OSU Physics Department
Comprehensive Examination #112

Monday, January 9 and Tuesday, January 10, 2012

Winter ~~Fall~~ 2012 Comprehensive Examination

PART 1, Monday, January 9, 9:00am

General Instructions

This Fall 2012 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, January 9, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, January 10, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Consider the following normalized wave function for a free particle with mass m at $t = 0$

$$\psi(\vec{r}, t=0) = \frac{1}{\pi^{\frac{3}{2}}\sigma^3} e^{i\vec{k_0}\cdot\vec{r}} e^{-\frac{|\vec{r}|^2}{2\sigma^2}} \tag{1}$$

ERRATUM: The above equation has an error, and is not normalized and indeed has the wrong dimensionality. The equation should be:

$$\psi(\vec{r}, t=0) = \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{i\vec{k_0}\cdot\vec{r}} e^{-\frac{|\vec{r}|^2}{2\sigma^2}} \tag{2}$$

Because of this mistake, certain portions of the exam were graded more leniently than would otherwise be the case. Unfortunately, no students caught the mistake.

(a) What is the expectation value of the energy of this particle?

(b) What is the expectation value of the momentum of this particle?

(c) What is the uncertainty in its momentum?

(d) Solve for the wave function $\psi(\vec{k}, t = 0)$ in momentum space.

(e) Write down an expression for wave function as a function of time and space. If your expression involves an integral, you need not perform this integral.

For this problem, the following integrals over all space may be helpful:

$$\iiint e^{-u^2} d\vec{u} = \pi^{\frac{3}{2}} \tag{3}$$

$$\iiint u^2 e^{-u^2} d\vec{u} = \frac{1}{2}\pi^{\frac{3}{2}} \tag{4}$$

Consider the following normalized wave function for a free particle with mass m at $t = 0$

$$\psi(\vec{r}, t = 0) = \frac{1}{\pi^{\frac{3}{2}}\sigma^3} e^{i\vec{k_0}\cdot\vec{r}} e^{-\frac{|\vec{r}|^2}{2\sigma^2}} \tag{5}$$

ERRATUM: The above equation has an error, and is not normalized and indeed has the wrong dimensionality. The equation should be:

$$\psi(\vec{r}, t = 0) = \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{i\vec{k_0}\cdot\vec{r}} e^{-\frac{|\vec{r}|^2}{2\sigma^2}} \tag{6}$$

Because of this mistake, certain portions of the exam were graded more leniently than would otherwise be the case. Unfortunately, no students caught the mistake.

(a) What is the expectation value of the energy of this particle?

Solution:

Since we're looking at a free particle, we only need to find its Laplacian and then integrate.

$$\vec{\nabla}\psi = i\vec{k_0}\psi - \frac{\vec{r}}{\sigma^2}\psi \tag{7}$$

$$\nabla^2\psi = -k_0^2\psi - 2i\frac{\vec{k_0}\cdot\vec{r}}{\sigma^2}\psi + \frac{r^2}{\sigma^4}\psi - \frac{3}{\sigma^2}\psi \tag{8}$$

The 3 in the above equation was a little subtle, and comes from the fact that $\vec{\nabla}\cdot\vec{r} = 3$, while the obvious (wrong) guess would be that it should be 1.

$$\langle H\rangle = \int \psi^* \left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi d\vec{r} \tag{9}$$

$$= -\frac{\hbar^2}{2m}\int \psi^* \left(-k_0^2\psi - 2i\frac{\vec{k_0}\cdot\vec{r}}{\sigma^2}\psi + \frac{r^2}{\sigma^4}\psi - \frac{3}{\sigma^2}\psi\right) d\vec{r} \tag{10}$$

$$= \frac{\hbar^2 k_0^2}{2m} + \frac{3\hbar^2}{2m\sigma^2} + \frac{\hbar^2}{2m}\frac{1}{\sigma^2}\left(2i\int \vec{k_0}\cdot\vec{r}|\psi|^2 d\vec{r} - \int \frac{r^2}{\sigma^2}|\psi|^2 d\vec{r}\right) \tag{11}$$

$$= \frac{\hbar^2 k_0^2}{2m} + \frac{3\hbar^2}{2m\sigma^2} - \frac{\hbar^2}{2m}\frac{1}{\pi^{\frac{3}{2}}\sigma^3}\frac{1}{\sigma^2}\int \frac{r^2}{\sigma^2} e^{-\frac{r^2}{\sigma^2}} d\vec{r} \tag{12}$$

$$\vec{u} = \frac{\vec{r}}{\sigma} \qquad d\vec{u} = \frac{d\vec{r}}{\sigma^3} \tag{13}$$

$$= \frac{\hbar^2 k_0^2}{2m} + \frac{3\hbar^2}{2m\sigma^2} - \frac{\hbar^2}{2m}\pi^{-\frac{3}{2}}\frac{1}{\sigma^2}\int u^2 e^{-u^2} d\vec{u} \tag{14}$$
$$= \frac{\hbar^2 k_0^2}{2m} + \frac{3\hbar^2}{2m\sigma^2} - \frac{\hbar^2}{2m}\pi^{-\frac{3}{2}}\frac{1}{\sigma^2}\int u^2 e^{-u^2} d\vec{u} \tag{15}$$
$$= \frac{\hbar^2 k_0^2}{2m} + \frac{3\hbar^2}{2m\sigma^2} - \frac{\hbar^2}{2m}\frac{1}{\sigma^2}\frac{1}{2} \tag{16}$$
$$\langle H \rangle = \frac{\hbar^2 k_0^2}{2m} + \frac{5\hbar^2}{4m\sigma^2} \tag{17}$$

This is just the ordinary plane-wave energy plus a small correction due to the gaussian envelope.

(b) What is the expectation value of the momentum of this particle?

Solution:

This is much like the last one...

$$\vec{p} = -i\hbar\vec{\nabla} \tag{18}$$
$$\vec{\nabla}\psi = i\vec{k_0}\psi - \frac{\vec{r}}{\sigma^2}\psi \tag{19}$$
$$\vec{p}\psi = \hbar\vec{k_0}\psi - i\hbar\frac{\vec{r}}{\sigma^2}\psi \tag{20}$$
$$\langle \vec{p} \rangle = \iiint \psi^* \vec{p}\psi d\vec{r} \tag{21}$$
$$= \iiint |\psi|^2 \left(\hbar\vec{k_0} - i\hbar\frac{\vec{r}}{\sigma^2}\right) d\vec{r} \tag{22}$$
$$= \hbar\vec{k_0} \tag{23}$$

This one comes out just like it would be for a pure plane wave! $\ddot{\smile}$ A common mistake on this problem (as well as elsewhere) was to omit the $\vec{\ }$ on vectors, as well as to omit the $\cdot$ in dot products. It is likely that students knew that they were working with vectors, but we can only grade students on what they actually write.

(c) What is the uncertainty in its momentum?

Solution:

To find the uncertainty in momentum, we will use the standard deviation, which we can get from the usual formula.

$$\Delta p = \sqrt{\langle p^2\rangle - |\langle \vec{p}\rangle|^2} \tag{24}$$

$$= \sqrt{\left(\hbar^2 k_0^2 + \frac{\hbar^2}{2\sigma^2}\right) - \hbar^2 k_0^2} \tag{25}$$

$$= \frac{\hbar}{\sqrt{2}\sigma} \tag{26}$$

Where I used the previous solutions for the expectation values of energy and momentum (and the fact that energy is $\frac{p^2}{2m}$).

(d) Solve for the wave function $\psi(\vec{k}, t=0)$ in momentum space.

Solution:

We just need to find the Fourier transform of the wave function. Note that I'm going to do a transform to $\vec{k}$ space, which is one sort of momentum space, but you could also do a transform to $\vec{p}$ space, which differs in dimensions by a factor of $\hbar$, which would change $(2\pi)^{\frac{3}{2}}$ to $(2\pi\hbar)^{\frac{3}{2}}$.

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint \psi(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}} d\vec{k} \tag{27}$$

$$\psi(\vec{k}, t=0) = \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint \psi(\vec{r}, t=0) e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \tag{28}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{i\vec{k_0}\cdot\vec{r}} e^{-\frac{|\vec{r}|^2}{2\sigma^2}} e^{-i\vec{k}\cdot\vec{r}} d\vec{r} \tag{29}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{-\frac{|\vec{r}|^2}{2\sigma^2} - i(\vec{k}-\vec{k_0})\cdot\vec{r}} d\vec{r} \tag{30}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \iiint \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{-\frac{1}{2}\left(\frac{\vec{r}}{\sigma} - i(\vec{k}-\vec{k}_0)\sigma\right)^2 - \frac{1}{2}|\vec{k}-\vec{k}_0|^2\sigma^2} d\vec{r} \tag{31}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}|\vec{k}-\vec{k}_0|^2\sigma^2} \iiint \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} e^{-\frac{1}{2}\left(\frac{\vec{r}}{\sigma} - i(\vec{k}-\vec{k}_0)\sigma\right)^2} d\vec{r} \tag{32}$$

We now just do a little change of variables to make the integral match one we know the solution to (because it's provided). The change of variables involves a complex integrand, but this is okay because at the limits (at ∞) the integrand is zero (which is real) and the contour of the integral doesn't matter, since there are no singularities in the integrand.

$$\vec{u} = \frac{1}{\sqrt{2}}\left(\frac{\vec{r}}{\sigma} - i(\vec{k}-\vec{k}_0)\sigma\right) \tag{33}$$

$$d\vec{u} = \frac{d\vec{r}}{\sqrt{2}\sigma} \tag{34}$$

$$\psi(\vec{k}, t=0) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}|\vec{k}-\vec{k}_0|^2\sigma^2} \frac{1}{\sqrt{\pi^{\frac{3}{2}}\sigma^3}} 2^{\frac{3}{2}}\sigma^3 \iiint e^{-u^2} d\vec{u} \qquad (35)$$

$$= \frac{1}{(\pi)^{\frac{3}{4}}} e^{-\frac{1}{2}|\vec{k}-\vec{k}_0|^2\sigma^2} \qquad (36)$$

(e) Write down an expression for wave function as a function of time and space. If your expression involves an integral, you need not perform this integral.

Solution:

Since a plane wave is an energy eigenstate of the Hamiltonian, we can write the time-dependence in momentum-space very easily in momentum space:

$$\psi(\vec{k}, t) = \psi(\vec{k}, t=0) e^{-i\frac{\hbar k^2}{2m}t} \qquad (37)$$

and thus we can find the time-dependence of the spatial wave function using a Fourier transform:

$$\psi(\vec{r}, t) = \iiint \psi(\vec{k}, 0) e^{i\vec{k}\cdot\vec{r} - i\frac{\hbar k^2}{2m}t} d\vec{k} \qquad (38)$$

Where the function $\psi(\vec{k}, 0)$ is your answer from the previous section. This comes out to:

$$\psi(\vec{r}, t) = \pi^{-\frac{3}{4}} \iiint e^{-\frac{1}{2}|\vec{k}-\vec{k}_0|^2\sigma^2 + i\vec{k}\cdot\vec{r} - i\frac{\hbar k^2}{2m}t} d\vec{k} \qquad (39)$$

At this point we have solved the problem. For extra fun, we could have taken an expansion under the assumption that $\frac{\hbar}{m}t \ll \sigma^2$, which would have shown how a wavepacket spreads out.

For this problem, the following integrals over all space may be helpful:

$$\iiint e^{-u^2} d\vec{u} = \pi^{\frac{3}{2}} \qquad (40)$$

$$\iiint u^2 e^{-u^2} d\vec{u} = \frac{1}{2}\pi^{\frac{3}{2}} \qquad (41)$$

Comp Exam Winter 2012

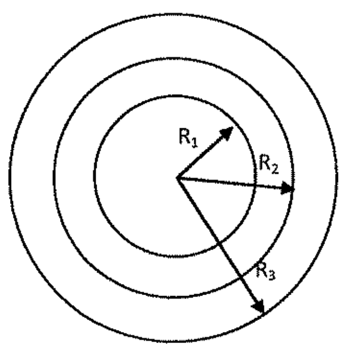
E&M Problem #1

Consider an infinitely extended cylindrical cable, in which the interior lead with radius R_1 is carrying a direct current I and the cable sheath, with interior and exterior radii R_2 and R_3, respectively, is carrying the same current I, but in the opposite direction. Magnetic permeability in the conducting regions is μ, while in the space between the lead and sheath, and outside the cable, it is μ_0.

Determine the vector potential and magnetic field in the regions:

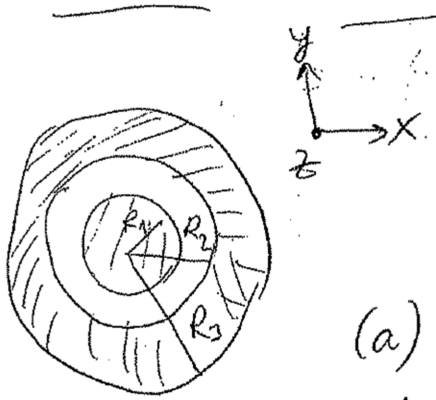
(a) $0 < r < R_1$
(b) $R_1 \leq r \leq R_2$
(c) $R_2 \leq r \leq R_3$
(d) Sketch the vector potential and magnetic field as a function of r (including the region with $r > R_3$). Discuss.

Assume the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Also assume that the vector potential is zero at $r = R_1$. Cross-section of the cable is shown below.



(1)

E & M Problem #2



I along Oz at $0<r<R_1$

I along $-Oz$ at $R_2<r<R_3$

(a) $0<r<R_1$

$$j_z = \frac{I}{\pi R_1^2}\ ;\ j_{x,y}=0$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{j} \Rightarrow \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \mu\vec{j} \Rightarrow \vec{\nabla}^2\vec{A} = -\mu\vec{j} \Rightarrow$$

$\vec{B} = \vec{\nabla}\times\vec{A}$

$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2 A$, with $\vec{\nabla}\cdot\vec{A} = 0 \Leftarrow$ gauge

$\vec{\nabla}^2\vec{A}$: only r-dependence (symmetry) $\frac{d}{d\varphi}, \frac{d}{dz} = 0$

$$\Rightarrow \vec{\nabla}^2 A_z = -\mu j_z$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_z}{dr}\right) = \frac{d^2A_z}{dr^2} + \frac{1}{r}\frac{dA_z}{dr} = -\frac{\mu I}{\pi R_1^2}$$

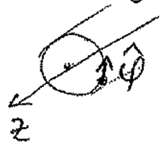
$$\frac{d}{dr}\left(r\frac{dA_z}{dr}\right) = -\frac{\mu I}{\pi R_1^2}r \Rightarrow r\frac{dA_z}{dr} = -\frac{\mu I}{\pi R_1^2}\frac{r^2}{2} + C_1 \Rightarrow$$

$$A_z(r) = -\frac{\mu I}{4\pi R_1^2}r^2 + C_1\ln r + C_2$$

(2)

$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$ since only A_z is present $\Rightarrow$
(and $A_z \neq f(\varphi)$)

$$\vec{B} = -\frac{\partial A_z}{\partial r}\hat{\varphi} \Rightarrow B_\varphi = \frac{\mu I}{2\pi R_1^2} r - \frac{C_1}{r}$$



$\hat{\varphi}$ ↑ unit vector

Since B_φ needs to be finite at $r=0 \Rightarrow \underline{C_1 = 0}$

If. $A_z\Big|_{r=R_1} = 0 \Rightarrow C_2 = \frac{\mu I}{4\pi} \Rightarrow$

$$A_z(r) = \frac{\mu I}{4\pi}\left(1 - \frac{r^2}{R_1^2}\right) \; ; \; B_\varphi = \frac{\mu I}{2\pi R_1^2} r$$

(b) $R_1 \leq r \leq R_2$

no current $\Rightarrow \vec{\nabla}^2 A_z = 0 \Rightarrow A_z(r) = C_3 \ln r + C_4$

$B_\varphi = -\frac{C_3}{r}$ $\Rightarrow A_z, H_\varphi$ are continuous

Determine C_3, C_4 from boundary conditions at

$r = R_1 \Rightarrow$ 1) $A_z(R_1) = \underbrace{\frac{\mu I}{4\pi}(1-1)}_{=0} = C_3 \ln R_1 + C_4$

2) $\overbrace{\frac{\mu I}{2\pi R_1^2} \cdot R_1 \cdot \frac{1}{\mu}}^{=H} = -\frac{C_3}{R_1 \mu_0} \Rightarrow C_3 = -\frac{\mu_0 I}{2\pi} \Rightarrow$

(3)

$$C_4 = -C_3 \ln R_1 = \frac{\mu_0 I}{2\pi} \ln R_1$$

So, $$A_z(r) = -\frac{\mu_0 I}{2\pi} \ln r + \frac{\mu_0 I}{2\pi} \ln R_1 =$$

$$= \frac{\mu_0 I}{2\pi} \ln \frac{R_1}{r} \quad ; \quad B_\varphi = \frac{\mu_0 I}{2\pi r}$$

(c) $R_2 \le r \le R_3$

In this region, $j_z = \frac{-I}{\pi (R_3^2 - R_2^2)}$

Then, $\vec{\nabla}^2 A_z = \frac{\mu I}{\pi (R_3^2 - R_2^2)} \Rightarrow \frac{d}{dr}\left(r \frac{dA_z}{dr}\right) = \frac{\mu I r}{\pi (R_3^2 - R_2^2)}$

$\frac{1}{r}\frac{d}{dr}\left(r \frac{dA_z}{dr}\right)$

$$r \frac{dA_z}{dr} = \frac{\mu I r^2}{2\pi (R_3^2 - R_2^2)} + C_5 \Rightarrow$$

$$A_z(r) = \frac{\mu I r^2}{4\pi (R_3^2 - R_2^2)} + C_5 \ln r + C_6$$

$$B_\varphi(r) = -\frac{\partial A_z}{\partial r} = -\frac{\mu I r}{2\pi (R_3^2 - R_2^2)} - \frac{C_5}{r}$$

(4)

Boundary conditions at $r = R_2$: A_z, H_φ are continuous

1) A_z: $$\frac{\mu_0 I}{2\pi} \ln \frac{R_1}{R_2} = \frac{\mu I R_2^2}{4\pi (R_3^2 - R_2^2)} + C_5 \ln R_2 + C_6$$

H_φ: $$\frac{\mu_0 I}{2\pi R_2} \cdot \frac{1}{\mu_0} = -\frac{\mu I R_2}{2\pi (R_3^2 - R_2^2)} \cdot \frac{1}{\mu} - \frac{C_5}{\mu R_2} \Rightarrow$$

$$C_5 = -\mu R_2 \left[\frac{I R_2}{2\pi (R_3^2 - R_2^2)} + \frac{I}{2\pi R_2} \right] = -\mu R_2 \frac{I R_3^2}{2\pi (R_3^2 - R_2^2) R_2} = -\frac{\mu I R_3^2}{2\pi (R_3^2 - R_2^2)}$$

$$B_\varphi (r) = \frac{\mu I}{2\pi r} \frac{R_3^2 - r^2}{R_3^2 - R_2^2} \Leftarrow$$

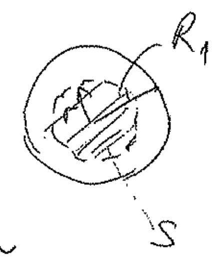
So,

$$C_6 = \frac{\mu_0 I}{2\pi} \ln \frac{R_1}{R_2} - \frac{\mu I R_2^2}{4\pi (R_3^2 - R_2^2)} + \frac{\mu I R_3^2}{2\pi (R_3^2 - R_2^2)} \ln R_2$$

$$A_z (r) = \frac{\mu I}{4\pi (R_3^2 - R_2^2)} (r^2 - R_2^2) + \frac{\mu_0 I}{2\pi} \ln \frac{R_1}{R_2} + \frac{\mu I R_3^2 \ln R_2}{2\pi (R_3^2 - R_2^2)} - \frac{\mu I R_3^2}{2\pi (R_3^2 - R_2^2)} \ln r = \frac{\mu I}{4\pi} \left[\frac{r^2 - R_2^2}{R_3^2 - R_2^2} - \frac{2\mu_0}{\mu} \ln \frac{R_1}{R_2} - \frac{2 R_3^2}{R_3^2 - R_2^2} \ln \frac{r}{R_2} \right]$$

(5)

Of course, one could approach this problem from finding $\vec{B}$ and then $\vec{A}$:



for example, for $r < R_1 \Rightarrow$

$$B_\varphi \cdot 2\pi r = \mu \underbrace{\frac{I}{\pi R_1^2}}_{j} \overbrace{S}^{\pi r^2} \Rightarrow \quad B_\varphi = \frac{\mu I}{2\pi R_1^2} r$$

Then, $A_z = -\int B_\varphi \, dr = -\frac{\mu I}{2\pi R_1^2}\frac{r^2}{2} + C$

$\vec{\nabla} \times \vec{A} = \vec{B}$

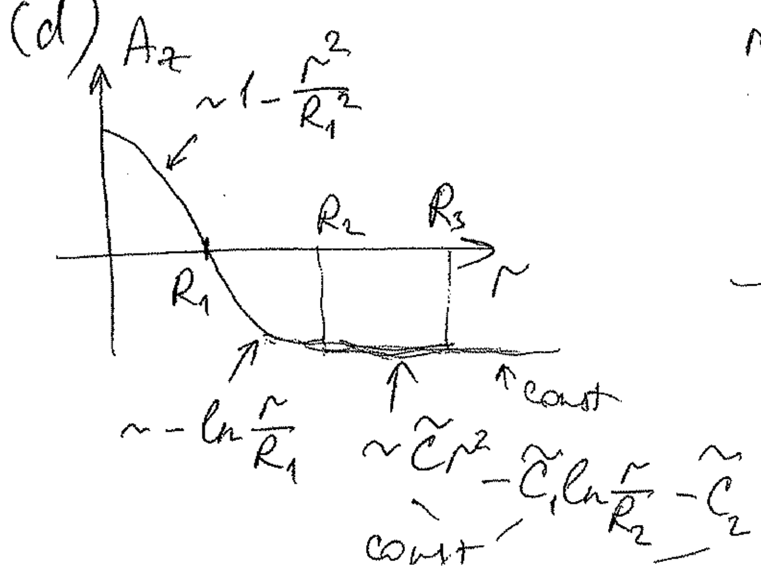
$-\frac{\partial A_z}{\partial r} = B_\varphi$

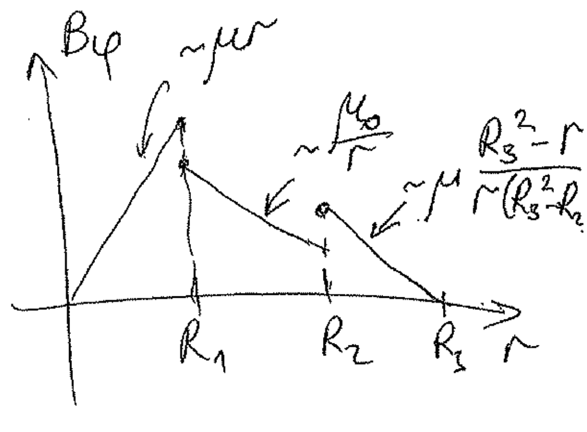
At $r = R_1 \Rightarrow A = 0 \Rightarrow C = \frac{\mu I}{4\pi} \Rightarrow$

$$A_z = \frac{\mu I}{4\pi}\left(1 - \frac{r^2}{R_1^2}\right)$$

Similarly for other regions

(d)



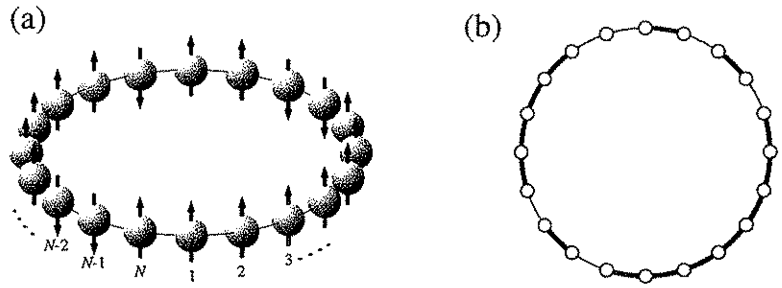


Suppose that N Ising spins are arranged along a ring (Plot (a) in the figure below). Assume that the energy of this system is is given by:

$$\mathcal{H} = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} \qquad (\sigma_{N+1} \equiv \sigma_1),$$

where σ_i takes the value $+1$ or -1. Assume that $N \gg 1$, and that it is an *even* number.

The physical situation in the system may be represented graphically in the form of a ring with N "dots", each of which symbolizes an Ising spin, as shown in Plot (b) below. The dots in are connected with "links" – some of them are "heavy" and some of them are "light". A "heavy" link means that the two spins it connects have the same orientation (both "up", $\uparrow\uparrow$, or both "down", $\downarrow\downarrow$). A "light" link means that the two spins at its ends have opposite orientations ($\uparrow\downarrow$, or $\downarrow\uparrow$).



(a) (2 pts.) Show that in such a ring the number of "light" links, $n_{\rm light}$, may only assume *even* values (0 is considered as an even number).

(b) (3 pts.) What is the energy of a state corresponding to a given $n_{\rm light}$ value, and what is the multiplicity factor of such state?

(c) (5 pts.) Show that the *exact* solution for the the partition function of the system is:

$$\mathcal{Z} = 2^N \left\{ [\cosh(J/kT)]^N + [\sinh(J/kT)]^N \right\}$$

Hint: Note that the partition function **is not** an expansion of a binomial raised to m-th power. However, you may want to express it in terms of a sum of the m-th powers of *two* binomials: $(x+1)^m + (x-1)^m$ (where x is a generic variable, and m is a generic integer power exponent).

(d) (5 pts.) Find the asymptotic form of the partition function in the large N limit (*Hint:* for $m \to \infty$, $\lim[\sinh(x)/\cosh(x)]^m = \dots$).

(e) (5 pts.) Find the temperature-dependent Helmholtz free energy F, and the heat capacity of the system in the large N limit.

S3.1

PROBLEM 3 – solution:

a) The "trick" with a ring is that when you make a full turn, you have to return to the original orientation of Spin #1. Hence, the # of "light" links (↑↓ or ↓↑) must be an even number.

The # of ↑↓ (or ↓↑) pairs may be therefore 0, 2, 4, 6, ... N−2, N.

Each "heavy" link contributes −J to the total energy; and each "light" link contributes +J. So, the total energy for m "light" links will be

$$E_{tot} = -J\cdot(N-m) + Jm = -J(N-2m) \quad \text{with } m = 0, 2, 4, \dots N.$$

The multiplicity of a state with a given m is

$2 \times \frac{N!}{m!(N-m)!}$ (2 is the degeneracy due to the "global flip").

b)
$$Z = \sum_{m=0,2,4..}^{N} \frac{2N!}{m!(N-m!)} e^{\frac{J(N-2m)}{kT}} = 2e^{\frac{JN}{kT}} \sum_{m=0,2,4..}^{N} \frac{N!}{m!(N-m)!} e^{-\frac{2Jm}{kT}}$$

$$= 2e^{\frac{JN}{kT}} \sum_{m=0,2,4..}^{N} \frac{N!}{m!(N-m)!} \left(e^{-\frac{2J}{kT}}\right)^m$$

This is not a binomial expansion (it would be if m = 0, 1, 2, 3, ..., N, but here odd m-s are missing).

S3.2

However, here one can use the following trick:

$$(1+x)^N = \quad = \sum_{m=0,1,2,3..}^{N} \frac{N!}{m!(N-m)!} x^m$$

$$(1-x)^N = [1+(-x)]^N = \sum_{m=0,1,2,3..}^{N} \frac{N!}{m!(N-m)!} (-1)^m x^m$$

Add up: $(1+x)^N + (1-x)^N = 2 \sum_{m=0,2,4...}^{N} \frac{N!}{m!(N-m)!} x^m$ All terms with odd m-s cancel out!

So, one can write Z as:

$$Z = 2e^{\frac{JN}{kT}} \cdot \frac{1}{2}\left[\left(1+e^{-\frac{2J}{kT}}\right)^N + \left(1-e^{-\frac{2J}{kT}}\right)^N\right]$$

$$= e^{\frac{JN}{kT}}\left\{\left[e^{-\frac{J}{kT}}\left(e^{\frac{J}{kT}}+e^{-\frac{J}{kT}}\right)\right]^N + \left[e^{-\frac{J}{kT}}\left(e^{\frac{J}{kT}}-e^{-\frac{J}{kT}}\right)\right]^N\right\}$$

$$= e^{\frac{JN}{kT}} \cdot e^{-\frac{JN}{kT}}\left\{\left[2\cosh\left(\frac{J}{kT}\right)\right]^N + \left[2\sinh\left(\frac{J}{kT}\right)\right]^N\right\}$$

$$= 2^N\left\{\left[\cosh\left(\frac{J}{kT}\right)\right]^N + \left[\sinh\left(\frac{J}{kT}\right)\right]^N\right\}$$

$$= 2^N\left[\cosh\left(\frac{J}{kT}\right)\right]^N \cdot \left\{1 + \left[\tanh\left(\frac{J}{kT}\right)\right]^N\right\}$$

S3.3

Note that $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is always < 1

And for any number $y < 1$, $\lim_{N\to\infty} y^N \longrightarrow 0$

Therefore, the asymptotic form of $Z(N\to\infty)$ is:

$$Z = 2^N \left[\cosh\left(\frac{J}{kT}\right)\right]^N$$

The Helmholtz free energy is $F = -kT \ln Z$, so that in the present case:

$$F = -kTN \ln\left[2\cosh\left(\frac{J}{kT}\right)\right]$$

To calculate heat capacity, one can use: $S = -\left(\dfrac{\partial F}{\partial T}\right)_x$, and then $C = T\left(\dfrac{\partial S}{\partial T}\right)_x$

(these are general formulae; and in the present case there are no other variables than T, so one can use ordinary derivatives:

$$S = -\frac{dF}{dT} = kN \ln\left[2\cosh\left(\frac{J}{kT}\right)\right] + kTN \frac{2\sinh\left(\frac{J}{kT}\right)}{2\cosh\left(\frac{J}{kT}\right)} \cdot \left(-\frac{J}{kT^2}\right)$$

$$= kN \ln\left[2\cosh\left(\frac{J}{kT}\right)\right] - \frac{JN}{T} \tanh\left(\frac{J}{kT}\right)$$

S3.4

and $C = T\frac{dS}{dT} = T\frac{d}{dT}\left\{kN\ln\left[2\cosh\left(\frac{J}{kT}\right)\right] - \frac{JN}{T}\tanh\left(\frac{J}{kT}\right)\right\}$

$$= T\left\{kN\cdot\frac{2\sinh\left(\frac{J}{kT}\right)}{2\cosh\left(\frac{J}{kT}\right)}\left(-\frac{J}{kT^2}\right) + \frac{JN}{T^2}\tanh\left(\frac{J}{kT}\right) - \frac{JN}{T}\frac{1}{\cosh^2\left(\frac{J}{kT}\right)}\left(\frac{J}{kT}\right)\right\}$$

$$= T\left\{\frac{-JN}{T^2}\tanh\left(\frac{J}{kT}\right) + \frac{JN}{T^2}\tanh\left(\frac{J}{kT}\right) + \frac{J^2N}{kT^3}\frac{1}{\cosh^2\left(\frac{J}{kT}\right)}\right\}$$

$$= \frac{NJ^2}{kT^2}\cdot\frac{1}{\cosh^2\left(\frac{J}{kT}\right)} = kN\left(\frac{J}{kT}\right)^2\frac{1}{\cosh^2\left(\frac{J}{kT}\right)}$$

In the above, we used: $\frac{d}{dx}\tanh(x) = \frac{d}{dx}\left[\frac{\sinh(x)}{\cosh(x)}\right] =$
$= \frac{\cosh(x)}{\cosh(x)} - \frac{\sinh(x)}{\cosh^2(x)} = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$

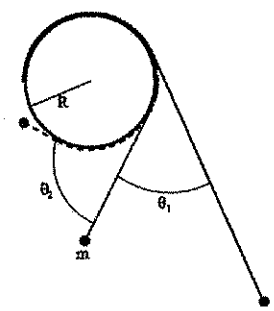
An alterative way would be to use: $F = U - TS$, so $U = F + TS = F - T\frac{dF}{dT}$ and then $C = \frac{dU}{dT}$. Using the result for the entropy S obtaind above, we get U:

$$U = -kTN\ln\left[2\cosh\left(\frac{J}{kT}\right)\right] + kTN\ln\left[2\cosh\left(\frac{J}{kT}\right)\right] - JN\tanh\left(\frac{J}{kT}\right)$$

$= -JN\tanh\left(\frac{J}{kT}\right)$, from which $C = \frac{dU}{dT} = \frac{NJ^2}{kT^2}\cdot\frac{1}{\cosh^2\left(\frac{J}{kT}\right)}$

— the same result as from the former method.

A pendulum consists of a mass m attached to an extension-less string of length l. The fixed end of the string is connected to a fixed vertical disk of radius R with $R < l/\pi$, as shown in the figure below. The motion of the pendulum occurs in the plane of the figure. The string is attached to the disk in such a way that for the oscillations of interest here the string is always tangent to the disk.



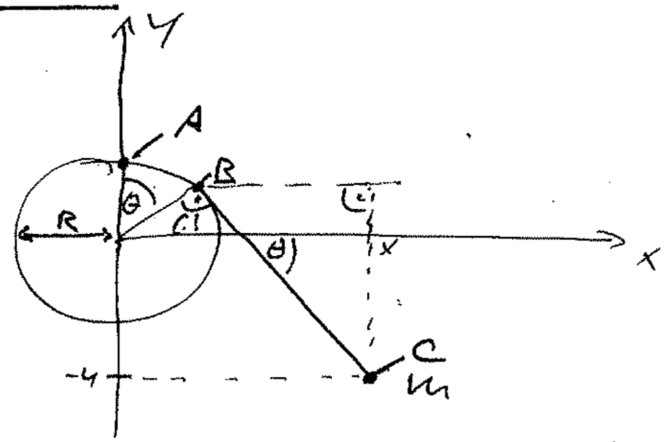
(a) Find the equation of motion for this pendulum.

(b) Calculate the period of oscillations for small oscillations.

(c) In the small oscillations regime find the line about which the angular motion extends equally in either direction (i.e. $\theta_1 = \theta_2$).

problem 4

a)



the actual location of the attachment of the string doesn't matter as long as it is far enough back ⇒ place at point A for convenience.

$\Rightarrow \overline{BC} = \ell - R\theta$

$x = \overline{BC}\cos\theta + R\sin\theta = (\ell - R\theta)\cos\theta + R\sin\theta$

$y = R\cos\theta - \overline{BC}\sin\theta = R\cos\theta - (\ell - R\theta)\sin\theta$

$\dot{x} = -R\dot{\theta}\cos\theta - (\ell - R\theta)\sin\theta \cdot \dot{\theta} + R\dot{\theta}\cos\theta$

$= (R\theta - \ell)\dot{\theta}\sin\theta$

$\dot{y} = -R\dot{\theta}\sin\theta - (\ell - R\theta)\dot{\theta}\cos\theta + R\dot{\theta}\sin\theta$

$= (R\theta - \ell)\dot{\theta}\cos\theta$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\ell - R\theta)^2\dot{\theta}^2$

$U = mgy = mg(R\cos\theta - (\ell - R\theta)\sin\theta)$

$$L = T - U$$
$$= \frac{1}{2} m (l - R\theta)^2 \dot{\theta}^2 - mg\left(R\cos\theta - (l - R\theta)\sin\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(m (l - R\theta)^2 \dot{\theta} \right)$$
$$= 2m(l - R\theta)(-R)\dot{\theta}^2 + m(l - R\theta)^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m(l - R\theta)(-R)\dot{\theta}^2 - mg\left[-R\sin\theta + R\sin\theta - (l - R\theta)\cos\theta\right.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Leftrightarrow -2Rm(l - R\theta)\dot{\theta}^2 + m(l - R\theta)^2\ddot{\theta} + mR(l - R\theta)\dot{\theta}^2 - mg(l - R\theta)\cos\theta = 0$$

$$\Leftrightarrow \boxed{(l - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0}$$

b) Define small angle $\epsilon = \theta - \theta_0$ for oscillat. centered around angle θ_0.

$\dot{\epsilon} = \dot{\theta}$, $\ddot{\epsilon} = \ddot{\theta}$, $\cos\theta = \cos(\epsilon + \theta_0)$

$(\dot{\epsilon})^2 \approx 0$ (small)^2 $\quad = \cos\epsilon\cos\theta_0 - \sin\epsilon\sin\theta_0$

$\theta = \theta_0 + \epsilon \approx \theta_0$ $\quad \approx \cos\theta_0 - \epsilon\sin\theta_0$

$$\Rightarrow (l - R\theta_0)\ddot{\epsilon} - g\cos\theta_0 + \epsilon g\sin\theta_0 = 0$$

$$\ddot{\epsilon} + \frac{g \sin\Theta_0}{l - R\Theta_0}\epsilon = \frac{g\cos\Theta_0}{l - R\Theta_0}$$

$$\ddot{\epsilon} + \omega^2 \epsilon = \text{RHS}$$

$$\omega = \sqrt{\frac{g\sin\Theta_0}{l - R\Theta_0}} \quad \text{for small oscillations}$$

c) by inspection of figure:
for $\Theta_0 = \frac{\pi}{2}$ pendulum reduces to simple pendulum for small oscillations, with $\Theta_1 = \Theta_2$.

explicit calculation:

$$\epsilon(t) = A\sin(\omega t + \delta) + \epsilon_0$$

with $\epsilon_0 = \dfrac{g\cos\Theta_0}{l - R\Theta_0}$

$$\epsilon_0 = 0 \quad \text{for } \Theta_0 = \frac{\pi}{2}$$

Consider a system consisting of two simple harmonic oscillators that are weakly coupled. The Hamiltonian of the combined system is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}x_1^2 + \frac{m\omega^2}{2}x_2^2 + \frac{m\gamma^2}{2}(x_1 - x_2)^2. \tag{42}$$

(a) Write down the eigenvalues and eigenstates of this Hamiltonian (Equation 43) when $\gamma = 0$. Please write your eigenstates in terms of the single-particle simple harmonic oscillator solutions.

(b) Use first-order perturbation theory to solve for corrections to the energy of the ground state and first excited state when $\gamma \ll \omega$.

(c) Solve for the exact eigenvalues and eigenstates of this Hamiltonian (Equation 43) when $\gamma \neq 0$.

(d) Check that in the limit of small coupling your exact solution approaches your approximate answer from perturbation theory.

(e) How would the set of energy eigenstates change if the two particles involved were identical spin-$\frac{1}{2}$ fermions? What will be the spin degeneracy of each state? Note that for this question you need to answer for *all* eigenstates, not just the first few.

Consider a system consisting of two simple harmonic oscillators that are weakly coupled. The Hamiltonian of the combined system is

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(a) Write down the eigenvalues and eigenstates of this Hamiltonian (Equation 43) when $\gamma = 0$. Please write your eigenstates in terms of the single-particle simple harmonic oscillator solutions.

Solution:
The hamiltonian (without the γ term) separates, so we have products of single-particle eigenstates, and the energy is the sum of their energies.

$$\Psi_{n_1 n_2}(x_1, x_2) = \varphi_{n_1}(x_1)\varphi_{n_2}(x_2) \tag{44}$$
$$\text{or} \tag{45}$$
$$|n_1 n_2\rangle = |n_1\rangle_1 |n_2\rangle_2 \tag{46}$$
$$\text{and} \tag{47}$$
$$E_{n_1 n_2} = \hbar\omega\,(n_1 + n_2 + 1) \tag{48}$$

(b) Use first-order perturbation theory to solve for corrections to the energy of the ground state and first excited state when $\gamma \ll \omega$.

Solution:
Unfortunately, there are a huge number of degeneracies, with the only non-degenerate state being the ground state. So let's handle the ground state first, and then tackle the first excited state later.

$$\Delta E_{00} = \left\langle 00 \left| \frac{m\gamma^2}{2}(x_1 - x_2)^2 \right| 00 \right\rangle \tag{49}$$
$$= \frac{m\gamma^2}{2} \langle 0 | \langle 0 | (x_1 - x_2)^2 | 0 \rangle_1 | 0 \rangle_2 \tag{50}$$
$$= \frac{m\gamma^2}{2} \langle 0 | \langle 0 | x_1^2 - 2x_1x_2 + -x_2^2 | 0 \rangle_1 | 0 \rangle_2 \tag{51}$$
$$= \frac{m\gamma^2}{2} \left(\langle 0 | x_1^2 | 0 \rangle_1 + \langle 0 | x_2^2 | 0 \rangle_2 - 2 \langle 0 | x_1 | 0 \rangle_1 \langle 0 | x_2 | 0 \rangle_2 \right) \tag{52}$$
$$= m\gamma^2 \langle 0 | x_1^2 | 0 \rangle_1 \tag{53}$$

At this stage, we just need to compute a few expectation values. For later use, we'll compute more general matrix elements than we need for the ground state. We can do this most easily by converting x_1 and x_2 into raising and lowering operators.

$$\langle n|x|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a+a^\dagger|n'\rangle \tag{54}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\delta_{n'(n+1)} + \sqrt{n'+1}\delta_{(n'+1)n}\right) \tag{55}$$

This tells us that an x connects neighboring states, and its expectation value ($n = n'$) is zero, which is also clear from symmetry considerations, since the square of the wave function is even, and x is odd.

$$\langle n|x^2|n'\rangle = \frac{\hbar}{2m\omega}\langle n|(a+a^\dagger)^2|n'\rangle \tag{56}$$

$$= \frac{\hbar}{2m\omega}\left\langle n\left|a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}\right|n'\right\rangle \tag{57}$$

$$= \frac{\hbar}{2m\omega}\left(\delta_{nn'}(2n+1) + \delta_{n(n'+2)}\sqrt{(n'+1)(n'+2)} + \delta_{(n+2)n'}\sqrt{n'(n'-1)}\right) \tag{58}$$

So now we've got all the matrix elements we'll need to handle degenerate perturbation theory... but first, let's finish the ground state.

$$\Delta E_{00} = \frac{m\gamma^2}{2}\left(\langle 0|x_1^2|0\rangle_1 + \langle 0|x_2^2|0\rangle_2 - 2\langle 0|x_1|0\rangle_1\langle 0|x_2|0\rangle_2\right) \tag{59}$$

$$= m\gamma^2\frac{\hbar}{2m\omega}(2\cdot 0+1) \tag{60}$$

$$= \frac{1}{2}\frac{\hbar\gamma^2}{\omega} \tag{61}$$

Alas, now we must bite the bullet and use degenerate perturbation theory to handle the first excited state. Degenerate perturbation theory requires us to diagonalize the perturbation hamiltonian in the subspace of degenerate states, so we'll begin by enumerating that subspace, which is defined by $n_1 + n_2 = n_{tot}$, so the total degeneracy is $n_{tot}+1$. Actually, the first excited state is pretty simple, as there are only two states involved, $|01\rangle$ and $|10\rangle$.

$$\left\langle 01\left|\frac{m\gamma^2}{2}(x_1-x_2)^2\right|01\right\rangle = \frac{m\gamma^2}{2}\langle 01|x_1^2 + x_2^2 - 2x_1x_2|01\rangle \tag{62}$$

$$= \frac{m\gamma^2}{2}\left(\langle 0|x_1^2|0\rangle_1 + \langle 1|x_2^2|1\rangle_2 - 2\langle 0|x_1|0\rangle_1\langle 1|x_2|1\rangle_2\right) \tag{63}$$

$$= \frac{\hbar\gamma^2}{4\omega}(1+3+0) \tag{64}$$

$$= \frac{\hbar\gamma^2}{\omega} \tag{65}$$

And for the off-diagonal matrix elements, we find

$$\left\langle 01 \left| \frac{m\gamma^2}{2}(x_1 - x_2)^2 \right| 10 \right\rangle = \frac{m\gamma^2}{2} \langle 01 | x_1^2 + x_2^2 - 2x_1x_2 | 10 \rangle \quad (66)$$

$$= \frac{m\gamma^2}{2} \left(\langle 0 | x_1^2 | 1 \rangle_1 + \langle 1 | x_2^2 | 0 \rangle_2 - 2 \langle 0 | x_1 | 1 \rangle_1 \langle 1 | x_2 | 0 \rangle_2 \right) \quad (67)$$

$$= \frac{\hbar\gamma^2}{4\omega}(0 + 0 - 2) \quad (68)$$

$$= -\frac{1}{2}\frac{\hbar\gamma^2}{\omega} \quad (69)$$

We end up with a simple matrix that looks like

$$\Delta H_{ij} = \frac{\hbar\gamma^2}{\omega} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad (70)$$

Using symmetry between the two particles, it's simple to diagonalize this matrix, and we find its two eigenstates

$$\Delta H \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) = \frac{\hbar\gamma^2}{\omega}\left(1 \mp \frac{1}{2}\right)\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \quad (71)$$

which tells us that in the end

$$\Delta E_{00} = \frac{1}{2}\frac{\hbar\gamma^2}{\omega} \quad (72)$$

$$\Delta E_{1+} = \frac{3}{2}\frac{\hbar\gamma^2}{\omega} \quad (73)$$

$$\Delta E_{1-} = \frac{1}{2}\frac{\hbar\gamma^2}{\omega} \quad (74)$$

$$(75)$$

(c) Solve for the exact eigenvalues and eigenstates of this Hamiltonian (Equation 43) when $\gamma \neq 0$.

Solution:

To solve for the exact solution, we need to find the normal modes of the coupled system. There are a variety of ways to solve this problem, which is commonly seen in classical mechanics. In this case, symmetry makes things simple. Because the x_1 and x_2 coordinates are completely symmetric, we can immediately see that the normal mode coordinates must be $q_+ \equiv \frac{x_1+x_2}{\sqrt{2}}$ and $q_- \equiv \frac{x_1-x_2}{\sqrt{2}}$, and we just need to perform a change of coordinates into these variables. Here q_+ is proportional to the center-of-mass coordinate, and q_- is a relative coordinate, although not the usual one. You could choose any set of coordinates that are proportional to q_+ and q_-, so long as you do the transformation properly. I just chose a length-preserving transformation, so that the masses would come out the same in the new coordinate system, so I wouldn't have to deal with total and reduced masses.

$$x_1 = \frac{q_+ + q_-}{\sqrt{2}} \tag{76}$$

$$x_2 = \frac{q_+ - q_-}{\sqrt{2}} \tag{77}$$

$$x_1^2 = \frac{1}{2}(q_+^2 + q_-^2 + 2q_+q_-) \tag{78}$$

$$x_2^2 = \frac{1}{2}(q_+^2 + q_-^2 - 2q_+q_-) \tag{79}$$

$$x_1^2 + x_2^2 = q_+^2 + q_-^2 \tag{80}$$

$$p_1 = i\hbar\frac{\partial}{\partial x_1} \tag{81}$$

$$= i\hbar\left(\frac{\partial q_+}{\partial x_1}\frac{\partial}{\partial q_+} + \frac{\partial q_-}{\partial x_1}\frac{\partial}{\partial q_-}\right) \tag{82}$$

$$= i\hbar\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial q_+} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial q_-}\right) \tag{83}$$

$$= \frac{1}{\sqrt{2}}(p_+ + p_-) \tag{84}$$

$$p_2 = \frac{1}{\sqrt{2}}(p_+ - p_-) \tag{85}$$

$$p_1^2 + p_2^2 = p_+^2 + p_-^2 \tag{86}$$

$$T = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} \tag{87}$$

$$= \frac{p_+^2}{2m} + \frac{p_-^2}{2m} \tag{88}$$

$$V = \frac{1}{2}m\omega^2 x_1^2 + \frac{1}{2}m\omega^2 x_2^2 + \frac{1}{2}m\gamma^2(x_1 - x_2)^2 \tag{89}$$

$$= \frac{1}{2}m\omega^2 q_+^2 + \frac{1}{2}m\left(\omega^2 + 2\gamma^2\right)q_-^2 \tag{90}$$

We can check that we did the change of variables properly, since the energies should be unchanged if $\gamma = 0$. We can also see that we chose a good coordinate transform, since in the new coordinate system the hamiltonian is once again uncoupled simple harmonic oscillators. So the energy eigenstates are simply:

$$E_{n_+n_-} = \hbar\omega\left(n_+ + \frac{1}{2}\right) + \hbar\sqrt{\omega^2 + 2\gamma^2}\left(n_- + \frac{1}{2}\right) \tag{91}$$

(d) Check that in the limit of small coupling your exact solution approaches your approximate answer from perturbation theory.

Solution:

Let's start with the ground state energy. The exact answer is:

$$\Delta E_{00} = \left(\frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\sqrt{\omega^2 + 2\gamma^2}\right) - \hbar\omega \tag{92}$$

$$= \frac{1}{2}\hbar\left(\sqrt{\omega^2 + 2\gamma^2} - \omega\right) \tag{93}$$

$$= \frac{1}{2}\hbar\omega\left(\sqrt{1 + 2\frac{\gamma^2}{\omega^2}} - 1\right) \tag{94}$$

$$\approx \frac{1}{2}\hbar\omega\left(1 + \frac{\gamma^2}{\omega^2} - 1\right) \tag{95}$$

$$= \frac{1}{2}\hbar\frac{\gamma^2}{\omega} \tag{96}$$

This is indeed the same answer that we found using perturbation theory, as indeed it must be, since first-order perturbation theory by construction gives the correct energy to first order.

Now for the first excited state energy. The exact answer is:

$$\Delta E_{1+} = \left(\frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\sqrt{\omega^2 + 2\gamma^2}\right) - 2\hbar\omega \tag{97}$$

$$= \frac{3}{2}\hbar\left(\sqrt{\omega^2 + 2\gamma^2} - \omega\right) \tag{98}$$

$$= \frac{3}{2}\hbar\frac{\gamma^2}{\omega} \tag{99}$$

Once again, we see that we did perturbation theory correctly.

$$\Delta E_{1+} = \left(\frac{3}{2}\hbar\omega + \frac{1}{2}\hbar\sqrt{\omega^2 + 2\gamma^2}\right) - 2\hbar\omega \tag{100}$$

$$= \frac{1}{2}\hbar\left(\sqrt{\omega^2 + 2\gamma^2} - \omega\right) \tag{101}$$

$$= \frac{1}{2}\hbar\frac{\gamma^2}{\omega} \tag{102}$$

And one last success story. It can be a useful trick to solve the same problem two ways, since it lets us see whether we made a mistake. I think it also can help us to understand perturbation theory just a bit better.

(e) How would the set of energy eigenstates change if the two particles involved were identical spin-$\frac{1}{2}$ fermions? What will be the spin degeneracy of each state? Note that for this question you need to answer for *all* eigenstates, not just the first few.

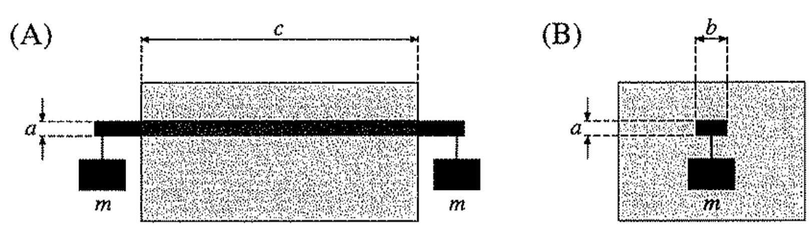
Solution:

If these were identical spin-$\frac{1}{2}$ fermions, they'd have to have antisymmetric wavefunctions. They would also have spin degrees of freedom. Since the spin degrees of freedom don't couple (in this case) to the spatial degrees of freedom, we can separate out the spin wavefunction from the spatial one. The spin one will either be symmetric or antisymmetric (triplet or singlet), and the spatial states must then be the opposite, so the total wavefunction is antisymmetric.

Figuring out which states are symmetric (or antisymmetric) is slightly tricky when we do our change of coordinates. We mean antisymmetric (or symmetric) under exchange of the two particles, which means when we switch x_1 and x_2. When we do this, q_+ doesn't change, but q_- flips sign. So when the wavefunction is antisymmetric under inversion of q_-, it is antisymmetric under particle exchange. Which means that when n_- is odd, we have an antisymmetric state. So each of our spatial states will be a valid wavefunction, but the states with n_- odd will be triplet states (triply-degenerate in spin), while the others will be singlets (and thus having no spin degeneracy).

A steel bar of rectangular cross section (height a and width b) is placed on a block of ice of width c, with the bar ends extending a trifle as shown in the figure below. A weight of mass m is hung from each end of the bar. The entire system is at 0° C. As a result of the pressure exerted by the bar, the ice melts beneath the bar and re-freezes above the bar. Heat is therefore liberated above the bar, conducted through the metal, and then absorbed by the ice below the bar (we assume that this is the most important way in which heat reaches the ice immediately beneath the bar in order to melt it). Based on the Claussius-Clapeyrone Equation, $dp/dT = l/(T\Delta v)$, Find an approximate expression for the speed with which the bar thus sinks through the ice. In addition to the data given above (a, b, m, and the ice temperature T), you can use the following parameters:

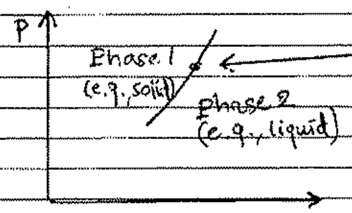
- l, the latent heat of fusion per unit mass of ice at 0° C;
- ρ_i and ρ_w, the densities of the ice and water, respectively, at 0° C;
- g, the acceleration due to gravity;
- κ_s, the thermal conductivity coefficient of of steel (if in a material there is a temperature gradient dT/dz along a z axis, the heat power P_{heat} flowing through a unit cross section area perpendicular to the z axis is related to this gradient via the κ coefficient: $P_{\text{heat}} = \kappa(dT/dz)$).



S6.1

PROBLEM 6 – solution:

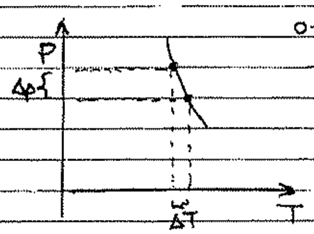
The Claussius – Clapeyron Equation gives the slope of the boundary between two phases in a T–P phase diagram:



$$\frac{dP}{dT} = \frac{\Delta S_{12}}{\Delta V_{12}} = \frac{L_{12}}{T\,\Delta V_{12}}$$

where $\Delta S_{12} = S_2 - S_1$ is the change of entropy (per mole, or per unit mass) associated with a transformation from Phase 1 to Phase 2 at temperature T, and $\Delta V_{12} = V_2 - V_1$ is the volume change (per mole, or unit mass, respectively) associated with the transformation. The heat absorbed by the substance $T \cdot \Delta S_{12} = L_{12}$ is the "latent heat" of the transition.

For ice → water transition ΔV_{12} is known to be negative (an anomalous effect), so that the slope of the phase boundary is negative.



Therefore, an increase in pressure by Δp leads to a decrease of the melting temperature by $\Delta T \cong \frac{T \cdot \Delta V_{12}}{L_{12}} \Delta p$

S6.2

The pressure produced by the bar therefore melts the ice underneath. The water travels up and re-freezes above the top surface of the bar. When re-freezing, the water gives away latent heat, which travels down due to the lower temperature of the bottom rod's surface.

The temperature difference between the top and the bottom surface can be found from the Claussius-Clapeyron Equation: the pressure exerted by the bar is $2mg/b \cdot c$; the specific volumes of the two phases are $v_{ice} = 1/\rho_{ice}$, and $v_{water} = 1/\rho_{water}$; therefore:

$$\Delta T = \frac{T(1/\rho_i - 1/\rho_w) \cdot 2mg}{L_{12} \cdot b \cdot c}$$

The temperature gradient in the bar is $\Delta T/a$, so that the heat power flowing downward is $P = \kappa \cdot \left(\frac{\Delta T}{a}\right) \cdot b \cdot c$.

The mass of ice melted by this power in a time unit is $\Delta M = P/L_{12}$. This mass corresponds to a volume of $\Delta M/\rho_i$ melted per time unit, or

S6.3

an ice "layer" of volume $b \cdot c \cdot \Delta z$, such that $\rho_i b \cdot c \cdot \Delta z = \mathcal{P}/L_{12}$. The thickness of this layer, Δz, is the distance traveled downward by the bar per time unit — in other words, it is the speed with which the bar "sinks" in the ice block.

Taking everything together, we find that the "sinking speed" is:

$$v_{sinking} = \Delta z = \frac{\mathcal{P}}{\rho_i b \cdot c \cdot L_{12}} = \frac{K \cdot b \cdot c \cdot \Delta T}{\rho_i b \cdot c \cdot L_{12} \cdot a} =$$

$$= \frac{K \cdot b \cdot c \cdot 2mgT\,(1/\rho_i - 1/\rho_w)}{a \cdot b \cdot c \cdot L_{12}^2 \cdot b \cdot c \cdot \rho_i}$$

$$= \frac{2mgTK}{abc\,L_{12}^2\,\rho_i}\left[\frac{1}{\rho_i} - \frac{1}{\rho_w}\right].$$

We ignored the weight of the bar, which also contributes to the pressure — so, to be more accurate, $2m$ in the above should be replaced by $(2m + abc \cdot \rho_{bar})$, where ρ_{bar} is the density of the bar material.

Two stars of mass m_1,m_2 are in a circular orbit (with separation $r = r_0$) when star 1 begins losing mass spherically symmetrically. Assume that the velocity of the ejected gas is much greater than the orbital velocities of the stars and that there is no interaction of this gas with the companion star 2.

Part 1: Consider the case in which star 1 is losing mass at a slow, constant rate $dm_1/dt = -\dot{m}$, such that the mass-loss timescale $\tau \sim -m_1/(dm_1/dt)$ is much greater than the orbital period (stellar wind). In this part only, take $m_1 >> m_2$.

(a) After star 1 begins to loose mass, will the distance between the 2 stars increase, decrease, or stay the same?

(b) Obtain a formula for the change over time of the distance between the 2 stars

$$dr/dt = f(r, \dot{m}, \dots),$$

justifying any assumptions made in your calculation.

Part 2: Now consider the case in which the mass-loss timescale τ is much less than the orbital period (supernova). The total mass lost by star 1 is Δm and the mass ejection can be assumed to happen instantaneously. The object of this part is to find the conditions under which the final orbit will still be bound.

(c) What are the kinetic and potential energies of the binary stars in the center of mass reference system. Express both in terms of the reduced mass $\mu_0 = m_1 m_2/(m_1 + m_2)$ and the total mass $M = m_1 + m_2$ and assume a velocity v_0.

(d) A mass m moves in a circular orbit (centered about the origin) in the field of an attractive central force with potential energy $U = kr^n$. Prove the virial theorem $T = nU/2$.

(e) Use the virial theorem to determine the velocity v_0 of the binary stars in the center of mass reference system.

(f) What are the kinetic and potential energies of the binary stars in the center of mass reference system after the instantaneous mass loss of star 1? Justify your result.

(g) Obtain the condition (involving only Δm and the initial masses m_1 and m_2) under which the final orbital will still be bound.

problem 7

part 1 $m_1 \gg m_2$

a) mass loss → distance increases

b) mass loss rate small → orbits will slowly spiral out and remain approximately circular.

$m_1 \gg m_2$: star 2 orbits start with angular momentum $l_2 = m_2 r^2 \omega$ = constant since ~~spherically symmetric~~ mass loss exerts no torque.

radial balance: $\frac{G m_1 m_2}{r^2} \approx m_2 r \omega^2 \approx \frac{l_2^2}{m_2 r^3}$

$$\Rightarrow r \simeq \frac{l_2^2}{G m_2^2 m_1}$$

$$\frac{dr}{dt} = \frac{l_2^2}{G m_2^2}\left(\frac{-1}{m_1^2}\right)\frac{dm_1}{dt} \quad \text{with } m_1 = \frac{l_2^2 r^2}{G r m_2^2}$$

$$\boxed{\frac{dr}{dt} = \frac{G \dot{m}}{l_2^2/m_2^2} \cdot r^2}$$

part 2

a) 2 body problem reduces to effective one body problem with

$$E_{kin} = \frac{1}{2}\mu_0 V_0^2 \quad E_{pot} = -\frac{G \mu_0 M_0}{r_0} \quad \text{where}$$

$$\mu_0 = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M_0} \qquad M_0 = m_1 + m_2$$

d) circular orbits, $U = kr^n$

radial balance: $-\frac{mv^2}{r} = -\frac{dU}{dr} = -nkr^{n-1}$

$$E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}nkr^n = \frac{n}{2}E_{pot} \quad \square$$

e) $n = -1 \Rightarrow E_{pot} = -\frac{G\mu_0 M_0}{r_0} = -\mu_0 V_0^2$

$$\Rightarrow V_0^2 = \frac{GM_0}{r_0}, \quad V_0 = \sqrt{\frac{GM_0}{r_0}}$$

f) $M_F = m_1 - \Delta m + m_2$

$$\mu_F = \frac{(m_1 - \Delta m)m_2}{M_F}$$

$E^F_{kin} = \frac{1}{2}\mu_F V_0^2$ — no force on star 1, r_0, V_0 remain unchanged

$$E^F_{pot} = -\frac{G\mu_F M_F}{r_0}$$

g) $E_F = \frac{1}{2}\mu_F V_0^2 - \frac{G\mu_F M_F}{r_0} = \frac{1}{2}\frac{G\mu_F M_0}{r_0} - \frac{G\mu_F M_F}{r_0}$

$$= \frac{G\mu_F}{r_0}\left(\frac{M_0}{2} - M_F\right) \overset{!}{<} 0 \quad \leftarrow \text{to remain in orbit}$$

$$\Rightarrow M_0 < 2M_F$$

$$\Leftrightarrow m_1 + m_2 < 2(m_1 - \Delta m + m_2)$$

$$\Rightarrow \boxed{2\Delta m < m_1 + m_2}$$

Problem #2

1) A hemisphere-shaped grounding electrode is buried in the ground, so that its flat is level with the surface. Lightning strikes, which send current I through the electrode. A man happens to be walking towards the electrode. The length of the man's step is l, and the distance between the man's nearer foot and the electrode is r_0.

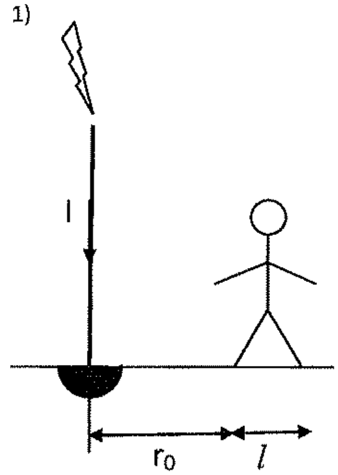
   (a) Find the current density in the ground with conductivity σ as a function of distance r from the center of the electrode;
   (b) Find the electric field as a function of r;
   (c) Find the voltage experienced by the man per step;
   (d) Calculate the voltage per step for $\sigma = 0.01$ (Ohm m)$^{-1}$, $I = 100$ kA, $r_0 = 2$ m, and $l = 1$ m.

2) Now let's consider a similar problem but in a slightly different geometry. An overhead conductor which broke and landed on the ground is carrying a current I. The length of the conductor on the ground is L.

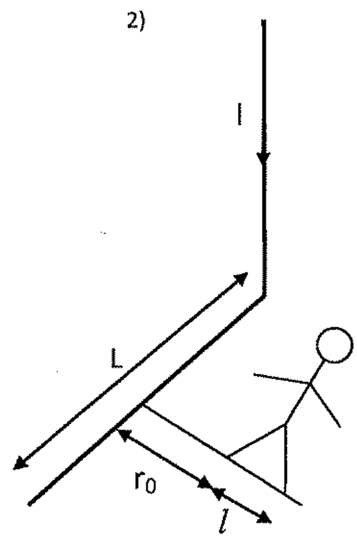
   (a) Find the voltage per step experienced by a man walking up to the conductor at a right angle with respect to the wire, close to the middle of the wire (see figure);
   (b) Calculate the voltage per step using $\sigma = 0.01$ (Ohm m)$^{-1}$, $I = 1$ kA, $r_0 = 2$ m, $l = 1$ m, and L = 500 m.

   In both parts of the problem, assume the conductivity of the ground is much lower than that of the grounding electrode or conductor.

1)

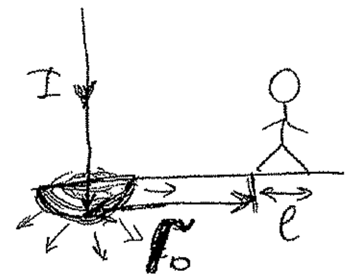


2)



E & M Problem #8

(1)



1) hemisphere-like grounding electrode

(a) $j_r = \dfrac{I}{2\pi r^2}$ ← the current from the grounding electrode flows out uniformly in all directions (hemisphere)

↑ distance from the wire

(b) electric field $E_r = \dfrac{j_r}{\sigma}$ ← conductivity of the ground

$$E_r = \frac{I}{2\pi r^2 \sigma}$$

(c) Voltage per step is then $V_{step} = \int_{r_0}^{r_0+\ell} E_r \, dr =$

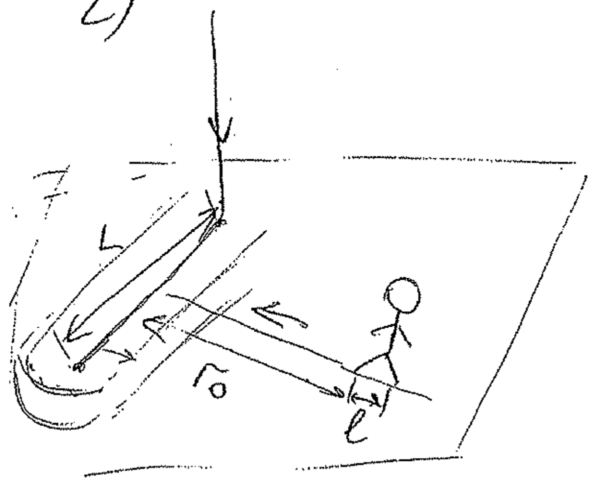
$$= \frac{I}{2\pi\sigma} \int_{r_0}^{r_0+\ell} \frac{dr}{r^2} = \frac{I}{2\pi\sigma}\left(\frac{1}{r_0} - \frac{1}{r_0+\ell}\right) = \frac{I\ell}{2\pi\sigma r_0 (r_0+\ell)}$$

If $I = 10^5 A$, $\sigma = 10^{-2} \frac{1}{\text{Ohm}.m}$, $r_0 = 2m$, $\ell = 1m \Rightarrow$

$$V_{step} = \frac{10^5}{0.12\pi} \approx 2.65 \cdot 10^5 \, V = 265 \, kV$$

(2)

2)



(a) $j_r = \frac{I}{\pi r L}$ ← half-cylinder

(b) $E_r = \frac{I}{\pi r L \sigma}$

(c) $V_{step} = \int_{r_0}^{r_0+\ell} \frac{I}{\pi r L \sigma} dr =$

$$= \frac{I}{\pi L \sigma} \ln \frac{r_0+\ell}{r_0}$$

If $I = 10^3 A$, $\sigma = 10^{-2} \frac{1}{\text{Ohm}\cdot\text{m}}$, $r_0 = 2m$, $\ell = 1m$, $L = 500\,m$ ⇒

$$V_{step} = \frac{10^3}{5\pi} \ln \frac{3}{2} \simeq 10^3\, 0{,}026\ V = 26\ V$$

Harmonic oscillator: $[a, a^\dagger] = 1$

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}}$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}}$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle$$

4 Electromagnetism

Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Magnetic dipole field:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}$$

Energy density: $U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
Poynting vector: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

General solutions of Laplace's equation

in cylindrical coordinates (independent of z):

$$\Phi(\rho,\phi) = a_o \log(\rho) + \sum_{n=1}^{\infty}\left(\frac{a_n}{\rho^n} + b_n\rho^n\right)(c_n\cos n\phi + d_n \sin n\phi)$$

in spherical coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(with azimuthal symmetry)

5 Useful math formulas

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$
$$\int_{-\infty}^{\infty} e^{ixy}dy = 2\pi\delta(x)$$
$$\int_0^{\infty} x^n e^{-x}dx = n!\,,\ \text{integer } n$$

$$(1+x)^n = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^k$$
$$\log(n!) \approx \frac{1}{2}\log(2\pi n) + n\log(n) - n$$
$$\sin(x\pm y) = \sin x\cos y \pm \cos x \sin y$$
$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\frac{1}{|\mathbf{x}-r'\hat{\mathbf{z}}|} = \sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \qquad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

Legendre polynomials:

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l}{dx^m}$$

Spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\phi}$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

1 Physical constants

fine structure constant : $\alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c} \approx \dfrac{1}{137}$

Rydberg energy : $E_o = \dfrac{m_e e^4}{2\hbar^2 (4\pi\epsilon_o)^2} = \dfrac{m_e c^2 \alpha^2}{2}$

Bohr magneton : $\mu_B = \dfrac{e\hbar}{2m_e}$

Bohr radius : $a_o = \dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$

2 Vector calculus relationships

Triple products:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Product rules:

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\
&\quad + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \\
\nabla\cdot(\phi\mathbf{A}) &= \phi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\phi \\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) + \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + \\
&\quad + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}
\end{aligned}$$

Second derivatives:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

Green's theorem:

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)dV = \oint_S (\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S}$$

Spherical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}} \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} \\
&\quad + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} \\
&\quad + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{z}} \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

3 Quantum mechanics

Raising and lowering operators for ang. momentum:

$$J_\pm = J_x \pm iJ_y$$
$$J_\pm\left|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\left|j,m\pm1\right\rangle$$

Perturbation theory for nondegenerate states:

$$E_n \approx E_n^o + \langle n|V|n\rangle + \sum_{m\neq n}\frac{\left|\langle n|V|m\rangle\right|^2}{E_n - E_m} + \cdots$$