OSU Physics Department
Comprehensive Examination #113

Monday, April 2 and Tuesday, April 3, 2012

Spring 2012 Comprehensive Examination

PART 1, Monday, April 2, 9:00am

General Instructions

This Spring 2012 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, April 2, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, April 3, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Consider an "gravitational hydrogen atom" with inverse-square interaction

$$V = -\frac{\alpha\hbar^2}{2m_e}\frac{1}{r^2} \tag{1}$$

In this problem, we will examine how this "atom" differs from the real hydrogen atom.

(a) Write down the Hamiltonian, and separate out the rotational kinetic energy, expressing the Hamiltonian in terms of L^2 and r.

(b) Write down the eigenstates and eigenvalues of L^2 and L_z for this system.

(c) Find a limit on the values of angular momentum l that can be bound by this "atom" as a function of the constant α.

(d) Given that an eigenstate has an energy E and angular quantum numbers l and m, solve for its behavior in the limit of large radius.

(e) In the limit of small radius, the eigenstates have a power law dependence on radius:

$$\psi(\mathbf{r}) \propto r^\nu Y_{lm}(\theta, \varphi)$$

Solve for the power ν, as a function of α and the angular momentum quantum numbers l and m.

Consider an "gravitational hydrogen atom" with inverse-square interaction

$$V = -\frac{\alpha\hbar^2}{2m_e}\frac{1}{r^2} \tag{2}$$

In this problem, we will examine how this "atom" differs from the real hydrogen atom.

(a) Write down the Hamiltonian, and separate out the rotational kinetic energy, expressing the Hamiltonian in terms of L^2 and r.

Solution:

$$\begin{aligned} H &= -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{\alpha\hbar^2}{2m_e}\frac{1}{r^2} && (3)\\ &= -\frac{\hbar^2}{2m_e}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{L^2}{2m_e r^2} - \frac{\alpha\hbar^2}{2m_e}\frac{1}{r^2} && (4)\\ &= -\frac{\hbar^2}{2m_e}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \left(\frac{L^2}{2m_e} - \frac{\alpha\hbar^2}{2m_e}\right)\frac{1}{r^2} && (5) \end{aligned}$$

(b) Write down the eigenstates and eigenvalues of L^2 and L_z for this system.

Solution:
The eigenstates of L^2 and L_z are

$$\psi_{nlm} = R(r)Y_{lm}(\theta,\phi) \tag{6}$$

and the eigenvalues are

$$\begin{aligned} L^2\psi_{nlm} &= \hbar^2 l(l+1)\psi_{nlm} && (7)\\ L_z\psi_{nlm} &= \hbar m\psi_{nlm} && (8)\\ & && (9) \end{aligned}$$

(c) Find a limit on the values of angular momentum l that can be bound by this "atom" as a function of the constant α.

Solution:

$$\begin{aligned} H\psi_{lm} &= E\psi_{lm} && (10)\\ -\frac{\hbar^2}{2m_e}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi_{lm}) + \left(\frac{L^2}{2m_e} - \frac{\alpha\hbar^2}{2m_e}\right)\frac{\psi_{lm}}{r^2} &= E\psi_{lm} && (11)\\ -\frac{\hbar^2}{2m_e}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi_{lm}) + \left(\frac{\hbar^2 l(l+1)}{2m_e} - \frac{\alpha\hbar^2}{2m_e}\right)\frac{\psi_{lm}}{r^2} &= E\psi_{lm} && (12)\\ -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial r^2}(r\psi_{lm}) + \frac{\hbar^2}{2m_e}\left(l(l+1) - \alpha\right)\frac{r\psi_{lm}}{r^2} &= Er\psi_{lm} && (13) \end{aligned}$$

Clearly if $\alpha <= l(l+1)$ there can be no bound state, since the effective potential is never negative.

(d) Given that an eigenstate has an energy E and angular quantum numbers l and m, solve for its behavior in the limit of large radius.

Solution:

$$H\psi_{lm} = E\psi_{lm} \tag{14}$$

$$-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial r^2}(r\psi_{lm}) + \frac{\hbar^2}{2m_e}\left(l(l+1)-\alpha\right)\frac{r\psi_{lm}}{r^2} = Er\psi_{lm} \tag{15}$$

We can define $u \equiv r\psi_{lm}/Y_{lm}$.

$$-\frac{\hbar^2}{2m_e}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m_e}\left(l(l+1)-\alpha\right)\frac{u}{r^2} = Eu \tag{16}$$

At large radius, the $1/r^2$ term is negligible, and we can see that

$$-\frac{\hbar^2}{2m_e}\frac{d^2u}{dr^2} \approx Eu \tag{17}$$

$$\frac{d^2u}{dr^2} \approx -\frac{2m_eE}{\hbar^2}u \tag{18}$$

$$\frac{d^2u}{dr^2} \approx -\frac{2m_eE}{\hbar^2}u \tag{19}$$

$$u \approx e^{-\sqrt{-\frac{2m_eE}{\hbar^2}}r} \tag{20}$$

This is sort of boring, since it's the same as the hydrogen atom.

FIXME: Need to give ψ and include angular bits.

(e) In the limit of small radius, the eigenstates have a power law dependence on radius:

$$\psi(\mathbf{r}) \propto r^\nu Y_{lm}(\theta,\varphi)$$

Solve for the power ν, as a function of α and the angular momentum quantum numbers l and m.

Solution:
Once again, we can use the radial equation:

$$-\frac{\hbar^2}{2m_e}\frac{d^2u}{dr^2} + \frac{\hbar^2}{2m_e}\left(l(l+1)-\alpha\right)\frac{u}{r^2} = Eu \tag{21}$$

If u is a power law r^n at small radius, this reduces to

$$-\frac{\hbar^2}{2m_e} n(n-1) r^{n-2} + \frac{\hbar^2}{2m_e} \left(l(l+1) - \alpha\right) r^{n-2} \approx E r^n \tag{22}$$

$$-\frac{\hbar^2}{2m_e} n(n-1) r^{n-2} + \frac{\hbar^2}{2m_e} \left(l(l+1) - \alpha\right) r^{n-2} \approx 0 \tag{23}$$

where in the second step we used the fact that r is small to eliminate higher powers. We thus find that

$$n(n-1) = l(l+1) - \alpha \tag{24}$$

$$n = \frac{1 \pm \sqrt{1 - 4\left(\alpha - l(l+1)\right)}}{2} \tag{25}$$

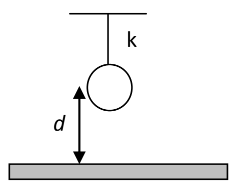
$$\psi(\mathbf{r}) \approx r^{n-1} Y_{lm}(\theta, \varphi) \tag{26}$$

Interestingly, this means that the power-law dependence at small radius will involve a non-integer power in the general case. Weird!

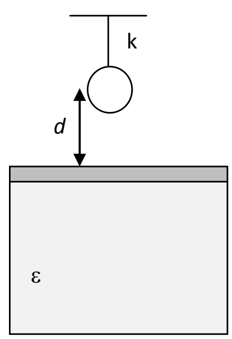
Problem #2 E&M

A small sphere is suspended in air above a horizontal conducting surface with a large area by an elastic insulating thread with an elastic constant *k*. Initially, there is no charge on the sphere, and it is in the equilibrium at a distance *d* from the surface.

(a) Determine the charge *q* which must be given to the sphere to reduce the distance between the sphere and the conducting plane by x.



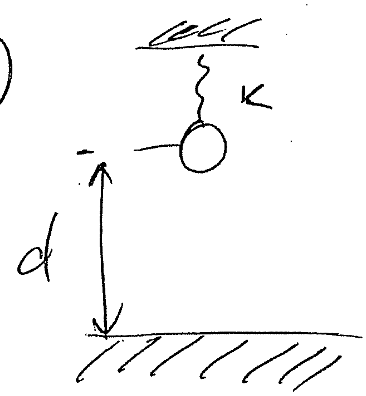
(b) Now let's fill the space below the conducting plane with a dielectric characterized by a relative dielectric constant $\varepsilon > 1$. How would this change the charge *q* of part (a)?



(1)

Problem # 2

(a)

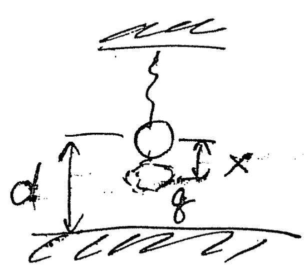


"small" sphere $\Rightarrow$ ignore mass; treat as a point charge

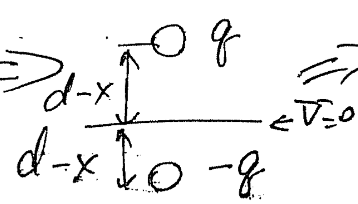
"large" area - conducting surface $\Rightarrow$ assume infinite $\Rightarrow$ provides an infinite supply of charge $\Rightarrow$ grounded $\Rightarrow$ $V = 0$

$\uparrow$ potential

Uniqueness theorem allows one to model electrostatic interaction of a point charge with a conducting plane as that of a point charge and its image (method of images):



$\Rightarrow$



$\Rightarrow$ $F = -\dfrac{q^2}{4\pi\varepsilon_0 \cdot 4(d-x)^2}$

$\uparrow$ electrostatic force

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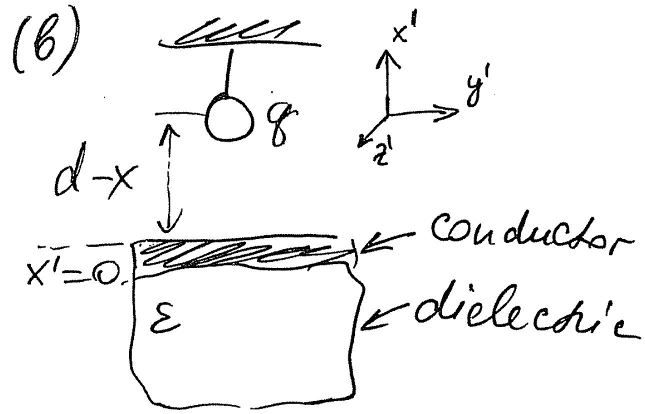
In equilibrium, $-\dfrac{q^2}{4\pi\varepsilon_0\, 4(d-x)^2} = -Kx$

$\uparrow$ electrostatic force $\qquad$ $\uparrow$ restoring force

$$q = \pm 4(d-x)\sqrt{\pi\varepsilon_0 K x}$$

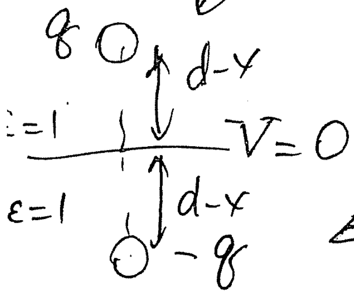
(b)



Since we are still having an infinite conducting plane $\Rightarrow V=0$ and all induced charge is on the surface

The situation can be modelled again using the method of images, to satisfy the boundary condition $\Rightarrow \dfrac{q}{\sqrt{(d-x)^2+y'^2+z'^2}} + \dfrac{q'}{\sqrt{(d-x)^2+y'^2+z'^2}} = 0$ at $x'=0$

should be valid for any $y', z' \Rightarrow$ $q = -q'$ $\Rightarrow$ yielding the same result as in part (a)



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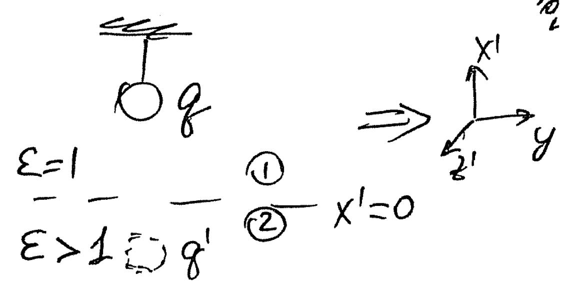
Because of the same boundary condition related to the presence of grounded conductor, it doesn't matter how thick the conducting plane (slab) is (all charge is on the surface!) and what's underneath. The situation changes if the thickness $\to 0$, so we lose the $V=0$ condition and have two dielectrics :



then the method of images can still be used, with the boundary conditions $\left.\frac{\partial V_{①}}{\partial y'}\right|_{x'=0} = \left.\frac{\partial V_{②}}{\partial y'}\right|_{x'=0}$ and $\left.\frac{\partial V_{①}}{\partial x'}\right|_{x'=0} = \varepsilon \left.\frac{\partial V_{②}}{\partial x'}\right|_{x'=0}$

conservation of E_τ

conservation of D_n

which yields (see Jackson) $q' = -q\,\frac{\varepsilon-1}{\varepsilon+1}$ and

image charge

from $\frac{qq'}{4\pi\varepsilon_0 \cdot 4(d-x)^2} = -Kx \Rightarrow q = \underbrace{4(d-x)\sqrt{\pi\varepsilon_0 K x}}_{\text{as in (a)}}\ \underbrace{\sqrt{\frac{\varepsilon+1}{\varepsilon-1}}}_{>1}$

Note: if $\varepsilon \to \infty$ (metal) $\Rightarrow$ the result is the same as in (a)

Assume that the gas of conducting electrona in a metal is fully isotropic, and is described by the "Fermion-in-a-$L \times L \times L$-box" model. Let the internal potential of free electrons in a metal be $-w$, and the Fermi potential μ_0 at zero T be $-\Phi$ when measured from the free vacuum, as shown in the figure below. The Fermi energy in typical metals is of the order of a few eV. At finite temperatures electrons having higher energies at the upper "tail" of the Fermi distribution can escape to the exterior. The effect becomes significant at temperatures above 1000 K, and it plays an important role in vacuum electronics.

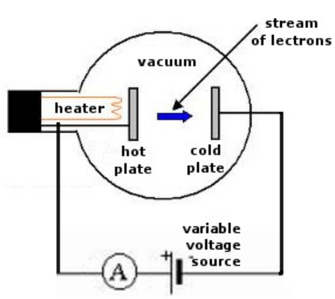
If two parallel plates are made of the same metal, kept in vacuum, and one of them is heated to a high temperature, the escaping electrons may reach the other plate, producing a so-called "thermionic current" which can be measured by an ammeter connected between the two plates. In order to suppress the thermionic current, a small negative voltage $-\Delta V$ (often called a "reverse bias voltage") has to be applied to the "cold plate".

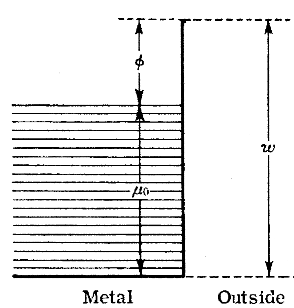
(a) If z is the axis perpendicular to the plate's surface, what is the minimum value of the wavevector component $(k_z)_{\min}$ the electron needs to have to pass over the energy barrier w, and escape to the vacuum?

(b) By integrating in k-space over the states of electrons capable of escaping, find number of electrons exiting the the metal per unit time (up to a multiplicative factor).

> *Hints:* note that the probability that an electron with a wavevector $\vec{k}$ emerges from the metal is proportional to the Fermi-Dirac *occupancy factor* of the state, **and** to the frequency with which such electron strikes the surface from inside—given by the z component of its velocity. Also, note that at moderate temperatures of 1000-2000 K, the work function $\Phi \gg k_B T$.

(c) Assume that if no "bias voltage" is applied ($\Delta V = 0$), all escaping electrons are collected by the "cold" plate. So, the current $I(\Delta V = 0)$ measured by the ammeter is proportional to the number of escaping electrons you have determined in (b). However, if a non-zero voltage ΔV is applied, the height of the effective energy barrier increases to $w + q\Delta V$ (where q is the electron's charge). How does the current depend on the "bias voltage" ΔV, relative to its value at $\Delta V = 0$?





"GRADUATE THERMAL" SOLUTION: (1)

(a) Electron's kinetic energy before exiting: $E_B = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$

" " " after exiting: $E_A = \hbar^2/_{2m}(k_x^2 + k_y^2 + k_z'^2)$

The x and y momentum components do not change! From energy conservation, it follows that $E_A = E_B - W \Rightarrow \frac{\hbar^2}{2m}k_z'^2 = \frac{\hbar^2}{2m}k_z^2 - W$

And the momentum component k_z' after exiting must be >0, so that we get: $k_z > \sqrt{\frac{2mW}{\hbar^2}}$

(b) Electrons with $k_z > \sqrt{2mW}/\hbar$ exit the "box", and their number per unit time is equal to the # of electrons "striking" the box face "from inside":

$$N = 4\left(\frac{L}{\pi}\right)^3 \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{\sqrt{2mW}/\hbar}^{\infty} \frac{v_z}{2L} \frac{1}{\exp\left(\frac{E-\mu}{k_BT}\right)+1} dk_z$$

(only the octants with $k_z > 0$ count)

v_z, the velocity component in the z direction is $v_z = \frac{\hbar k_z}{m}$

For temperatures ~1000-2000K the value of μ can be taken with a very good approximation as $\mu \approx \mu_0$. Since $E > W$ for the exiting electrons, $E - \mu > \phi$

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Φ is normally 2 – 5 eV, k_BT for 1000–2000 K is 0.12 – 0.23 eV, so that $(E-\mu)/k_BT$ is $\gtrsim 10$, e^{10} is 22,000, and the "1" in the denominator of the Fermi-Dirac function can be dropped.

Hence, the integrand in the integration over k_z can be taken as:

$$\frac{N_z}{2L} \cdot \frac{1}{\exp\left(\frac{E-\mu}{k_BT}\right)+1} \cong \frac{\hbar k_z}{2mL}\, e^{\frac{\mu_0}{kT}} \cdot e^{-\frac{\hbar^2}{2mk_BT}\left(k_x^2+k_y^2+k_z^2\right)}$$

If we factor out all constant coefficients, we get:

$$N \propto \exp\left(\frac{\mu_0}{kT}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{\hbar^2 k_x^2}{2mk_BT}\right) dk_x \int_{-\infty}^{+\infty} \exp\left(-\frac{\hbar^2 k_y^2}{2mk_BT}\right) dk_y \int_{\sqrt{2mW}/\hbar}^{\infty} k_z \exp\left(-\frac{\hbar^2 k_z^2}{2mk_BT}\right) dk_z$$

The first two integrals are of the

$$\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi} \quad \text{type:}$$

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{\hbar^2 k_x^2}{2mk_BT}\right) dk_x = \int_{-\infty}^{+\infty} \exp\left[-\left(\frac{\hbar k_x}{\sqrt{2mk_BT}}\right)^2\right] \left(\frac{\sqrt{2mk_BT}}{\hbar}\right) d\left(\frac{\hbar k_x}{\sqrt{2mk_BT}}\right)$$

$$= \frac{\sqrt{2\pi m k_BT}}{\hbar}\,;$$ the same for the integral over k_y.

3

The integral over k_z is:

$$\int_{\sqrt{2mW}/\hbar}^{\infty} k_z\, e^{-\frac{\hbar^2 k_z^2}{2mk_BT}} = \left[-\frac{mk_BT}{\hbar^2}\, e^{-\frac{\hbar^2 k_z^2}{2mk_BT}}\right]_{\sqrt{2mW}/\hbar}^{\infty}$$

$$= \frac{mk_BT}{\hbar^2}\, e^{-W/k_BT}$$

Therefore: $N \propto \left(\frac{\sqrt{2\pi m k_BT}}{\hbar}\right)^2 \frac{mk_BT}{\hbar^2}\, e^{\frac{\mu_0 - W}{k_BT}}$

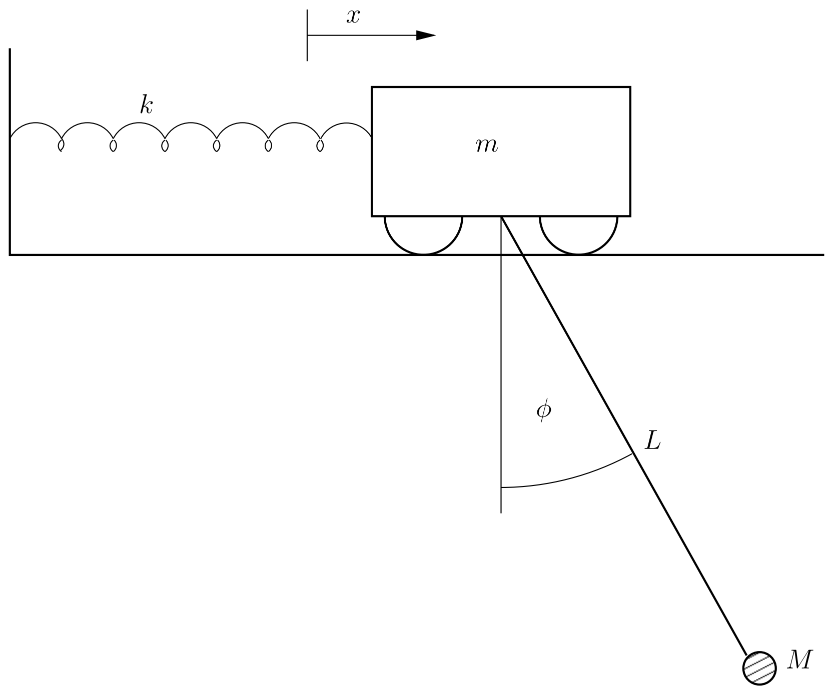
Or: $N = \text{const}\cdot T^2 e^{-\Phi/k_BT}$ (known as the "Richardson equation")

(c) If negative "bias voltage" is applied, then the barrier height increases from $w \rightarrow w + q\cdot\Delta V$, and for reaching the other plate, the electron's k_z component has to be $k_z > \sqrt{2m(w+q\Delta V)}/\hbar$.

The number N' of electrons reaching the "cold" plate then becomes $N' = \text{const}\, T^2 e^{-(\Phi+q\Delta V)/k_BT}$

If the <u>current</u> for $\Delta V < 0$ is I_0, then the current for $\Delta V > 0$ is: $I(\Delta V) = I_0 e^{-q\Delta V/kT}$, meaning that for, i.e., $T = 1000K$, it will fall e-times with ΔV increasing by $\Delta V = 0.116$ V.

A simple pendulum (mass M, mass-less string of length L) is suspended from a cart (mass m) that can move horizontally and is attached to the end of a harmonic spring (force constant k) (see Figure).



(a) Write the Lagrangian of the system in terms of two generalized coordinates x and ϕ (see Figure), where x is the extension of the spring from its equilibrium length.

(b) Find the Lagrange equations of motion.

(c) Simplify the equations of motions in the case that both x and ϕ are small.

(d) Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units) find the normal frequencies, and for each normal frequency find and describe the motion of the corresponding normal mode.

a) $U = \frac{1}{2}kx^2 - MgL\cos\phi$

$$\vec{r}_M = \begin{pmatrix} x + L\sin\phi \\ L\cos\phi \end{pmatrix} \qquad \vec{v}_M = \begin{pmatrix} \dot{x} + \dot{\phi}L\cos\phi \\ -\dot{\phi}L\sin\phi \end{pmatrix}$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + 2\dot{x}\dot{\phi}L\cos\phi + L^2\dot{\phi}^2)$$

$$\boxed{\begin{aligned} L = T - U = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}M(L^2\dot{\phi}^2 + 2\dot{x}\dot{\phi}L\cos\phi) \\ - \frac{1}{2}kx^2 + MgL\cos\phi \end{aligned}}$$

b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

$$\frac{\partial L}{\partial \dot{x}} = (m+M)\dot{x} + \dot{\phi}ML\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m+M)\ddot{x} + \ddot{\phi}ML\cos\phi - \dot{\phi}^2 ML\sin\phi$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{\phi}} = ML^2\dot{\phi} + \dot{x}ML\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = ML^2\ddot{\phi} + \ddot{x}ML\cos\phi - \dot{x}\dot{\phi}ML\sin\phi$$

$$\frac{\partial L}{\partial \phi} = -ML\dot{x}\dot{\phi}\sin\phi - MgL\sin\phi$$

$$\Rightarrow \boxed{\begin{aligned} (m+M)\ddot{x} + \ddot{\phi}ML\cos\phi - \dot{\phi}^2 ML\sin\phi + kx &= 0 \\ ML^2\ddot{\phi} + \ddot{x}ML\cos\phi + MLg\sin\phi &= 0 \end{aligned}}$$

c) x, ϕ small $\rightarrow$ linearize

$\cos\phi \approx 1 + O(\phi^2) \quad \sin\phi \approx \phi + O(\phi^3)$

ϕ small $\rightarrow$ $\dot{\phi}$ small $\rightarrow$ $\dot{\phi}^2 \rightarrow 0$

$$\Rightarrow \boxed{\begin{aligned} (m+M)\ddot{x} + ML\ddot{\phi} &= -kx \\ \ddot{x} + L\ddot{\phi} &= -g\phi \end{aligned}}$$

d) with $m = M = g = l = 1$; $k = 2$:

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{=\underline{\underline{M}}} \begin{pmatrix} \ddot{x} \\ \ddot{\phi} \end{pmatrix} = - \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}_{=\underline{\underline{K}}} \begin{pmatrix} x \\ \phi \end{pmatrix}$$

ansatz: $\vec{y}(t) = \begin{pmatrix} x(t) \\ \phi(t) \end{pmatrix} = \mathrm{Re}\left\{ \vec{y} e^{i\omega t} \right\}$

$\Rightarrow \left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}} \right) \vec{y} = 0$

solutions for $\left| \underline{\underline{K}} - \omega^2 \underline{\underline{M}} \right| = 0$

$$\begin{vmatrix} 2\omega^2 - 2 & \omega^2 \\ \omega^2 & \omega^2 - 1 \end{vmatrix} = 0$$

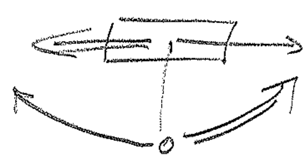
$\Leftrightarrow (2\omega^2 - 2)(\omega^2 - 1) - \omega^4 = 0$

$\omega^4 - 4\omega^2 + 2 = 0$

$\Rightarrow \omega_{1/2}^2 = 2 \pm \sqrt{2} \Rightarrow \boxed{\omega_{1/2} = \sqrt{2 \pm \sqrt{2}}}$

$\omega_1 = \sqrt{2+\sqrt{2}}$: $\underbrace{2(1+\sqrt{2})}_{>0} y_1 + \underbrace{2+\sqrt{2}}_{>0} y_2 = 0$

→ y_1, y_2 have opposite sign, out of phase :

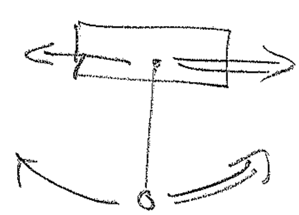


$$\frac{y_1}{y_2} = - \frac{2+\sqrt{2}}{2(1+\sqrt{2})} \cdot \frac{(1-\sqrt{2})}{(1-\sqrt{2})} = -\frac{1}{\sqrt{2}}$$

$\omega_2 = \sqrt{2-\sqrt{2}}$: $2\underbrace{(1-\sqrt{2})}_{<0} y_1 + \underbrace{(2-\sqrt{2})}_{>0} y_2 = 0$

→ wagon, pendulum move in phase :

$$\frac{y_1}{y_2} = - \frac{2-\sqrt{2}}{2(1-\sqrt{2})} \cdot \frac{(1+\sqrt{2})}{(1+\sqrt{2})} = \frac{1}{\sqrt{2}}$$



Consider a charged simple harmonic oscillator with charge q, frequency ω_0, and mass m interacting with a pulse of light. The electric field of the pulse of light is given by

$$\mathbf{E} = E_{\text{max}} \sin(\omega t)\, e^{-\frac{\gamma^2 t^2}{2}} \hat{\mathbf{x}} \tag{27}$$

The oscillator is prepared (at $\gamma t \ll 0$) in the state $|2\rangle$, which is its second excited state.

(a) Solve for the probability that the oscillator will be in each state $|n\rangle$ at $\gamma t \gg 0$. You may assume that the amplitude of the electric field E_{max} is small.

(b) Between which states are transitions possible? How would your answer change if you included higher-order effects?

(c) Sketch the probability of ending up in the lowest five states as a function of the frequency ω of the light pulse. Discuss how this relates to the expected behavior in the limit of a classical harmonic oscillator interacting with light—this problem nicely demonstrates Bohr's correspondence principle.

The following definite integral may be useful:

$$\int_{-\infty}^{\infty} e^{-p^2 x^2 \pm q x} dx = \frac{\sqrt{\pi}}{|p|} e^{\frac{q^2}{4p^2}} \tag{28}$$

Consider a charged simple harmonic oscillator with charge q, frequency ω_0, and mass m interacting with a pulse of light. The electric field of the pulse of light is given by

$$\mathbf{E} = E_{\max} \sin(\omega t)\, e^{-\frac{\gamma^2 t^2}{2}} \hat{\mathbf{x}} \tag{29}$$

The oscillator is prepared (at $\gamma t \ll 0$) in the state $|2\rangle$, which is its second excited state.

(a) Solve for the probability that the oscillator will be in each state $|n\rangle$ at $\gamma t \gg 0$. You may assume that the amplitude of the electric field $E_{\max}$ is small.

Solution:
The exact solution is determined by applying Schrödinger's equation:

$$H|\psi\rangle = -i\hbar \frac{\partial}{\partial t}|\psi\rangle \tag{30}$$

$$\frac{\partial}{\partial t}|\psi\rangle = i\frac{H}{\hbar}|\psi\rangle \tag{31}$$

where

$$H = H_0 + qEx \tag{32}$$

$$= \hbar\omega_0 \left(a^\dagger a + \frac{1}{2}\right) + qE(t)\sqrt{\frac{2\hbar}{m\omega_0}}\left(a^\dagger + a\right) \tag{33}$$

We will use first-order time-dependent perturbation theory, which is appropriate because we are assuming that $E_{\max}$ is small. We can begin with a summary of the first-order perturbation theory result for the amplitude of the n^{th} eigenstate:

$$c_n(t) = \frac{-i}{\hbar}\int_0^t \langle n|V(t')|2\rangle e^{-i\omega_0(2-n)t'} \tag{34}$$

where I assumed that we're looking at every state *but* the first excited state, which is populated at $t = 0$. For our light pulse, this comes out to:

$$c_n(t) = \frac{-i}{\hbar}\int_0^t \langle n|qE_{\max}\sin(\omega t')e^{-\frac{\gamma^2 t'^2}{2}} x|2\rangle e^{-i\omega_0(2-n)t'}dt' \tag{35}$$

$$= \frac{-iqE_{\max}}{\hbar}\langle n|x|2\rangle \int_0^t \sin(\omega t')e^{-\frac{\gamma^2 t'^2}{2}} e^{-i\omega_0(2-n)t'}dt' \tag{36}$$

At this point it is clear that we have two (mathematical) tasks: we just have to compute the matrix element $\langle n|x|2\rangle$ and the time integral in question.

Finding the matrix element

This is pretty easy if we do it using raising and lowering operators.

$$\begin{aligned} \langle n|x|2\rangle &= \sqrt{\frac{2\hbar}{m\omega}}\,\langle n|a + a^\dagger|2\rangle && (37)\\ &= \sqrt{\frac{2\hbar}{m\omega}}\left(\sqrt{3}\delta_{n3} + \delta_{n1}\right) && (38)\end{aligned}$$

Finding the time integral

The time integral is fairly straightforward, and there are a couple of ways we can perform it. An obvious option is to write the sin as a sum of exponentials, so the integral we need to find the final probabilities is

$$\begin{aligned} I(t) &\equiv \int_0^t \sin(\omega t')e^{-\frac{\gamma^2 t'^2}{2}} e^{\mp i\omega_0 t'}\,dt' && (39)\\ &= -i\int_0^t e^{-\frac{\gamma^2 t'^2}{2}} e^{\mp i\omega_0 t'}\left(e^{i\omega t'} - e^{-i\omega t'}\right)dt' && (40)\\ &= -i\int_0^t e^{-\frac{\gamma^2 t'^2}{2}}\left(e^{i(\omega\mp\omega_0)t'} - e^{-i(\omega\pm\omega_0)t'}\right)dt' && (41)\\ &= -i\sqrt{2\pi}\gamma\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right) && (42)\end{aligned}$$

In the last step, we used the definite integral provided with the problem. We could have done this by completing the square and converting it into a standard Gaussian integral if this hadn't been provided.

In any case, we can see that the time integral gives two peaks in ω, one centered at ω_0, and the other at $-\omega_0$. The width of the gaussians are both γ. This is the uncertainty principle in action: the uncertainty in frequency is determined by the width in time.

Putting it together

$$\begin{aligned} c_n(t) &= \frac{-iqE_{\max}}{\hbar}\langle n|x|2\rangle I(t) && (43)\\ &= -\frac{qE_{\max}}{\hbar}\sqrt{\frac{2\hbar}{m\omega}}\left(\sqrt{3}\delta_{n3} + \delta_{n1}\right)\sqrt{2\pi}\gamma\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right) && (44)\\ &= -qE_{\max}\gamma\sqrt{\frac{4\pi}{m\hbar\omega}}\left(\sqrt{3}\delta_{n3} + \sqrt{2}\delta_{n1}\right)\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right) && (45)\end{aligned}$$

Using the deltas, we get

$$c_1(t) = -qE_{\max}\gamma\sqrt{\frac{8\pi}{m\hbar\omega}}\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right) \tag{46}$$

and

$$c_3(t) = -qE_{\max}\gamma\sqrt{\frac{12\pi}{m\hbar\omega}}\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right) \tag{47}$$

and the rest are zero (except of course for c_2, which starts out at 1).

The probabilities of being in the first and third states are given by the squares of the amplitudes, so:

$$P_3 = q^2E_{\max}^2\gamma^2\frac{12\pi}{m\hbar\omega}\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right)^2 \tag{48}$$

$$P_1 = q^2E_{\max}^2\gamma^2\frac{8\pi}{m\hbar\omega}\left(e^{-\frac{(\omega+\omega_0)^2}{2\gamma^2}} - e^{-\frac{(\omega-\omega_0)^2}{2\gamma^2}}\right)^2 \tag{49}$$

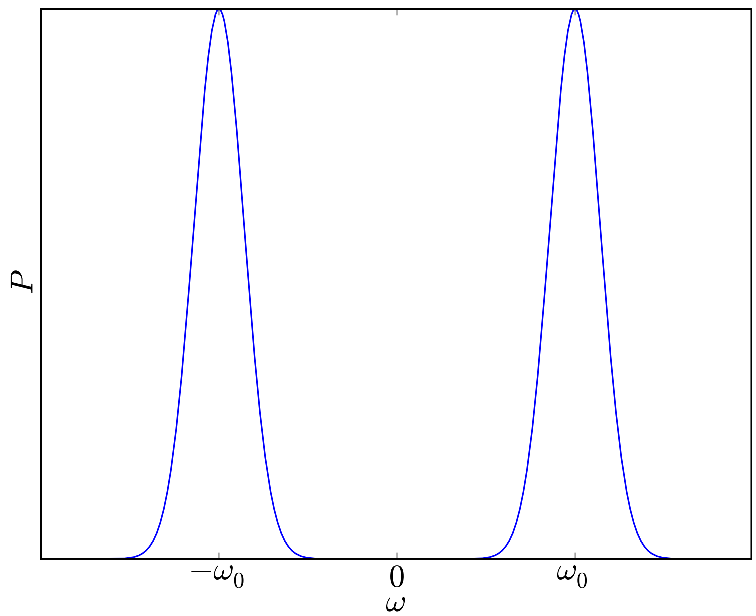
$$P_2 = 1 - P_1 - P_3 \tag{50}$$

(b) Between which states are transitions possible? How would your answer change if you included higher-order effects?

Solution:
We used first-order perturbation theory, and found that only non-zero probabilities were those of the first, second, and third excited states. The second excited state, of course, has non-zero probability because we start in that state, and there is always a chance that no transition will occur. The first and third excited states correspond to absorption and stimulated emission of a single photon. If we had used second-order perturbation theory, we would have seen transitions to the ground state and the fourth excited state, which would be two-photon processes. And in general, the n^{th} order of perturbation theory corresponds to n-photon processes. These naturally become more probable as we increase the intensity of the pulse.

(c) Sketch the probability of ending up in the lowest five states as a function of the frequency ω of the light pulse. Discuss how this relates to the expected behavior in the limit of a classical harmonic oscillator interacting with light—this problem nicely demonstrates Bohr's correspondence principle.

Solution:



We can see that the probability of a transition (either up or down) peaks as the light frequency matches that of the oscillator. We expect this in the classical limit, since an oscillator is moving with frequency ω_0, and will thus radiate at that frequency. Similarly, a classical forced and damped oscillator will absorb the most energy when it is forced at its natural (i.e. resonant) frequency. This also relates to why transitions can only happen between adjacent energy levels: otherwise we would see emission at frequencies other than ω_0, which would violate the correspondence principle in the classical limit.

The following definite integral may be useful:

$$\int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} dx = \frac{\sqrt{\pi}}{|p|} e^{\frac{q^2}{4p^2}} \tag{51}$$

The atmospheric pressure is known to decrease with increasing altitude.

(a) (4 pts.) Starting from the general equation for hydrostatic pressure, $\frac{dP}{dz} = -\rho g$, combine it with the ideal gas law, $PV = Nk_{\mathrm{B}}T$, to find the differential expression for dP/P, and then, by integrating it, to obtain the known "barometric formula", i.e., the equation describing the change of the atmospheric pressure with altitude for an isothermal (T = const.) atmosphere. Assume that the atmospheric air can be described to a good approximation as a single-component ideal gas consisting of molecules of mass M.

However, it's a well-known that the atmosphere is *not* isothermal. In fact, air temperature falls quite noticeably with increasing altitude. Imagine a packet of air which is being swirled around in the atmosphere (e.g., due to fluctuations, a bird flying nearby, etc.). We would expect it to always remain at the same pressure as its surroundings, otherwise it would be mechanically unstable. It is also plausible that the packet moves around too quickly to effectively exchange heat with its surroundings, since air is very a poor heat conductor. So, to a first approximation, the air in the packet is adiabatic. In a steady-state atmosphere, we expect that as the packet moves upwards, expands due to the reduced pressure, and cools adiabatically, its temperature always remains the same as that of its immediate surroundings. Thus we can use the adiabatic gas law to characterize the cooling of the atmosphere with increasing altitude.

(b) (8 pts.) Based on the ideal gas law and the equations for the internal energy and the entropy of an ideal diatomic gas consisting of N molecules:

$$U = \frac{5}{2}Nk_{\mathrm{B}}T \quad \text{and} \quad S = S_0 + Nk_{\mathrm{B}} \ln\left[\left(\frac{U}{U_0}\right)^{5/2}\left(\frac{V}{V_0}\right)\right]$$

and on the fact that for such gas the heat capacities at constant volume and at constant pressure are, respectively: $C_{\mathrm{V}} = \frac{5}{2}Nk_{\mathrm{B}}$ and $C_{\mathrm{P}} = \frac{7}{2}Nk_{\mathrm{B}}$, prove that the equation of state for an adiabatic process in such gas can be expresses in terms of its pressure and temperature in a differential form:

$$\frac{dT}{dp} = \frac{\gamma - 1}{\gamma}\frac{T}{P} \qquad \text{where} \qquad \gamma = \frac{C_{\mathrm{P}}}{C_{\mathrm{V}}}$$

(c) (8 pts.) Combining the above with the expression for dP/P you have obtained in (a), derive the equation for the rate of the temperature change with altitude, dT/dz, and express it terms of γ, M, g, and k_B—then, calculate its value, using $M = 4.65 \times 10^{-26}$ kg, $k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J/K, and assuming that for atmospheric air $\gamma = 1.4$. Express the result in Kelvins per kilometer.

The dT/dz derivative for dry air is referred to as the "adiabatic lapse rate" of the atmosphere. The observed value is lower, about 5 K/km, which is associated with the humidity of real atmospheric air. Needless to say, the knowledge of the atmospheric lapse rate is important in meteorology, aviation, and, of course, in atmospheric sciences in general.

Solution, Undergrad Thermal:

(a) $\frac{dP}{dZ} = -\rho g$ and $PV = Nk_BT$

$$\rho = \frac{\text{gas mass}}{\text{gas Volume}} = \frac{N \cdot M}{V} = \frac{NM}{Nk_BT/P} = \frac{MP}{k_BT}$$

So:

$$\frac{dp}{dz} = -\frac{Mg}{k_BT}p \Rightarrow \frac{dP}{P} = -\frac{Mg}{k_BT}dz \Rightarrow \boxed{p = P_0 e^{-\frac{Mgz}{k_BT}}}$$

(b)

From: $\left\{\begin{array}{l} S = S_0 + Nk_B \ln\left[\left(\frac{U}{U_0}\right)^{5/2}\left(\frac{V}{V_0}\right)\right] \\ \text{and } U = \frac{5}{2}Nk_BT \text{ and } PV = Nk_BT \end{array}\right\} \Rightarrow S = S_0 + Nk_B \ln\left(\frac{T^{7/2}p^{-1}}{T_0^{7/2}p_0^{-1}}\right)$

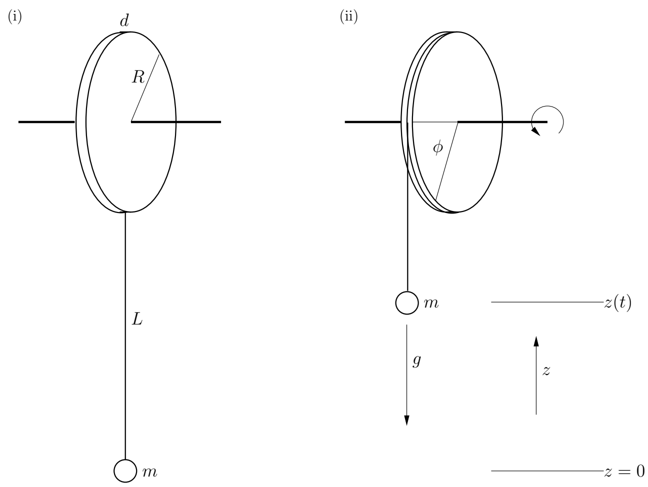
Adiabatic process: $S = \text{const} \Rightarrow T^{7/2}p^{-1} = \text{const}$

Differentiate: $7/2\, T^{5/2}p^{-1}dT - T^{7/2}p^{-2}dp = 0$

$\left.\begin{array}{l} \text{Which yields: } \frac{dT}{dp} = \frac{2}{7}\frac{T}{P} \\ \text{And: } (\gamma-1)/\gamma \\ = \left(\frac{C_P}{C_V}-1\right)\Big/\frac{C_P}{C_V} = \frac{C_P - C_V}{C_P} = \frac{\frac{7}{2}k_BN - \frac{5}{2}k_BN}{\frac{7}{2}k_BN} = \frac{2}{7} \end{array}\right\}$ So, indeed $\frac{dT}{dp} = \frac{\gamma-1}{\gamma}\frac{T}{P}$

(c) $\left.\begin{array}{l} \text{From (a): } \frac{dP}{P} = -\frac{Mg\,dz}{k_BT} \\ \text{From (b): } dT/dP = \frac{\gamma-1}{\gamma}\frac{T}{P} \Rightarrow \frac{dP}{P} = \frac{\gamma}{\gamma-1}\frac{dT}{T} \end{array}\right\} \Rightarrow \frac{\gamma}{\gamma-1}\frac{dT}{T} = -\frac{Mg\,dz}{k_BT}$

So: $\boxed{\frac{dT}{dz} = \frac{\gamma-1}{\gamma}\frac{Mg}{k_B}}$ $\left.\begin{array}{l} M = 4.65\times10^{-26}\text{ kg} \\ k_B = 1.3807\times10^{-23}\text{ J/K} \\ \gamma = 1.4;\ g = 9.81\ \frac{m}{s^2} \end{array}\right\}$ $\frac{dT}{dz} = 0.0094\text{ K/m} = 9.4\text{ K/km}$



Consider a mass m attached to a mass-less thread of length L to a point on the the circumference of a disk of radius R, as shown in Figure (i). The disk is free to rotate about a fixed axis through its center. At $t = 0$ the thread is wound around the disk and unwinds as the mass m is accelerated by gravity from its initial height $z = L$, as shown in Figure (ii). Neglect friction and assume $L >> R$, i.e. neglect the displacement of the mass m in the horizontal plane.

(a) Calculate the moment of inertia I of the disk having uniform density ρ, thickness d and mass M.

(b) Using energy conservation, calculate the maximum angular frequency of rotation ω_{max} of the disk.

(c) Write down the equations of motion for the angle $\phi(t)$ of the disk and $z(t)$ of the mass m.

(d) Find the solutions $\phi(t)$ and $z(t)$, and check against the result obtained in part b). What is the period T of the motion, i.e. the time for the system to exactly return to the initial state?

(e) Sketch the solutions $\phi(t)$ and $z(t)$ over more than one period. Comment on the change of linear and angular momentum, especially near the turning points.

(f) If you want to determine the earth's gravitational acceleration g from a measurement of the period T, what is the uncertainty δg if the relative uncertainties are $\delta T/T = 5 \times 10^{-3}$ and $\delta L/L = 10^{-2}$, and all other uncertainties are negligible?

a) $$I = \int_V \rho r^2 dV = \rho \cdot d \cdot 2\pi \int_0^R r^3 dr = \underbrace{\rho\, \overbrace{d \pi R^2}^{=V}}_{=M} \frac{1}{2} R^2 = \frac{1}{2} M R^2$$

b) $$U = mgz \quad (t=0,\ z=L) \quad \rightarrow U_{max} = mgL$$

$$T_{rot} = \frac{1}{2} I \omega^2 \quad (t=T_0,\ z=0) \qquad T_{rot,max} = \frac{1}{2} I \omega_{max}^2$$

$$T_{kin} = \frac{1}{2} m v^2 \qquad T_{kin,m} = \frac{1}{2} m v_{max}^2$$

$$|v| = |\omega| \cdot R \quad \text{(geometry)}$$

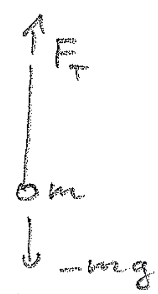
$$T_{rot,m} + T_{kin,m} = U_m$$

$$\frac{1}{2} I \omega_m^2 + \frac{1}{2} m v_m^2 = mgL$$

$$\frac{1}{2} \omega_m^2 (I + mR^2) = mgL$$

$$\omega_m = \sqrt{\frac{2mgL}{I + mR^2}}$$

c)



$$F_T \cdot R = I \dot{\omega} = \frac{1}{2} M R^2 \ddot{\phi}$$

using the coordinate system in figure: $-v_z = \omega R$

$$-\ddot{z} = \ddot{\phi} R$$

$$m\ddot{z} = -mg + F_T$$

$$m\ddot{z} = -mg + \frac{1}{2} M R \ddot{\phi}$$

$$= -mg - \frac{1}{2} M \ddot{z}$$

$$\Rightarrow \boxed{\ddot{z} = \frac{-mg}{\frac{M}{2} + m}} \Rightarrow \boxed{\ddot{\phi} = \frac{mg}{R\left(\frac{M}{2} + m\right)}}$$

d) Integrate $\ddot{z} = -\frac{mg}{\frac{M}{2}+m}$

$$\int_{\dot{z}=0}^{\dot{z}(t)} \longrightarrow \dot{z}(t) = -\frac{mg}{\frac{M}{2}+m}\,t$$

$$\rightarrow \int_{z(0)=L}^{z(t)} \longrightarrow z(t) = L - \frac{1}{2}\left(\frac{mg}{\frac{M}{2}+m}\right)t^2 \qquad ; 0 < t < T_0$$

with
$$\Rightarrow \dot{\phi} = -\frac{\dot{z}}{R} \rightarrow \phi(t) = \frac{1}{2}\,\frac{mg}{R\left(\frac{M}{2}+m\right)}\,t^2 \qquad ; 0 < t < T_0$$

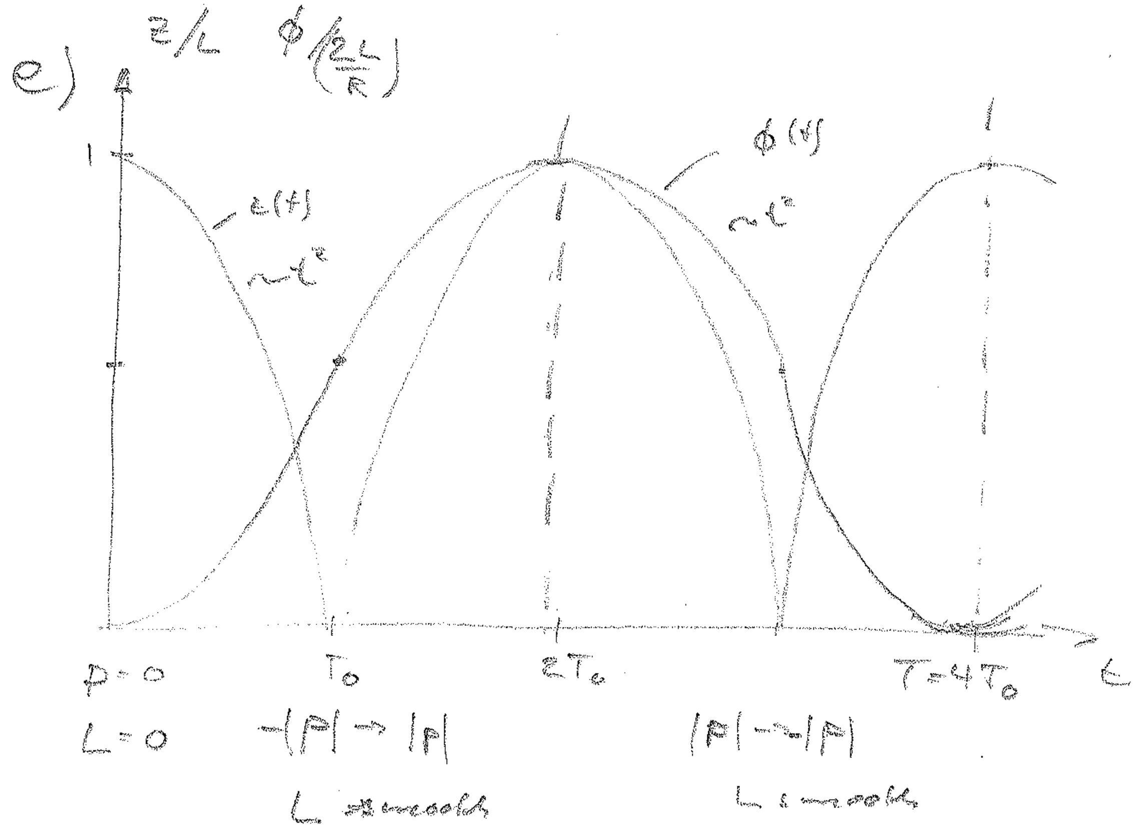
find ω_{max}: $z(T_0) = 0 \rightarrow T_0 = \sqrt{\frac{2L}{mg}\left(\frac{M}{2}+m\right)}$

$$\omega_{max} = -\dot{z}(T_0)\frac{1}{R} = \frac{1}{R}\,\frac{mg}{\frac{M}{2}+m}\sqrt{\frac{2L}{mg}\left(\frac{M}{2}+m\right)}$$

$$= \sqrt{\frac{2mgL}{\left(\frac{M}{2}+m\right)R^2}} = \sqrt{\frac{2mgL}{I + mR^2}} \qquad \square$$

solutions for $z(t)$, $\phi(t)$ for $t > T_0$ can be obtained by further integrating with appr. initial conditions (at $t = T_0$ $v \to -v$).
Symmetry considerations show that $\phi(t)$ continues to grow until $T = 2T_0$ (now $z = L$, $\phi = \frac{2L}{R}$, $\dot{\phi} = \omega = 0$) when the disk (jojo) reverses its sense of rotation (see e).
The system returns exactly to the initial state ($z = L$, $\phi = 0$) after $\underline{\underline{T = 4T_0}}$.

e)



The linear momentum p of the little mass m changes discontinuously at $t = T_0, 3T_0, \ldots$ Momentum $|2p|$ absorbed by axis.

The angular momentum is a diff'able function of time.

f) $$\frac{\delta g}{g} = \frac{1}{g}\left|\frac{\partial g}{\partial L}\right|\delta L + \frac{1}{g}\left|\frac{\partial g}{\partial T}\right|\delta T$$

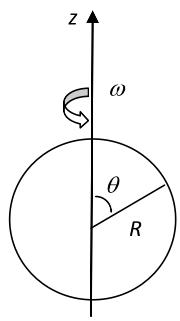
$$g = \frac{2L}{T_0^2}\,\frac{\frac{M}{2}+m}{m} = \frac{32\,L}{T^2}\,\frac{\frac{M}{2}+m}{m} = k\,\frac{L}{T^2}$$

$$\delta g = \frac{kL}{T^2}\frac{\delta L}{L} + \left|\frac{-2kL}{T^2}\right|\frac{\delta T}{T} \Rightarrow \frac{\delta g}{g} = \frac{\delta L}{L} + 2\,\frac{\delta T}{T}$$

with $\frac{\delta T}{T} = 5\cdot 10^{3}$; $\frac{\delta L}{L} = 10^{-2}$ $\Rightarrow$ $\frac{\delta g}{g} = 2\cdot 10^{-2}$

Problem #1 E&M

Consider magnetic induction due to a thin uniform spherical shell of radius R and charge q rotating about the z-axis with the frequency ω, as illustrated in the figure below.



(a) Determine the surface current density as a result of the rotation.

(b) By analogy to the electric field and potential, it is sometimes possible to introduce a magnetic scalar potential Φ_M such that the magnetic induction **B** = - $\nabla\Phi_M$. Show that if that is the case, the potential Φ_M satisfies the Laplace equation $\nabla^2\Phi_M = 0$.

(c) Demonstrate that the magnetic potential Φ_M inside(Φ_{in}) and outside (Φ_{out}) the sphere can be represented as:

$$\Phi_{in} = \sum_{m=1}^{\propto} b_m r^m P_m(\mathrm{COS}\,\theta)$$

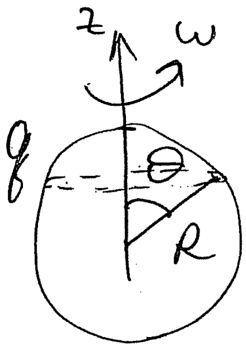
$$\Phi_{out} = \sum_{m=0}^{\propto} \frac{a_m}{r^{m+1}} P_m(\mathrm{COS}\,\theta)$$

(d) Apply Gauss' law and Ampere's law to determine boundary conditions appropriate for this problem.

(e) Now apply boundary conditions from part (d) to determine coefficients a_m and b_m and to fully determine Φ_{in} and Φ_{out}.

(1)

Problem #8



(a) Surface current density

$$K = \underset{\substack{\uparrow \\ \text{charge} \\ \text{density}}}{\sigma} \underset{\substack{\uparrow \\ \text{velocity} \\ \text{of} \\ \text{charge motion}}}{v} = \frac{q}{4\pi R^2} \cdot \omega R \sin\theta =$$

$$= \frac{q}{4\pi R} \omega \sin\theta = K(\theta)$$

(b) Introduce $\underbrace{\varphi_M}_{\text{magnetic scalar potential}}$, so that $\vec{B} = -\vec{\nabla}\varphi_M$

Gauss law $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \vec{\nabla}\varphi_M = \underbrace{\vec{\nabla}^2 \varphi_M = 0}$

(c) See any Math Physic or E&M book $\Rightarrow$ solve in spherical coordinates $\Leftarrow$ Laplace equation

$$\underbrace{\varphi_{in}}_{\substack{\text{inside} \\ \text{the sphere}}} = \sum_{m=1}^{\infty} b_m r^m P_m(\cos\theta) \; ; \; \underbrace{\varphi_{out}}_{\text{outside}} = \sum_{m=0}^{\infty} \frac{a_m}{r^{m+1}} P_m(\cos\theta)$$

$\Rightarrow \varphi_{in}$ is finite for $0 \le r \le R$

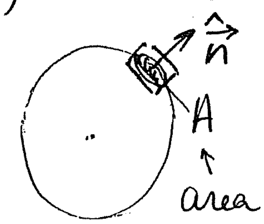
($\varphi_{out} \to 0$ as $r \to \infty$)

2

(d) Now need to find a_m & b_m

Boundary conditions:

1) Gauss's law $\Rightarrow$ $\oint_S \vec{B}\cdot d\vec{S} = 0 =$

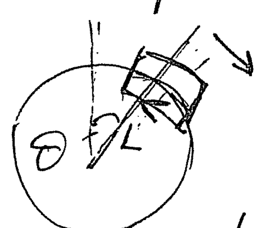


$$= \int_A \underbrace{(\vec{B}_{out} - \vec{B}_{in})\cdot\hat{n}}_{=0}\, dA \Rightarrow$$

$$\left.\frac{\partial \varphi_{out}}{\partial r}\right|_{r=R} = \left.\frac{\partial \varphi_{in}}{\partial r}\right|_{r=R}$$

2) Ampere's law $\Rightarrow$ $\oint_C \vec{B}\cdot d\vec{l} = \int_L (\vec{B}_{out} - \vec{B}_{in})\cdot\hat{\theta}\, dl = \mu_0 \underbrace{\int_L K\, dl}_{\text{current}}$

L ← length



$\hat{\theta}$ ← unit vector tangent to the surface in the direction of increasing θ

$\Downarrow$

$$\frac{1}{r}\left.\left(\frac{\partial \varphi_{out}}{\partial \theta} - \frac{\partial \varphi_{in}}{\partial \theta}\right)\right|_{r=R} = \mu_0 K(\theta)$$

$K(\theta)$ ← from part (a)

3

(d) Apply boundary conditions

$$\frac{\partial \varphi_{out}}{\partial r} = \sum_m (-m-1)\, a_m \frac{1}{r^{m+2}} P_m(\cos\theta)$$

$$\frac{\partial \varphi_{in}}{\partial r} = \sum_m m\, b_m\, r^{m-1} P_m(\cos\theta)$$

So, $-(m+1)\, a_m \frac{1}{R^{m+2}} = m\, b_m R^{m-1} \Rightarrow$

$$\boxed{b_m = -a_m \frac{m+1}{m} \frac{1}{R^{2m+1}}} \quad m \geq 1 \;;\; a_0 = 0$$

$$\frac{\partial \varphi_{out}}{\partial \theta} = \sum_m \frac{a_m}{r^{m+1}} \frac{dP_m(\cos\theta)}{d\theta}$$

$$\frac{\partial \varphi_{in}}{\partial \theta} = \sum_m \frac{-a_m}{R^{2m+1}} \frac{m+1}{m}\, r^m \frac{dP_m(\cos\theta)}{d\theta}$$

So,

$$\frac{1}{R} \sum_m a_m \frac{dP_m(\cos\theta)}{d\theta} \underbrace{\left[\frac{1}{R^{m+1}} + \frac{R^m}{R^{2m+1}} \frac{m+1}{m}\right]}_{= \frac{1}{R^{m+1}} \frac{2m+1}{m}} =$$

$$= \mu_0 \frac{q}{4\pi R}\, \omega \underbrace{\sin\theta}_{= -\frac{dP_1(\cos\theta)}{d\theta}}$$

$P_1(\cos\theta) = \cos\theta$

(4)

So, $$\sum_{m=1}^{\infty} \frac{2m+1}{m} \frac{a_m}{R^{m+1}} \frac{dP_m(\cos\theta)}{d\theta} =$$

$$= -\frac{\mu_0 q \omega}{4\pi} \frac{dP_1(\cos\theta)}{d\theta} \Longrightarrow$$

1) a_m for $m > 1 \; = 0$

2) for $m = 1$: $$\underbrace{\frac{2m+1}{mR^{m+1}}}_{=\frac{3}{R^2}} \underbrace{a_m}_{=1} \frac{dP_1}{d\theta} = -\frac{\mu_0 q \omega}{4\pi} \frac{dP_1}{d\theta}$$

$$3a_1 = -\frac{\mu_0 q \omega}{4\pi} R^2 \Longrightarrow a_1 = -\frac{\mu_0 q \omega}{12\pi} R^2$$

Finally,

$$\Phi_{out} = \frac{a_1}{r^2} P_1(\cos\theta) = -\frac{\mu_0 q \omega R^2}{12\pi} \frac{\cos\theta}{r^2}$$

$$\Phi_{in} = b_1 r P_1(\cos\theta) = -a_1 \cdot 2 \cdot \frac{1}{R^3} r \cos\theta =$$

$$= \frac{\mu_0 q \omega}{6\pi} \frac{r}{R} \cos\theta$$