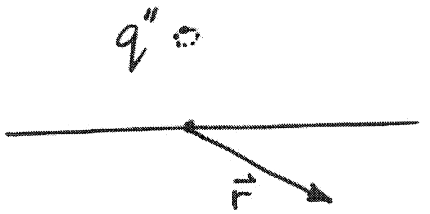
b) Upper region, $z > 0$.



$$\phi_{upper} = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{|\vec{r} - z_1\hat{z}|} + \frac{q'}{|\vec{r} + z_1\hat{z}|}\right)$$

②

Lower region $z<0$



$$\phi_{lower} = \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{|\vec{r}-z_1\hat{z}|}\right)$$

Now match boundary conditions. See if q' & q'' can be picked to satisfy the boundary conditions.

When $x \gg z_1$ at the interface,

$$E_x^{upper} = \frac{q+q'}{4\pi\epsilon_1 x^2}$$

$$E_x^{lower} = \frac{q''}{4\pi\epsilon_2 x^2}$$

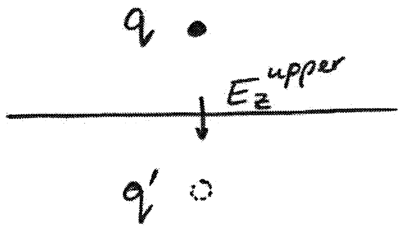
$$E_x^{upper} = E_x^{lower}$$

$$\frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2}$$

$$\boxed{\epsilon_2(q+q') = \epsilon_1 q''}$$ ——— ①

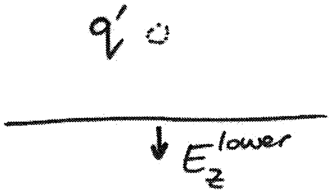
③

Directly below q, at the interface,



$$E_z^{upper} = \frac{-q}{4\pi\epsilon_1 z_1^2} + \frac{q'}{4\pi\epsilon_1 z_1^2}$$

$$= \frac{q'-q}{4\pi\epsilon_1 z_1^2}$$



$$E_z^{lower} = \frac{-q''}{4\pi\epsilon_2 z_1^2}$$

$$\epsilon_1 E_z^{upper} = \epsilon_2 E_z^{lower}$$

$$\boxed{q' - q = -q''} \qquad \text{———} \quad ②$$

Combining ① & ② to eliminate q''

$$q' - q = -\frac{\epsilon_2}{\epsilon_1}(q + q')$$

$$q' + \frac{\epsilon_2}{\epsilon_1}q' = q - \frac{\epsilon_2}{\epsilon_1}q$$

$$q'\left(1 + \frac{\epsilon_2}{\epsilon_1}\right) = q\left(1 - \frac{\epsilon_2}{\epsilon_1}\right)$$

$$q' = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right)q$$

Combining ① & ② to eliminate q'

$$\epsilon_2(q+q-q'') = \epsilon_1 q''$$

$$2\epsilon_2 q - \epsilon_2 q'' = \epsilon_1 q''$$

$$2\epsilon_2 q = \epsilon_1 q'' + \epsilon_2 q''$$

$$q'' = \left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right) q$$

c) Point A is in the lower region where

$$\phi^A = \frac{1}{4\pi\epsilon_2}\,\frac{1}{(z_1+z_2)}\,q''$$

$$= \frac{1}{4\pi\cancel{\epsilon_2}}\,\frac{1}{(z_1+z_2)}\,\frac{2\cancel{\epsilon_2}}{(\epsilon_1+\epsilon_2)}\,q = \frac{2q}{4\pi(\epsilon_1+\epsilon_2)(z_1+z_2)}$$

When the charge moves from z_1 to $z_1+\delta$

$$\Delta\phi = \frac{2q}{4\pi(\epsilon_1+\epsilon_2)}\left(\frac{1}{z_2+z_1+\delta} - \frac{1}{z_2+z_1}\right)$$

If $\delta \ll z_1+z_2$ then

$$\frac{1}{z_2+z_1+\delta} = \frac{1}{z_2+z_1}\left(1+\frac{\delta}{z_2+z_1}\right)^{-1} \approx \frac{1}{z_2+z_1}\left(1-\frac{\delta}{z_2+z_1}\right)$$

$$\Rightarrow \Delta\phi = \frac{2q}{4\pi(\epsilon_1+\epsilon_2)}\,\frac{\delta}{(z_1+z_2)^2}$$

Two identical spin-1/2 particles in free space undergo mutual spin-spin interactions described as

$$H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin angular momenta of the two particles, and α is the coupling constant. Spin measurements on the individual particles at $t = 0$ determine the initial state of the two-particle system as

$$|\psi(t = 0)\rangle = |+ -\rangle$$

indicating that the particle 1 is spin-up and the particle 2 is spin-down.

(a) Find the state vector at an arbitrary time t.

(b) What is the probability that the particle 2 is spin-up at t?

(c) Find how the expectation value of S_z evolves in time.

(d) What about the expectation value of $S_x = S_{1x} + S_{2x}$?

P4. Two identical spin-1/2 particles in free space undergo mutual spin-spin interactions described as

$$H = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin angular momenta of the two particles, and α is the coupling constant. Spin measurements on the individual particles determine the initial state of the two-particle system at $t = 0$ as

$$|\psi(t = 0)\rangle = |+\,-\rangle$$

indicating that the particle 1 is spin-up and the particle 2 is spin-down.

(a) Find the state vector at an arbitrary time t.

(i) Energy eigenstates

The four uncoupled states, $|+\,+\rangle, |+\,-\rangle, |-\,+\rangle, |-\,-\rangle$, form a set of basis vectors, but they are not the energy eigenstates. The Hamiltonian can be rewritten as

$$H = \frac{\alpha}{2}[S^2 - S_1^2 - S_2^2] = \frac{1}{2}\alpha\left(S^2 - \frac{3}{2}\hbar^2\right),$$

thus eigenstates of the total angular momentum S^2 are simultaneously energy eigenstates of the two-particle system. For $s_1 = s_2 = \frac{1}{2}$, s can take only two values: $s = s_1 + s_2 = 1$ or $s = |s_1 - s_2| = 0$. The energy eigenstates are

for triplet ($s = 1$) with energy eigenvalue $E_1 = \frac{1}{2}\alpha\left[s(s+1)\hbar^2 - \frac{3}{2}\hbar^2\right] = \frac{1}{4}\alpha\hbar^2$

$$|s = 1, m_s = 1\rangle = |+\,+\rangle$$

$$|s = 1, m_s = 0\rangle = \frac{1}{\sqrt{2}}(\,|+\,-\rangle + |-\,+\rangle\,)$$

$$|s = 1, m_s = -1\rangle = |-\,-\rangle$$

and for singlet ($s = 0$) with energy eigenvalue $E_0 = -\frac{3}{4}\alpha\hbar^2$

$$|s = 0, m_s = 0\rangle = \frac{1}{\sqrt{2}}(\,|+\,-\rangle - |-\,+\rangle\,)$$

(ii) Time evolution

The time-dependent state vector is obtained by applying time-evolution operator to the initial state:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_1 t}|1{,}0\rangle + e^{-\frac{i}{\hbar}E_0 t}|0{,}0\rangle\right) = \frac{e^{-\frac{i}{\hbar}E_1 t}}{\sqrt{2}}(\,|1{,}0\rangle + e^{i\alpha\hbar t}|0{,}0\rangle\,)$$

(b) What is the probability that the particle 2 is in the spin-up state at t?

The state vector can be expressed in terms of the uncoupled states:

$$|\psi(t)\rangle = \frac{e^{-\frac{i}{\hbar}E_1 t}}{\sqrt{2}}\left(\,|1,0\rangle + e^{i\alpha\hbar t}|0,0\rangle\,\right)$$

$$= \frac{e^{-\frac{i}{\hbar}E_1 t}}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(\,|+-\rangle + |-+\rangle\,) + e^{i\alpha\hbar t}\frac{1}{\sqrt{2}}(\,|+-\rangle - |-+\rangle\,)\right]$$

$$= \frac{1}{2}e^{-\frac{i}{\hbar}E_1 t}\left[\left(1 + e^{i\alpha\hbar t}\right)|+-\rangle + \left(1 - e^{i\alpha\hbar t}\right)|-+\rangle\right] = c_{2-}(t)|+-\rangle + c_{2+}(t)|-+\rangle$$

The probability for the particle 2 being spin-up at t is

$$|c_{2+}(t)|^2 = \frac{1}{4}\left|1 - e^{i\alpha\hbar t}\right|^2 = \sin^2\left(\frac{\alpha\hbar}{2}t\right)$$

(c) Find how the expectation value of S_z evolve in time.

$\langle S_z\rangle(t) = \langle\psi(t)|S_z|\psi(t)\rangle$

$$S_z|\psi(t)\rangle = \frac{e^{-\frac{i}{\hbar}E_1 t}}{\sqrt{2}}\left(S_z\,|1,0\rangle + S_z|0,0\rangle\,\right) = 0$$

Thus, $\langle S_z\rangle(t) = 0$

(d) What about the expectation value of $S_x = S_{1x} + S_{2x}$?

$\langle S_x\rangle(t) = \langle\psi(t)|S_x|\psi(t)\rangle$

$$S_x|+-\rangle = S_{1x}|+-\rangle + S_{2x}|+-\rangle = \frac{\hbar}{2}(|--\rangle + |++\rangle) = \frac{\hbar}{2}(|1,1\rangle + |1,-1\rangle)$$

Similarly, $S_x|-+\rangle = \frac{\hbar}{2}(|1,1\rangle + |1,-1\rangle)$

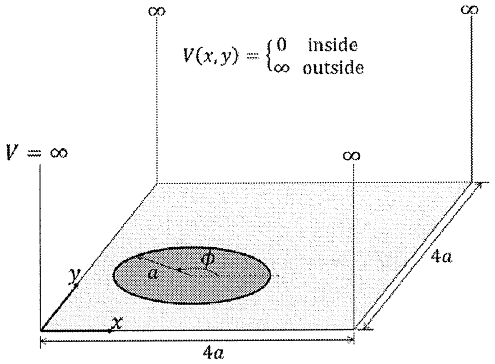
Thus, $S_x|1,0\rangle = \frac{1}{\sqrt{2}}(S_x\,|+-\rangle + S_x|-+\rangle\,) = \frac{\hbar}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$

and $S_x|0,0\rangle = \frac{1}{\sqrt{2}}(S_x\,|+-\rangle - S_x|-+\rangle\,) = 0$

Then,

$$\langle S_x\rangle(t) = \langle\psi(t)|S_x|\psi(t)\rangle = \frac{e^{\frac{i}{\hbar}E_1 t}}{\sqrt{2}}\left(\,\langle 1,0| + e^{-i\alpha\hbar t}\langle 0,0|\,\right)\frac{\hbar}{2}e^{-\frac{i}{\hbar}E_1 t}\left(|1,1\rangle + e^{i\alpha\hbar t}|1,-1\rangle\right) = 0$$

We consider air hockey in the quantum limit. Imagine that a hockey puck (a solid disk of radius a and mass m) is placed in a hockey table (2-dimensional infinite potential well of two equal sides $4a$, i.e., the puck can move only in a $2a \times 2a$ square) as shown in the figure below.



(a) Write down the Schrödinger equation in terms of (x, y, ϕ) coordinates. Find the energy eigenvalues and eigenfunctions.

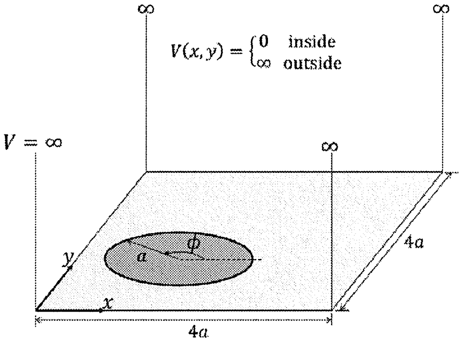
(b) Identify the energy eigenvalues and eigenfunctions of the five lowest energy levels. Note that an energy level can be degenerate. Determine the degree of degeneracy for the five energy levels.

(c) Someone tilts the table a little so that the potential linearly increases along the x-axis, i.e., $V(x, y) = 0$ at one edge and $V(x, y) = V_0$ at the other end at the potential well, where V_0 is small compared to the energy eigenvalues acquired in (b). Make first order corrections to the three lowest energy levels. The following integrals may be useful:

$$\int x \sin^2 x \, dx = \frac{x^2}{4} - \frac{1}{8}\cos(2x) - \frac{1}{4}x\sin(2x)$$

$$\int x \sin(x) \sin(2x) \, dx = \frac{1}{18}[9\cos(x) - \cos(3x) + 12x\sin^3(x)]$$

P5. We consider air hockey in the quantum limit. Imagine that a hockey puck (a solid disk of radius a and mass m) is placed in a hockey table (2-dimensional infinite potential well of two equal sides $4a$, i.e., the puck can move only in a $2a \times 2a$ square) as shown in the figure below.



(a) Write down the Schrödinger equation in terms of (x, y, ϕ) coordinates. Find the energy eigenvalues and eigenfunctions.

Since the translational motion is confined within a $2a \times 2a$ square, we can write the effective potential as

$$V_{eff}(x,y) = \begin{cases} 0 & 0 < x, y < 2a \\ \infty & \text{otherwise} \end{cases}$$

The Hamiltonian consists of translational and rotational energies in addition to the potential:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{L_z^2}{2I} + V_{eff}(x,y)$$

where $p_x = \frac{\hbar}{i}\frac{d}{dx}$, $p_y = \frac{\hbar}{i}\frac{d}{dy}$, and $L_z = \frac{\hbar}{i}\frac{d}{d\phi}$ with moment of inertia $I = \frac{1}{2}ma^2$.

Then, the Schrodinger equation is written as

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dx^2} + \frac{2}{a^2}\frac{d^2}{d\phi^2}\right) + V_{eff}(x,y)\right]\psi(x,y,\phi) = E\,\psi(x,y,\phi)$$

Applying separation of variable and the boundary conditions, $\psi(x=0) = \psi(x=2a) = \psi(y=0) = \psi(y=2a) = 0$ and $\psi(\phi) = \psi(\phi + 2\pi)$, we obtain the normalized eigenfuctions,

$$\psi_{n_x,n_y,m_z}(x,y,\phi) = X_{n_x}(x)\, Y_{n_y}(y)\, \Phi_{\mathrm{m_z}}(\phi)$$

where $X_{n_x}(x) = \frac{1}{\sqrt{a}}\sin\left(\frac{\pi n_x}{2a}x\right)$, $Y_{n_x}(x) = \frac{1}{\sqrt{a}}\sin\left(\frac{\pi n_y}{2a}y\right)$, and $\Phi_{\mathrm{m_z}}(\phi) = \frac{1}{\sqrt{2\pi}}e^{im_z\phi}$

with the three quantum numbers, $n_x, n_y = 1,2,3,\cdots$ and $m_z = 0, \pm1, \pm2, \cdots$.

The corresponding energy eigenvalues are

$$E_{n_x,n_y,m_z} = \frac{\hbar^2}{2m}\left(\frac{\pi n_x}{2a}\right)^2 + \frac{\hbar^2}{2m}\left(\frac{\pi n_y}{2a}\right)^2 + \frac{\hbar^2}{ma^2}m_z^2 = \frac{\pi^2\hbar^2}{8ma^2}\left(n_x^2 + n_y^2 + \frac{8}{\pi^2}m_z^2\right)$$

(b) Identify the energy eigenvalues and eigenfunctions of the five lowest energy levels. Note that an energy level can be degenerated. Determine the degree of degeneracy for the five energy levels.

Energy level	n_x	n_y	m_z	$E_{n_x,n_y,m_z}/\frac{\pi^2\hbar^2}{8ma^2}$
1	1	1	0	2
2 (2-fold degeneracy)	1	1	±1	2.81
3 (2-fold degeneracy)	1	2	0	5
	2	1	0	
4 (2-fold degeneracy)	1	1	±2	5.24
5 (4-fold degeneracy)	1	2	±1	5.81
	2	1	±1	

The eigenfunctions of the five levels are

Level-1

$$\psi_{110}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{2a}y\right)$$

Level-2

$$\psi_{11\pm1}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{2a}y\right)e^{\pm i\phi}$$

Level-3

$$\psi_{12\pm1}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)$$

$$\psi_{21\pm1}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}y\right)$$

Level-4

$$\psi_{11\pm2}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{2a}y\right)e^{\pm2i\phi}$$

Level-5

$$\psi_{12\pm1}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{2a}x\right)\sin\left(\frac{\pi}{a}y\right)e^{\pm i\phi}$$

$$\psi_{21\pm1}(x,y,\phi) = \frac{1}{a\sqrt{2\pi}}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}y\right)e^{\pm i\phi}$$

(c) Someone tilts the table a little so that the potential linearly increases along the x-axis, i.e., $V(x,y)=0$ at one edge and $V(x,y)=V_0$ at the other inside the potential well, where V_0 is negligibly small compared to the energy eigenvalues acquired in (b). Make first order corrections to the three lowest energy levels. The following integrals may be useful:

$$\int x\sin^2 x\,dx=\frac{x^2}{4}-\frac{1}{8}\cos(2x)-\frac{1}{4}x\sin(2x)$$

$$\int x\sin(x)\sin(2x)\,dx=\frac{1}{18}[9\cos(x)-\cos(3x)+12x\sin^3(x)]$$

The perturbation Hamiltonian can be written as

$$H_1=\begin{cases}\dfrac{V_0}{4}+\dfrac{V_0}{4a}x & 0<x,y<2a\\ \infty & \text{otherwise}\end{cases}$$

(i) The first order energy correction for the ground state (level-1) is

$$\Delta E_{110}=\langle 110|H_1|110\rangle=\int_0^{2\pi}\int_0^{2a}\int_0^{2a}\psi_{110}^*H_1\psi_{110}\,dxdyd\phi$$

$$=\frac{V_0}{4}+\frac{V_0}{4a^2}\int_0^{2a}x\sin^2\left(\frac{\pi}{2a}x\right)dx$$

$$=\frac{V_0}{4}+\frac{V_0}{4a^2}\left(\frac{2a}{\pi}\right)^2\int_0^{\pi}\eta\sin^2(\eta)\,d\eta,\qquad \text{where } \eta=\frac{\pi}{2a}x$$

$$=\frac{V_0}{4}+\frac{V_0}{4a^2}\left(\frac{2a}{\pi}\right)^2\left[\frac{\eta^2}{4}-\frac{1}{8}\cos(2\eta)-\frac{1}{4}\eta\sin(2\eta)\right]_0^{\pi}=\frac{V_0}{2}$$

(ii) Level-2

This energy level is degenerated with two rotational states, but the perturbation affects only translational energy. The two states have the same first order energy correction.

$$\Delta E_{11\pm1}=\langle 11\pm1|H_1|11\pm1\rangle=\int_0^{2\pi}\int_0^{2a}\int_0^{2a}\psi_{11\pm1}^*H_1\psi_{11\pm1}\,dxdyd\phi$$

$$=\frac{V_0}{4}+\frac{V_0}{4a^2}\int_0^{2a}x\sin^2\left(\frac{\pi}{2a}x\right)dx=\frac{V_0}{2}$$

(ii) Level-3

We have to apply degenerated perturbation theory for this level. First, we calculate the four matrix elements:

$$\langle 120|H_1|120\rangle = \frac{V_0}{4} + \frac{V_0}{4a^2}\int_0^{2a} x\sin^2\left(\frac{\pi}{2a}x\right)dx = \frac{V_0}{2}$$

$$\langle 210|H_1|210\rangle = \frac{V_0}{4} + \frac{V_0}{4a^2}\int_0^{2a} x\sin^2\left(\frac{\pi}{a}x\right)dx$$

$$= \frac{V_0}{4} + \frac{V_0}{4a^2}\left(\frac{a}{\pi}\right)^2\left[\frac{\eta^2}{4} - \frac{1}{8}\cos(2\eta) - \frac{1}{4}\eta\sin(2\eta)\right]_0^{2\pi} = \frac{V_0}{2}$$

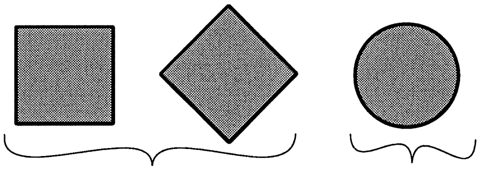
$\langle 120|H_1|210\rangle = \langle 210|H_1|120\rangle = 0$ because the y-axis basis $Y_{n_y}(y)$ are orthogonal. The perturbation is diagonalized in this 2 dimensional subspace,

$$H_1 = \begin{pmatrix} V_0/2 & 0 \\ 0 & V_0/2 \end{pmatrix}$$

i.e.,

$$\Delta E_{120} = \Delta E_{210} = \frac{V_0}{2}$$

Consider a protein that has two conformations, square and circular. The square conformation has a multiplicity of two—meaning that there are two distinguishable ways that the protein may assume a square conformation that is identical in every other way.



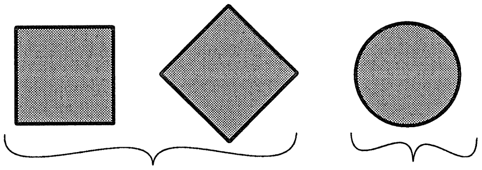
(a) What is the entropy of a system consisting of N protein molecules, of which $N_\square$ are square and $N_\bigcirc$ are circular? Please only consider the entropy arising from the conformation of the molecules (e.g. neglect entropy of orientation or position). Moreover, you may assume that the protein has no internal degrees of freedom in either conformation, apart from the above-mentioned multiplicity of two in the square conformation.

(b) Derive an expression for the ratio $N_\square/N_\bigcirc$ if the energies and volumes of each configuration are identical, using the Second Law and your expression for the entropy of the proteins.

(c) Now consider the scenario where the energies the two conformations differ ($\varepsilon_\square \neq \varepsilon_\bigcirc$), but their volumes are identical ($V_\square = V_\bigcirc$). Derive an expression for $N_\square/N_\bigcirc$ as a function of temperature and pressure.

(d) Finally, consider the scenario where $\varepsilon_\square \neq \varepsilon_\bigcirc$, and $V_\square \neq V_\bigcirc$. Derive an expression for $N_\square/N_\bigcirc$.

Consider a protein that has two conformations, square and circular. The square conformation has a multiplicity of two—meaning that there are two distinguishable ways that the protein may assume a square conformation that is identical in every other way.



(a) What is the entropy of a system consisting of N protein molecules, of which $N_\square$ are square and $N_\bigcirc$ are circular? Please only consider the entropy arising from the conformation of the molecules (e.g. neglect entropy of orientation or position). Moreover, you may assume that the protein has no internal degrees of freedom in either conformation, apart from the above-mentioned multiplicity of two in the square conformation.

Solution:
The entropy is most easily found using the formula

$$\begin{aligned}
S &= -Nk_B \sum_i^{\text{single-particle microstates}} P_i \ln P_i && (34)\\
&= -Nk_B \left(P_{\square,1}\ln P_{\square,1} + P_{\square,2}\ln P_{\square,2} + P_\bigcirc \ln P_\bigcirc\right) && (35)\\
&= -Nk_B \left(\frac{P_\square}{2}\ln\frac{P_\square}{2} + \frac{P_\square}{2}\ln\frac{P_\square}{2} + P_\bigcirc \ln P_\bigcirc\right) && (36)\\
&= -Nk_B \left(P_\square \ln\frac{P_\square}{2} + P_\bigcirc \ln P_\bigcirc\right) && (37)\\
&= -Nk_B \left((1-P_\bigcirc)\ln\frac{1-P_\bigcirc}{2} + P_\bigcirc \ln P_\bigcirc\right) && (38)
\end{aligned}$$

where $P_\bigcirc = N_\bigcirc/(N_\square + N_\bigcirc)$.

We could also have obtained the same answer by the $S = k_B \ln W$ definition, using Stirling's approximation and a bit more math.

(b) Derive an expression for the ratio $N_\square/N_\bigcirc$ if the energies and volumes of each configuration are identical, using the Second Law and your expression for the entropy of the proteins.

Solution:

The Second Law states that the entropy of a system plus that of its surroundings cannot decrease. Thus this combined entropy must be at a maximum for a system in equilibrium with its surroundings. We have already established the entropy of the system, so we need only find how the surroundings will change in entropy and then maximize the total. Since there is no energetic or volume difference between the states, the surroundings aren't changed when $N_{\bigcirc}$ changes, so the entropy of the surroundings doesn't change, and our life is easy.

$$\begin{aligned}
\frac{dS_{sys}}{dP_{\bigcirc}} &= 0 && (39)\\
-\frac{1}{Nk_B}\frac{dS_{sys}}{dP_{\bigcirc}} &= 0 && (40)\\
0 &= -\ln\frac{1-P_{\bigcirc}}{2} - 1 + \ln P_{\bigcirc} + 1 && (41)\\
&= \ln P_{\bigcirc} - \ln\frac{1-P_{\bigcirc}}{2} && (42)\\
&= \ln\frac{2P_{\bigcirc}}{1-P_{\bigcirc}} && (43)\\
1 &= \frac{2P_{\bigcirc}}{1-P_{\bigcirc}} && (44)\\
\frac{N_{\square}}{N_{\bigcirc}} &= 2 && (45)
\end{aligned}$$

Of course, it is also correct to get this answer by arguing that each microstate is equally probable in this case. But that wouldn't help as much for answering the following questions.

(c) Now consider the scenario where the energies the two conformations differ ($\varepsilon_{\square} \neq \varepsilon_{\bigcirc}$), but their volumes are identical ($V_{\square} = V_{\bigcirc}$). Derive an expression for $N_{\square}/N_{\bigcirc}$ as a function of temperature and pressure.

Solution:

This problem is much like the last, with the difference that a change in $P_{\bigcirc}$ will now change the surroundings, since there is an energy difference (and energy is conserved, as the First Law requires). The change in entropy of the surroundings is actually quite simple using the First Law:

$$\begin{aligned}
\Delta U_{surr} &= Q + W && (46)\\
&= \int T dS_{surr} - \int p dV_{surr} && (47)\\
&= T\Delta S_{surr} - p\Delta V_{surr} && (48)
\end{aligned}$$

Here I have used the thermodynamic identity, and the fact that the surroundings are at a given pressure and temperature (as stated in the problem), and can be taken to be much larger than the system (so T and p are constant). Since the volume doesn't change, the work term vanishes (but will be relevent in the next section), and we have an expression:

$$\Delta S_{surr} = \frac{\Delta U_{surr}}{T} \tag{49}$$

$$\frac{dS_{surr}}{dP_{\bigcirc}} = \frac{1}{T}\frac{dU_{surr}}{dP_{\bigcirc}} \tag{50}$$

$$= -\frac{1}{T}\frac{dU_{sys}}{dP_{\bigcirc}} \tag{51}$$

$$= -\frac{N}{T}\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) \tag{52}$$

Now we will invoke the Second Law as we did before, and maximize the total entropy

$$0 = \frac{dS_{sys}}{dP_{\bigcirc}} + \frac{dS_{surr}}{dP_{\bigcirc}} \tag{53}$$

$$= -Nk_B \ln\frac{2P_{\bigcirc}}{1-P_{\bigcirc}} - \frac{N}{T}\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) \tag{54}$$

$$\ln\frac{2N_{\bigcirc}}{N_{\Box}} = \beta\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) \tag{55}$$

$$\frac{N_{\Box}}{N_{\bigcirc}} = 2e^{\beta\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right)} \tag{56}$$

Clearly this is just the familiar Boltzmann factor, along with the extra degeneracy term.

(d) Finally, consider the scenario where $\varepsilon_{\Box} \neq \varepsilon_{\bigcirc}$, and $V_{\Box} \neq V_{\bigcirc}$. Derive an expression for $N_{\Box}/N_{\bigcirc}$.

Solution:

This last solution is essentially like the previous one, except that we don't drop the $-p\Delta V$ term that showed up in Equation 48.

$$0 = \frac{dS_{sys}}{dP_{\bigcirc}} + \frac{dS_{surr}}{dP_{\bigcirc}} \tag{57}$$

$$= -Nk_B \ln\frac{2P_{\bigcirc}}{1-P_{\bigcirc}} - \frac{N}{T}\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) + \frac{Np}{T}\left(V_{\bigcirc} - V_{\Box}\right) \tag{58}$$

$$\ln\frac{2N_{\bigcirc}}{N_{\Box}} = \beta\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) - p\beta\left(V_{\bigcirc} - V_{\Box}\right) \tag{59}$$

$$\frac{N_{\Box}}{N_{\bigcirc}} = 2e^{\beta\left(\left(\varepsilon_{\bigcirc} - \varepsilon_{\Box}\right) - p\left(V_{\bigcirc} - V_{\Box}\right)\right)} \tag{60}$$

$$\tag{61}$$

As you can see we now have the enthalpy sitting in where the energy normally lives in the Boltzmann factor. We could alternatively convert the multiplicity factor of 2 into an entropy difference (which would move up into the exponential), which would put the Gibbs free energy into the Boltzmann factor. Yay.

In the Bohr model of the hydrogen atom ground state, the electron moves in a circular orbit of radius $a_0 = 0.53\times10^{-10}$ m around the proton. The proton is assumed to be rigidly fixed in space. Since the electron is accelerating, a classical analysis suggests that it will continuously radiate energy, and therefore the radius of the orbit should shrink with time.

> **Question:** Assuming that the electron is a point charge and always in a nearly circular orbit, and that the rate of radiation of energy is sufficiently well approximated by classical, nonrelativistic electrodynamics, how long is the fall time of the electron, i.e., the time for the electron to spiral into the origin?

- When answering the above question, you are encouraged to make appropriate estimates and approximations. It is acceptable to have a final answer that is off by a factor 10.
- Dimensional analysis can give you an equation for energy radiated by an orbiting charge. Such an equation would be correct to within a multiplicative constant. Note that the energy radiated by a charge moving in a circular orbit depends on

ω, the angular frequency of the orbit,

p, the effective dipole moment of the orbit

c, the speed of light

ε_0, the vacuum dielectric constant

①

Solution to Problem 7

Soln in SI units.

Start by calculating energy radiated for a given dipole moment and ω.

Looking for power P in terms of $p_0, \omega, \epsilon_0, c$

$$[P] = \frac{E}{T}$$

dipole moment

Note that $\left[\dfrac{p_0^2}{4\pi\epsilon_0}\right] = E\,L^3$ ——— ①

(compare to $\dfrac{e^2}{4\pi\epsilon_0 r}$ which has dimension = E)

Now combine c & ω to get $\dfrac{1}{TL^3}$

The L^3 must come from c

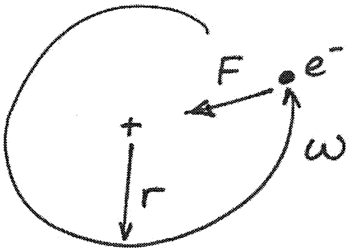
$$\Rightarrow \left[\frac{\omega^4}{c^3}\right] = \frac{1}{TL^3} \qquad \text{——— ②}$$

Combining ① & ②

$$\boxed{P \sim \frac{p_0^2}{4\pi\epsilon_0}\,\frac{\omega^4}{c^3}} \qquad \text{——— ③}$$

②

Now find P for a particular orbit



Central force $\vec{F} = \dfrac{-e^2}{4\pi\epsilon_0 r^2}\,\hat{r}$

For a "stable" orbit

$$F = ma$$

$$\frac{e^2}{4\pi\epsilon_0 r^2} = m r \omega^2$$

$$\omega = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r^3}}$$

Substitute into Eq ③ with $p_0 \approx er$

$$P \sim \frac{e^2 r^2}{4\pi\epsilon_0 c^3}\left(\frac{e^2}{4\pi\epsilon_0 m r^3}\right)^2$$

$$P \sim \left(\frac{e^2}{4\pi\epsilon_0}\right)^3 \frac{1}{r^4}\,\frac{1}{c^3 m^2}$$

③

Total energy of the electron is

$$U = K.E. + P.E. \qquad \text{where} \quad P.E. = \frac{-e^2}{4\pi\epsilon_0 r}$$

From Viral Thm we know $K.E. = -\frac{1}{2} P.E.$

$$\Rightarrow \quad U = \frac{-e^2}{8\pi\epsilon_0 r}$$

We also know that $\frac{dU}{dt} = -P$

Now our goal is to find $\frac{dr}{dt}$

Note that $\frac{dU}{dt} = \frac{dU}{dr}\frac{dr}{dt}$

$$\Rightarrow \quad \frac{dr}{dt} = -P\left(\frac{dU}{dr}\right)^{-1}$$

$$= -P\left(\frac{e^2}{8\pi\epsilon_0 r^2}\right)^{-1}$$

$$\frac{dr}{dt} \overset{*}{\sim} \left(\frac{e^2}{4\pi\epsilon_0}\right)^3 \frac{-1}{r^4 c^3 m^2} \left(\frac{e^2}{4\pi\epsilon_0}\right)^{-1} \left(\frac{1}{2r^2}\right)^{-1}$$

$$\sim \; -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2}{r^2 c^3 m^2}$$

$$\sim \; -\left(\frac{e^2}{4\pi\epsilon_0 r}\right)^2 \frac{2c}{m^2 c^4}$$

Note that dimensions are correct. Good check.

Now we know that $\frac{dr}{dt} = \frac{\text{const}}{r^2}$

$$\Rightarrow \int r^2 dr = \int \text{const}\, dt$$

$$\left[\frac{r^3}{3} \right]_{a_0}^{0} = \text{const}\ \tau$$

where τ is the time it takes for the e^- to reach the origin.

$$\frac{a_0^3}{3} = \text{const}\ \tau$$

$$\tau = \frac{a_0^3}{3\ \text{const}}$$

$$\sim \frac{a_0^3}{3} \left(\frac{4\pi\epsilon_0}{e^2} \right)^2 \frac{c^3 m^2}{2}$$

$$\tau \sim \frac{(0.5\times10^{-10})^3}{3} \left(\frac{12\times10^{-11}}{(1.6\times10^{-19})^2} \right)^2 \frac{(3\times10^{8})^3 (10^{-30})^2}{2}$$

$$\sim \frac{0.125\times10^{-30}}{\cancel{3}} \left(\frac{10^{-10}}{\cancel{3}\times10^{-38}} \right)^2 \frac{\cancel{3}^{3}\times10^{24}\ 10^{-60}}{2}$$

$$\sim \frac{0.125\times10^{-66}}{2} (10^{28})^2 \sim 0.06\times10^{-66}\times10^{56}$$

$$= 6\times10^{-12}\ \text{s}$$

A *skyhook* is an orbiting 'elevator' of length L extending radially from a fixed location just above the ground at the equator of the Earth. It is not attached to the earth (of radius R), so the only force is gravitational. Thus the angular velocity of all locations on the elevator is the same as that of the earth.

a) Idealize the orbiting elevator as a (thin) rod with uniform mass density ρ and cross sectional area A_e. Calculate the required length of the elevator relative to the radius of the earth, i.e. L/R, and give a quantitative estimate.

b) Estimate the maximum stress (force per unit area) induced within the rod. Could it be made from ordinary materials, which fail at a stress of $\sim 10^8$ N m^{-2} ?

Explain any approximations made. All answers need only be accurate to one significant figure (less than 10% error).

Possibly relevant constants and parameters for the earth:

$$G \approx 6.7 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

For the earth:

$$R \approx 6.4 \times 10^6\ \text{m} \qquad M \approx 6.0 \times 10^{24}\ \text{kg} \qquad g \approx 9.8\ \text{ms}^{-2}$$

Use appropriate common sense estimates for any other numbers you might possibly need.

a) sky hook (elevator) : linear density ϱ

gravitational force:

$$F_g = \int_{x=R}^{R+L} \frac{GM\varrho}{x^2}\, dx = GM\varrho\left[-\frac{1}{x}\right]_R^{R+L} = \frac{GM\varrho L}{R(R+L)}$$

centre of mass acceleration:

$$a = \frac{1}{\varrho L}\int_R^{R+L} \varrho x \omega^2\, dx = \frac{\omega^2}{2L}\left[(R+L)^2 - R^2\right] = \left(R+\frac{L}{2}\right)\omega^2$$

in equilibrium:

$$F_g = \varrho L a$$

$$\Leftrightarrow \quad \frac{GM}{R^2\left(1+\frac{L}{R}\right)} = \omega^2 R\left(1+\frac{L}{2R}\right)$$

$$\Leftrightarrow \quad \frac{GM}{\omega^2 R^3} = 1 + \frac{3}{2}\frac{L}{R} + \frac{1}{2}\left(\frac{L}{R}\right)^2$$

$$\Rightarrow \left(\frac{L}{R}\right)^2 + 3\left(\frac{L}{R}\right) + 2\left(1 - \frac{GM}{\omega^2 R^3}\right) = 0$$

if $L \gg R$ (*)

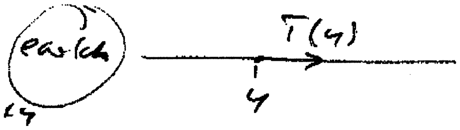
$$\frac{L}{R} \approx \sqrt{\frac{2GM}{\omega^2 R^3}} = \left(\frac{2\cdot 6.7\, 10^{-11}\, \cancel{m^3}\, 6\cdot 10^{24}\, \cancel{kg}}{(2\pi)^2 (24\cdot 3600)^{-2}\, \cancel{kg\, s^2}\, s^{-2}\, (6.4)^3\, 10^{18}\, \cancel{m^3}}\right)^{1/2}$$

$$= \left(\frac{2\times 6.7\times 6\times (2.4)^2\times (3.6)^2 \times 10^{21}}{(2\pi)^2 (6.4)^3 \times 10^{18}}\right)^{1/2}$$

$$\approx \sqrt{\frac{1}{2}10^3} = \sqrt{5}\cdot 10 \approx 20$$

justifies (*)

b) tension in the rod as function of position $T(y)$:



$$T(y) = \int_R^{R+y} \frac{GM\rho}{x^2}\,dx - \int_R^{R+y} \rho\omega^2 x\,dx$$

$$= \frac{GM\rho y}{R(R+y)} - \rho y\left(R+\frac{y}{2}\right)\omega^2$$

$$= \rho\omega^2R^2\underbrace{\left(\frac{y}{R}\left[\frac{GM}{\omega^2R^3}\left(\frac{1}{1+\frac{y}{R}}\right)-\left(1+\frac{y}{2R}\right)\right]\right)}_{\sim 250 \text{ independent of } y \ (y\gg R)}$$

estimate:

$$= \frac{\rho\cdot(2\pi)^2(6.4)^2\cdot 10^{12}\cdot 2.5\cdot 10^2}{(2.4\cdot 3.6)^2\, 10^8}\,\frac{m^2}{s^2}$$

$$= \rho \times 10^8\,\frac{Nm}{kg}$$

a typical material has volume density

$\rho_V \sim 10^3\,kg/m^3$, with $\rho = \rho_V\cdot A$ (cross sectional area A)

$$\rightarrow \frac{T}{A} \approx 10^{11}\,\frac{N}{m^2} \gg 10^8\,\frac{N}{m^2}$$

$\Rightarrow$ ordinary materials can't support the sky hook.