OSU Physics Department
Comprehensive Examination #111

Monday, September 26 and Tuesday, September 27, 2011

Fall 2011 Comprehensive Examination

PART 1, Monday, September 26, 9:00am

General Instructions

This Fall 2011 Comprehensive Examination consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 26, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 27, at 9:00 am and 1:00 pm, respectively. Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit—especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed except when a numerical answer is required—calculators will then be provided by the person proctoring the exam. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work, separated by at least one empty page from your solutions. "Scratch" work will not be graded.

An electron under the influence of a uniform magnetic field **B** (directed along y-axis) has its spin initially (at $t = 0$) pointing in the positive x-direction. That is, it is in an eigenstate of S_x with eigenvalue $\hbar/2$. We need to calculate the probability of finding the electron with its spin pointing in the positive z-direction. (Ignore all Hamiltonian terms apart from the interaction of the magnetic dipole moment due to spin and the magnetic field, $H = -\boldsymbol{\mu}\cdot\mathbf{B} = \omega S_y$.)

For this:

(a) Present the initial state of the electron in terms of the eigenstates of the Hamiltonian H
(b) Write down time evolution of the state for an arbitrary time $t > 0$
(c) Find the probability of the electron to be in the state with spin in the positive z-direction at time t
(d) At what time(s) is the probability of finding the electron in the state with spin in the positive z-direction equal to 1?

(1)

QM Problem #1

(a) $|\Psi(0)\rangle = |S_x = \frac{\hbar}{2}\rangle$

$$H = -\vec{\mu}\cdot\vec{B} = \omega S_y$$

$\uparrow$ $\vec{B} = \hat{y}B$

Need to present $|\Psi(0)\rangle$ in terms of S_y eigenstates

S_z eigenstates

$$|S_x = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

$$|S_y = \pm\tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm i|\downarrow\rangle\right) \Rightarrow$$

$$|S_x = \tfrac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{\pi i}{4}}|S_y = \tfrac{\hbar}{2}\rangle + e^{\frac{\pi i}{4}}|S_y = -\tfrac{\hbar}{2}\rangle\right)$$

$\parallel$

$|\Psi(0)\rangle$

(b) $|\Psi(t)\rangle = e^{-\frac{i\omega t S_y}{\hbar}}|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{\pi i}{4} - i\frac{\omega t}{2}}|S_y = \tfrac{\hbar}{2}\rangle\right.$

$\left. + e^{\frac{\pi i}{4}}e^{\frac{i\omega t}{2}}|S_y = -\tfrac{\hbar}{2}\rangle\right) = \cos\left(\frac{\pi}{4} + \frac{\omega t}{2}\right)|\uparrow\rangle +$

(2)

$+ \sin\left(\frac{\pi}{4} + \frac{\omega t}{2}\right) |\downarrow\rangle$

(c) The probability of finding the electron in the state $|\uparrow\rangle$ is

$$\mathcal{P}_\uparrow(t) = \cos^2\left(\frac{\pi}{4} + \frac{\omega t}{2}\right)$$

(d) $\mathcal{P}_\uparrow = 1$ at times $t = \frac{3\pi}{2\omega}, \ldots \frac{\pi}{2\omega}(4n-1)$

A pair of circular metal plates of radius R are separated by a distance $h \ll R$ in vacuum. If the plates are connected to a DC source of voltage V, there is a uniform electric field

$$E = \frac{V}{h} \tag{1}$$

between the plates (neglecting the edge effects).

However, if this system is connected to an AC source with output voltage $V(t) = V_{\mathrm{ampl}} e^{i\omega t}$, by inserting this voltage to the "static" Eq. (1) one obtains a zeroth-order solution for the time-dependent electric field E_0:

$$E_0 = E_0(t) = \frac{V_{\mathrm{ampl}}}{h} e^{i\omega t} \tag{2}$$

which is exactly correct **only** on the axis of the system.

(a) Using qualitative reasoning, explain why Eq. (2) is not the exact solution for points other than those lying on the system axis;

Then, use the iterative method to calculate corrections to $E_0(t)$ at an arbitrary radius r from the center, to the fourth order in r – by performing the three following tasks:

(b) Write Maxwell's Equations for the electromagnetic field at all points between the plates *in integral form* (again, neglecting the edge effects);

(c) By choosing appropriate integration paths, use the above equations:

- For calculating the magnetic induction at an arbitrary radius r,
- And then for calculating the correction to the electric field $E_0(t)$ at r, and for showing that this correction is of the *second order* in r.

(d) Use an analogous procedure to calculate the B-field contributed by this second-order correction to the electric field, and then for calculating the resulting next correction term to $E_0(t)$, and for showing that this is a correction of *fourth order* in r.

The items (a) - (d) are priced 5 pts. each.

A pair of circular metal plates of radius R are separated by a distance $h \ll R$ in vacuum. If the plates are connected to a DC source of voltage V, there is a uniform electric field

$$E = \frac{V}{h} \tag{3}$$

between the plates (neglecting the edge effects).

However, if this system is connected to an AC source with output voltage $V(t) = V_{\mathrm{ampl}}e^{i\omega t}$, by inserting this voltage to the "static" Eq. (1) one obtains a zeroth-order solution for the time-dependent electric field E_0:

$$E_0 = E_0(t) = \frac{V_{\mathrm{ampl}}}{h}e^{i\omega t} \tag{4}$$

which is exactly correct **only** on the axis of the system.

(a) Using qualitative reasoning, explain why Eq. (2) is not the exact solution for points other than those lying on the system axis;

Solution:
When there is an AC input voltage, the electric field, changing in time, induces a magnetic field perpendicular to the E-field lines, and, at every point, it's tangent to a circular loop whos center lies on the system symmetry axis (see Fig. 1; the circular shape and the position of the loop center are dictated by the system symmetry). For the points lying on the axis the loop is reduced to a point, so the E-flux accross is zero and no magnetic fiels is induced at the axis. THe magnetic field, in turn, induces an electric field; an appropropriate loop shape, again dictated by the symmetry, for calculating the flux of the magnetic field, and the electric field induced is shown in Fig. 2. For poits on the symmetry axis there is zero flux because the loop area is reduced to zero, and consequently, no "extra" E-field. The E-field at all other points is the correction to the spatially uniform E-field resulting from the input voltage. By rerpeating the procedure for the induced E-field, one can obtain a higher-order corection; from that correction, yet another correction, etc. But in the current problem the task is to find only the first two correction tems to E.

Then, use the iterative method to calculate corrections to $E_0(t)$ at an arbitrary radius r from the center, to the fourth order in r – by performing the three following tasks:

(b) Write Maxwell's Equations for the electromagnetic field at all points between the plates *in integral form* (again, neglecting the edge effects);

Solution:
Maxwell's Equations in vacuum in differential form (it was not necessary to write them down in the exam solution):

$$\nabla \vec{D} = 0$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = -\frac{\partial \vec{D}}{\partial t}$$

The last two can be applied in integral form:

$$\oint \vec{E} d\vec{\ell} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s} \tag{5}$$

and

$$\oint \vec{H} d\vec{\ell} = \frac{\partial}{\partial t}\int \vec{D}\cdot d\vec{s}$$

But in vacuum $\vec{B} = \mu_0\vec{H}$, and $\vec{D} = \epsilon_0\vec{E}$, so that the latter equation can also be written in terms of E and B:

$$\oint \vec{B} d\vec{\ell} = \mu_0\epsilon_0\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{s} \quad\Longrightarrow\quad \oint \vec{B} d\vec{\ell} = \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{s} \tag{6}$$

(c) By choosing appropriate integration paths, use the above equations:

- For calculating the magnetic induction at an arbitrary radius r,
- And then for calculating the correction to the electric field $E_0(t)$ at r, and for showing that this correction is of the *second order* in r.

Solution:
In the zeroth order, $E_0 = (V_{\rm ampl}/h)e^{i\omega t}$. Since $E_0 = E_0(t)$, at radius r there is a B-field (denoted as B_1) that can be found from Eq. (4) and a geometry shown in Figure 1 below:

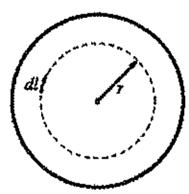


Figure 1: A loop for calculating the induced magnetic field.

$$B_1\cdot 2\pi r = \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}_0\cdot d\vec{s} = \frac{i\omega E_0}{c^2}\pi r^2 \quad\Longrightarrow\quad B_1 = \frac{i\omega r}{2c^2}E_0 = B_1(t) \tag{7}$$

This varying B_1-field will produce an electric field (denoted as E_2), which we find by applying Eq. (3) to the path depicted in Figure 2 below, keeping in mind that $E_2(r=0) = 0$, and that the electric field is vertical:

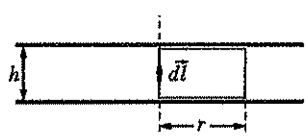


Figure 2: A loop for calculating the E-field, induced by the magnetic field.

$$-E_2h = -\frac{\partial}{\partial t}\int \vec{B}_1 d\vec{s} = \frac{\omega^2}{2c^2}\int_0^r rE_0hdr \quad \Longrightarrow \quad E_2 = -\frac{\omega^2 r^2}{4c^2}E_0 = -\frac{\omega^2 r^2}{4c^2}\left(\frac{V_{\text{ampl}}}{h}\right)e^{i\omega t} \tag{8}$$

Which is indeed a second-order correction term in r.

(d) Use an analogous procedure to calculate the B-field contributed by this second-order correction to the electric field, and then for calculating the resulting next correction term to $E_0(t)$, and for showing that this is a correction of *fourth order* in r.

Solution:
The electric field E_2 will contribute more B-field (denoted below as B_3):

$$B_3 \cdot 2\pi r = \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{E}_2 \cdot d\vec{s} \;=\; \frac{1}{c^2}\frac{\partial}{\partial t}\int_0^r \left(-\frac{\omega^2 r^2}{4c^2}\frac{V_{\text{ampl}}}{h}e^{i\omega t}\right)\cdot 2\pi r dr \;=\; \frac{i\omega^3}{16c^2}2\pi r^4$$

So that:

$$B_3 = \frac{i\omega^3 r^3}{16c^4}E_0 = B_3(t) \tag{9}$$

Which, in turn, contributes another electric field correction term (denoted below as E_4). Invoking again the Eq. (3) one gets:

$$-E_4h = -\frac{\partial}{\partial t}\int_0^r \left(-\frac{i\omega^3 r^3 E_0}{16c^4}\right)hdr \quad \Longrightarrow \quad E_4 = \frac{\omega^4 r^4}{64c^4}E_0 \tag{10}$$

which is indeed of the fourth order in r.

By combining Eqs. (2), (6) and (8), one obtains:

$$E(t,r) = E_0(t)\left(1 - \frac{\omega^2 r^2}{4c^2} + \frac{\omega^4 r^4}{64c^4}\right) \tag{11}$$

The items (a) - (d) are priced 5 pts. each.

Consider the earth's atmosphere. For this problem we will neglect the rotation of the earth, which we will also treat as a sphere. You may call the earth's radius R. For simplicity, we will assume that air is one part oxygen and four parts nitrogen at the earth's surface. You may call the mass of a single molecule of oxygen and nitrogen m_O and m_N respectively. The thickness of the atmosphere is small compared with the radius of the earth, so you may assume a gravitational force which is independent of height.

(a) Work out the probability per unit volume of a single molecule with mass m being between height h above the surface of the earth, provided it is in thermal equilibrium at temperature T.

(b) Assuming there are N molecules of a given element in the atmosphere, work out the number per unit volume present as a function of height above the surface of the earth.

(c) Given that the ratio of the concentration of oxygen and nitrogen molecules at sea level is 1:4, find ratio of the total number of molecules of oxygen in the atmosphere to the total number of nitrogen molecules in the atmosphere. You will need to know the ratio $\frac{m_N}{m_O} = \frac{7}{8}$

(d) What is the pressure, as a function of altitude, taking into account both nitrogen and oxygen. You may assume the entire atmosphere is at the same temperature (which it isn't).

(e) Verify that the pressure at any given altitude is equal to the combined weight per unit area of the atmosphere above that altitude.

Consider the earth's atmosphere. For this problem we will neglect the rotation of the earth, which we will also treat as a sphere. You may call the earth's radius R. For simplicity, we will assume that air is one part oxygen and four parts nitrogen at the earth's surface. You may call the mass of a single molecule of oxygen and nitrogen m_O and m_N respectively. The thickness of the atmosphere is small compared with the radius of the earth, so you may assume a gravitational force which is independent of height.

(a) Work out the probability per unit volume of a single molecule with mass m being between height h above the surface of the earth, provided it is in thermal equilibrium at temperature T.

Solution:
We use the standard formula for probability, which is the Boltzmann factor divided by partition function. Because the energy is separable between momentum and position coordinates, we can ignore the kinetic energy contribution. Note that since the change in height is small compared to the radius, we can ignore the $(R+h)^2$ aspect of integrating over volume.

$$V(h) = mgh \tag{12}$$

$$P(h) = \frac{e^{-\beta V(h)}}{Z} \tag{13}$$

$$= \frac{e^{-\beta mgh}}{Z} \tag{14}$$

$$Z = \int_0^\infty e^{-\beta mgh} 4\pi R^2 dh \tag{15}$$

$$= \frac{4\pi R^2}{\beta mg} \tag{16}$$

$$P(h) = \frac{\beta mg}{4\pi R^2} e^{-\beta mgh} \tag{17}$$

(b) Assuming there are N molecules of a given element in the atmosphere, work out the number per unit volume present as a function of height above the surface of the earth.

Solution:
This is actually quite easy given the probability we previously computed. The number of molecules per unit volume is just the probability per unit volume of each molecule times the number of molecules.

$$n(h) = NP(h) \tag{18}$$

$$= N\frac{\beta mg}{4\pi R^2} e^{-\beta mgh} \tag{19}$$

(c) Given that the ratio of the concentration of oxygen and nitrogen molecules at sea level is 1:4, find ratio of the total number of molecules of oxygen in the atmosphere to the total number of nitrogen molecules in the atmosphere. You will need to know the ratio $\frac{m_N}{m_O} = \frac{7}{8}$

Solution:

$$n_N(0) = 4n_O(0) \tag{20}$$
$$N_N \frac{\beta m_N g}{4\pi R^2} = 4N_O \frac{\beta m_O g}{4\pi R^2} \tag{21}$$
$$N_N m_N = 4N_O m_O \tag{22}$$
$$\frac{N_N}{N_O} = 4\frac{m_O}{m_N} \tag{23}$$
$$= 4\frac{8}{7} \tag{24}$$
$$= \frac{32}{7} \tag{25}$$

So in total, there is $\frac{32}{7}$ times as much nitrogen in the atmosphere as there is oxygen.

(d) What is the pressure, as a function of altitude, taking into account both nitrogen and oxygen. You may assume the entire atmosphere is at the same temperature (which it isn't).

Solution:
To find the pressure, we just use the ideal gas law, $p = k_B T n$, and add up the pressures of the two gas species. Or if you like, we add up their densities to find the total density.

$$p(h) = k_B T(n_N(h) + n_O(h)) \tag{26}$$
$$= k_B T N_N \frac{\beta m_N g}{4\pi R^2} e^{-\beta m_N g h} + k_B T N_O \frac{\beta m_O g}{4\pi R^2} e^{-\beta m_O g h} \tag{27}$$
$$= N_N \frac{m_N g}{4\pi R^2} e^{-\beta m_N g h} + N_O \frac{m_O g}{4\pi R^2} e^{-\beta m_O g h} \tag{28}$$
$$= \frac{g}{4\pi R^2} \left(N_N m_N e^{-\beta m_N g h} + N_O m_O e^{-\beta m_O g h}\right) \tag{29}$$

(e) Verify that the pressure at any given altitude is equal to the combined weight per unit area of the atmosphere above that altitude.

Solution:
To find the weight of the atmosphere above a given altitude, we need only find the total number of each

sort of molecule above that altitude.

$$N^O_{\text{above}}(h) = \int_h^\infty n_O(h) 4\pi R^2 dh \tag{30}$$
$$= \int_h^\infty N_O \frac{\beta m_O g}{4\pi R^2} e^{-\beta m_O g h} 4\pi R^2 dh \tag{31}$$
$$= \int_h^\infty N_O \beta m_O g e^{-\beta m_O g h} dh \tag{32}$$
$$= N_O \int_{\beta m_O g h}^\infty e^{-u} du \tag{33}$$
$$= N_O e^{-\beta m_O g h} \tag{34}$$

The nitrogen comes out the same way, so the weight above h is

$$W(h) = M_o g + M_N g \tag{35}$$
$$= m_O N_O^{\text{above}}(h) g + m_N N_N^{\text{above}}(h) g \tag{36}$$
$$= m_O g N_O e^{-\beta m_O g h} + m_N g N_N e^{-\beta m_N g h} \tag{37}$$

The surface area, of course, is just $4\pi R^2$, so we get a pressure of

$$p(h) = \frac{W(h)}{4\pi R^2} \tag{38}$$
$$= \frac{g}{4\pi R^2} \left(m_O N_O e^{-\beta m_O g h} + m_N N_N e^{-\beta m_N g h} \right) \tag{39}$$

which is the same as we worked out using the density.

(a) From Newton's law show that the equation of motion in a rotating reference frame with angular velocity ω is given by

$$m\ddot{\mathbf{r}} = \vec{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\omega} + m\,(\boldsymbol{\omega} \times \mathbf{r}) \times \boldsymbol{\omega}\,. \tag{40}$$

Hint: the time derivative of any one vector $\vec{Q}$ as measured in the inertial frame S_0 in terms of the corresponding derivative in the rotating frame S is given by

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\omega} \times \vec{Q}\,.$$

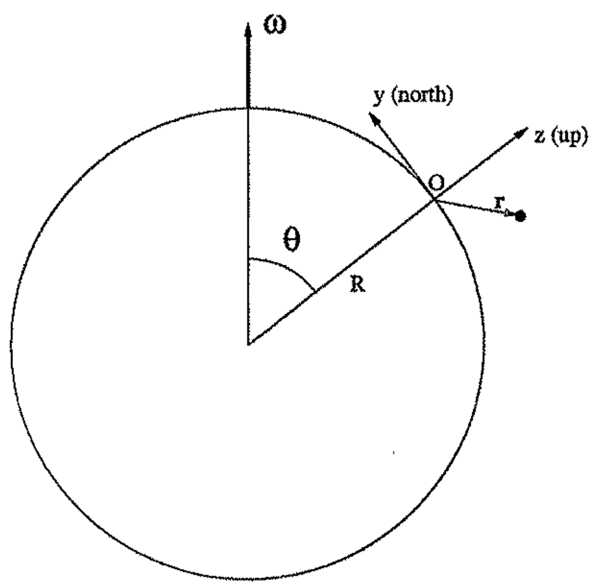
(b) Identify and describe the terms in equation 40.

Now consider a free falling object (mass m) on the surface of a spherical earth (radius R, mass M, angular velocity ω, no atmosphere) (see figure).

(c) The gravitational force on the surface of the non-rotating earth is given by

$$\vec{F}_{grav} = -\frac{GMm}{R^2}\hat{\mathbf{r}} = m\vec{g}_0\,.$$

Neglecting the Coriolis force, what is the effective gravitational acceleration $\vec{g}$ at a point (θ, ϕ) on the surface of the earth due to the rotation of the earth?
Give an order of magnitude estimate of this effect ($R \sim 12000\,\text{km}$, $g_0 = 10\,\text{m/s}^2$, 1 revolution in 24 hours).



(d) Due to the Coriolis force a free falling object will not fall straight down (straight down is defined as the direction of a plumb line which points in the direction of $\vec{g}$ and not $\vec{g_0}$). Give the direction of the deflection and derive a first order estimate for the deflection.

a) $m\ddot{\bar r}_0 = \bar F$ (1)

$$\left.\frac{d\bar r}{dt}\right|_0 = \dot{\bar r} + \bar\omega\times\bar r$$

$$\left.\frac{d\dot{\bar r}}{dt}\right|_0 = \ddot{\bar r} + \frac{d}{dt}(\bar\omega\times\bar r) + \bar\omega\times(\dot{\bar r} + \bar\omega\times\bar r)$$

$$\ddot{\bar r}_0 = \ddot{\bar r} + \bar\omega\times\dot{\bar r} + \bar\omega\times\dot{\bar r} + \bar\omega\times(\bar\omega\times\bar r)$$

$$\ddot{\bar r}_0 = \ddot{\bar r} + 2\,\bar\omega\times\dot{\bar r} + \bar\omega\times(\bar\omega\times\bar r)$$

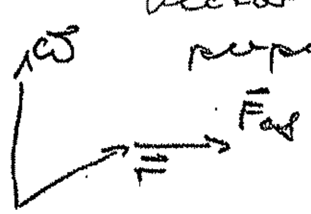
in (1)

$$m\ddot{\bar r} = \bar F - 2m(\bar\omega\times\dot{\bar r}) - m\bar\omega\times(\bar\omega\times\bar r)$$

$$m\ddot{\bar r} = \bar F + 2m(\dot{\bar r}\times\bar\omega) + m(\bar\omega\times\bar r)\times\bar\omega \quad ||$$

b) $m\underbrace{(\bar\omega\times\bar r)\times\bar\omega}_{} =: \vec F_{cf}$

vector in $\bar\omega, \bar r$ plane and perpendicular to $\vec\omega$, pointing outward

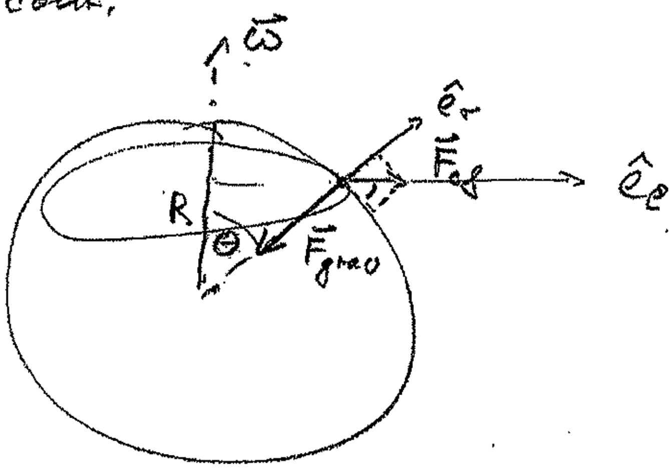


→ centrifugal acceleration.

$$2m(\dot{\bar r}\times\bar\omega) = \bar F_{cor}$$

0 if object is stationary, and perpendicular to motion of object
→ Coriolis force.

c) cont.



$$\vec{F}_{grav} = m\vec{g}_0 = -mg_0\hat{e}_r$$

$$\vec{F}_{eff} = \vec{F}_{grav} + \vec{F}_{cf} = m\vec{g}_0 + m(\vec{\omega}\times\vec{r})\times\vec{\omega}$$

on earth's surface $\vec{r} = R\cdot\hat{e}_r$

and $(\vec{\omega}\times\vec{r})\times\vec{\omega} = R\sin\Theta\,\omega^2\hat{e}_\varrho$

so $\vec{g} = \vec{g}_0 + \Omega^2 R\sin\theta\,\hat{e}_\varrho$

and magnitudes

$$g = g_0 - \Omega^2 R\sin^2\Theta$$

$R = 6000\,km = 6\cdot10^6\,m$

$$\Omega = \frac{2\pi\cdot1}{24h} \approx \frac{6}{24\cdot3600}s^{-1} = \frac{1}{14400}s^{-1}$$

$$\approx \frac{1}{1.5}\cdot10^{-4}s^{-1} = \frac{2}{3}\cdot10^{-4}s^{-} \approx 0{,}7\cdot10^{-4}s^{-1}$$

$\sin\Theta = 1$ for $\Theta = \frac{\pi}{2}$ (at the equator)

$\rightarrow \left(\Omega^2 R\sin\Theta\right)_{max} \approx 4\cdot10^{-2}\,\frac{m}{s^2}$ or about 0.4% of g_0, small but measurable.

d) cont.

the effect of $\vec{F}_{cf}$ is $m\vec{g}$ →

$$m\ddot{\vec{r}} = m\vec{g} + 2m(\dot{\vec{r}} \times \vec{\omega})$$

$$\ddot{\vec{r}} = \vec{g} + 2\dot{\vec{r}} \times \vec{\omega}$$

Using a coordinate system with z = up and y = north (figure on problem sheet)

$$\vec{\omega} = (0, \omega\sin\Theta, \omega\cos\Theta)$$

and $\dot{\vec{r}} \times \vec{\omega} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} 0 \\ \omega\sin\Theta \\ \omega\cos\Theta \end{pmatrix}$

$$\dot{\vec{r}} \times \vec{\omega} = \begin{pmatrix} \dot{y}\omega\cos\Theta - \dot{z}\omega\sin\Theta \\ -\dot{x}\omega\cos\Theta \\ \dot{x}\omega\sin\Theta \end{pmatrix}$$

and equations of motion:

$$\ddot{x} = 2\omega(\dot{y}\cos\Theta - \dot{z}\sin\Theta)$$

$$\ddot{y} = -2\omega\dot{x}\cos\Theta$$

$$\ddot{z} = -g + 2\omega\dot{x}\sin\Theta$$

first 0-th order: $\omega = 0$

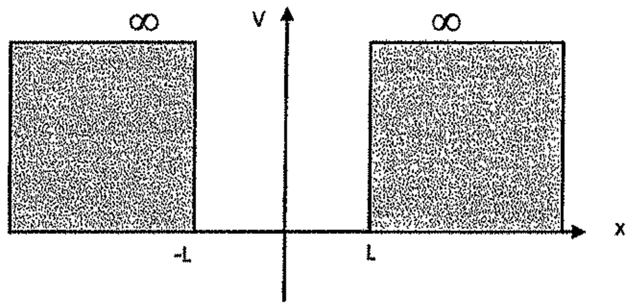
$\ddot{x} = 0 \quad \ddot{y} = 0$, $\ddot{z} = -g$ $\Big\}$ $x = x_0 = 0$, $y = y_0 = 0$ $\quad \dot{z} = -gt$, $z = z_0 - \frac{1}{2}gt^2$

1st order (use 0-th order result on RHS):

$\ddot{x} = -g\omega\sin\Theta\, t$, $\ddot{y} = 0$, $\ddot{z} = -g$ $\Big\}$ $x = \frac{1}{3}g\omega\sin\Theta \cdot t^3$, $y = 0$, $z = z_0 - \frac{1}{2}gt^2$

→ deflection in positive x-direction (East)

A particle of mass m and charge q moves in one dimension between the impenetrable walls of an infinite square-well potential, so that $V(x) = 0$ at $|x| < L$ and $V(x) = \infty$ at $|x| > L$.



(a) Write down wave functions and energy levels.

(b) The particle is in the ground state. At t=0, we instantaneously turn on a weak uniform time-independent electric field of strength $\mathcal{E}$.

- (i) Write down the perturbing potential due to the electric field interacting with the particle. Assume that the potential is equal to zero at $x = 0$.
- (ii) Calculate the first non-zero correction to the particle's ground-state energy and state.
- (iii) What is the probability of finding the particle in the first excited state?

(c) Consider now the case of a time-dependent electric field of the form $\mathcal{E} = \mathcal{E}_0 \exp(-t/\tau)$ at $t > 0$ and $\mathcal{E} = 0$ at $t < 0$ (i.e turned on instantaneously at $t = 0$ followed by exponential decay).

- (i) Calculate the transition probability from the ground state of the system to the first excited state in first-order time-dependent perturbation theory at times $t \gg \tau$.
- (ii) Under what condition does this probability approach that calculated in part (b)? Show and discuss.

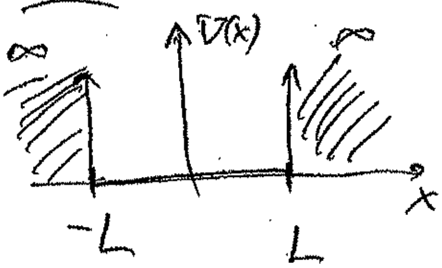
Hint: you may find the following useful:

$$\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) \sin\left(\frac{\pi m x}{L}\right) x\,dx = (-1)^m \frac{32mL^2}{(4m^2-1)^2\pi^2} \quad (m \text{ is an integer})$$

$$\sum_{m=1}^{\infty} \frac{m^2}{(4m^2-1)^5} = \frac{\pi^2(15-\pi^2)}{12288}$$

(1)

QM Problem # 5



$$V(x)=\begin{cases}0, & |x|<L\\ \infty, & |x|>L\end{cases}$$

(a) weak uniform electic field $\mathcal{E}$ $\Rightarrow$ the perturbing potential $H'=-q\mathcal{E}x$

<u>unperturbed wave functions:</u> $\Psi_n(x)=\frac{1}{2\sqrt{L}}\left(e^{i\frac{\pi n x}{2L}}+(-1)^{n+1}e^{-i\frac{\pi n x}{2L}}\right)$ (1) (even or odd due to symmetric potential V)

<u>unperturbed energy:</u> $E_n=\frac{\hbar^2\pi^2}{8mL^2}n^2$

<u>1st-order energy correction:</u>

$E_n^{(1)}=-q\mathcal{E}\langle\Psi_n|x|\Psi_n\rangle=0$ ← due to symmetry

<u>2nd-order energy correction:</u>

$$E_n^{(2)}=-\frac{8mL^2q^2\mathcal{E}^2}{\pi^2\hbar^2}\sum_{n'\neq n}\frac{|\langle\Psi_n|x|\Psi_{n'}\rangle|^2}{n'^2-n^2}$$

$\Rightarrow$ take $n=1$ $\Rightarrow$ (ground state)

Due to parity reasons, only $\langle \Psi_1 | x | \Psi_{2\nu} \rangle \neq 0$ ②

opposite parity $\nu = 1, 2, \ldots$

Then, $\langle \Psi_1 | x | \Psi_{2\nu} \rangle = \frac{1}{L} \int_{-L}^{L} \cos \frac{\pi n x}{2L} \sin \frac{\pi \nu x}{L} \, x \, dx \;(=)$

$$\Psi_1 = \frac{1}{\sqrt{L}} \cos \frac{\pi n x}{2L}$$

$$\Psi_{2\nu} = \frac{1}{\sqrt{L}} \sin \frac{2\pi \nu x}{2L} = \frac{1}{\sqrt{L}} \sin \frac{\pi \nu x}{L}$$

$n = 1$

$$(=) \frac{32 L \nu (-1)^{\nu+1}}{\pi^2 (4\nu^2 - 1)^2}$$

$$|\langle \Psi_1 | x | \Psi_{2\nu} \rangle|^2 = \frac{32^2 L^2 \nu^2}{\pi^4 (4\nu^2 - 1)^4}$$

$$\sum_{\nu=1}^{\infty} \frac{\nu^2}{(4\nu^2-1)^5} = \frac{\pi^2 (15 - \pi^2)}{12288}$$

$12288 = 12 \cdot 32^2$

$$E_1^{(2)} = -\frac{8 m L^4 q^2 \mathcal{E}^2}{\pi^6 \hbar^2} 32^2 \cdot \sum_{\nu=1}^{\infty} \frac{\nu^2}{(4\nu^2 - 1)^4} \frac{1}{4\nu^2 - 1} \;(=)$$

$$\frac{\nu^2}{(4\nu^2-1)^5}$$

$$(=) -\frac{8 m L^4 q^2 \mathcal{E}^2}{\pi^6 \hbar^2 \cdot 12} \pi^2 (15 - \pi^2) =$$

$$= -\frac{2}{3} \frac{m L^4 q^2 \mathcal{E}^2}{\pi^4 \hbar^2} (15 - \pi^2)$$

1st-order correction to the ground state: (3)

$$|\Psi_1\rangle^{(1)} = \frac{8mL^2 g\mathcal{E}}{\pi^2\hbar^2}\sum_{\nu=1}^{\infty}\frac{1}{4\nu^2-1}\langle\Psi_{2\nu}|x|\Psi_1\rangle\cdot|\Psi_{2\nu}\rangle =$$

$$= \frac{8mL^2 g\mathcal{E}}{\pi^2\hbar^2}\cdot\frac{32L}{\pi^2}\sum_{\nu=1}^{\infty}\frac{\nu}{(4\nu^2-1)^3}\underbrace{|\Psi_{2\nu}\rangle}_{=\frac{1}{\sqrt{L}}\sin\frac{\pi\nu x}{L}}$$

Probability of finding the particle in the state $n=2$:

$$\mathcal{P}_{1\to2} = |\langle\Psi_2|\Psi_1\rangle|^2 = |\langle\Psi_2|\Psi_1\rangle^{(1)}|^2 =$$

$$= \left|\underbrace{\frac{1}{\sqrt{L}}\sum_\nu\int_{-L}^{L}\sin\frac{\pi x}{L}\sin\frac{\pi\nu x}{L}\,dx}_{=L\delta_{\nu 1}}\frac{1}{\sqrt{L}}\frac{8\cdot32\, mL^3 g\mathcal{E}}{\pi^4\hbar^2}\frac{\nu}{(4\nu^2-1)^3}\right|^2 =$$

$$= \frac{(8\cdot32)^2 m^2L^6g^2\mathcal{E}^2}{\pi^8\hbar^4}\frac{1}{3^6} = \frac{8^4\cdot4^2}{3^6}\left(\frac{mL^3g\mathcal{E}}{\pi^4\hbar^2}\right)^2$$

(b) $\mathcal{E}(t) = \mathcal{E}_0 e^{-t/\tau}$ at $t > 0$ (4)

0 at $t < 0$

$$\mathcal{P}_{1\to n}(t) = \frac{q^2 \mathcal{E}_0^2}{\hbar^2} \left| \int_0^t e^{i\frac{E_n - E_1}{\hbar} t'} e^{-\frac{t'}{\tau}} \cdot \langle \Psi_n | x | \Psi_1 \rangle dt' \right|^2 =$$

only $\langle \Psi_{2\nu} | x | \Psi_1 \rangle$ are $\neq 0$ (from parity)

$\Vert \leftarrow$ from part (a)

$$\frac{32 L \nu (-1)^{\nu+1}}{\pi^2 (4\nu^2 - 1)^2} \overset{\nu = 1}{=} \frac{32 L}{\pi^2 \cdot 3^2}$$

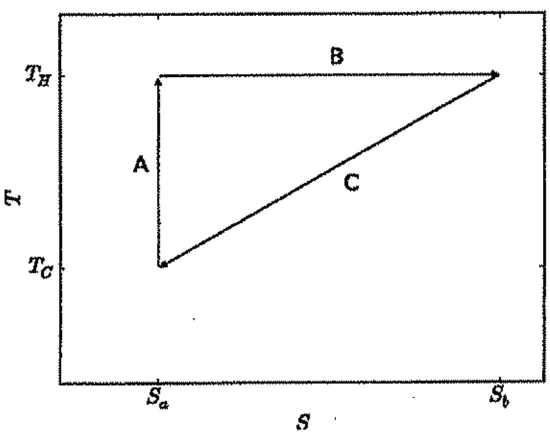
$$= \frac{q^2 \mathcal{E}_0^2}{\hbar^2} \cdot \frac{32^2 L^2}{\pi^4 3^4} \left| \frac{e^{i\frac{E_2 - E_1}{\hbar} t - \frac{t}{\tau}} - 1}{i\frac{E_2 - E_1}{\hbar} - \frac{1}{\tau}} \right|^2$$

$$E_2 - E_1 = \frac{\hbar^2 \pi^2}{8 m L^2} \cdot 3$$

At $t \gg \tau \Rightarrow$

$$\mathcal{P}_{1\to 2} = \frac{q^2 \mathcal{E}_0^2 \cdot 32^2 L^2}{3^4 \hbar^2 \pi^4} \frac{\tau^2}{\left(\frac{3 \hbar \pi^2}{8 m L^2} \tau\right)^2 + 1} \overset{\tau \to \infty}{\Longrightarrow} \frac{8^4 \cdot 4^2}{3^6} \left(\frac{m L^3 q \mathcal{E}_0}{\pi^4 \hbar^2}\right)^2$$

as in (b)



Consider a heat engine operated using the three-step cycle shown above.

(a) Describe in words each of the three steps of the process, including whether they involve compression or expansion. You may assume that the working material is an ordinary material that expands when its temperature is increased. If you would need more information to describe any of the steps, please say so.

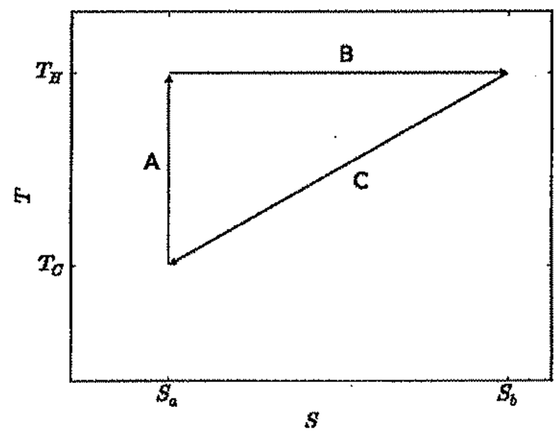
(b) Find the quantity of heat transfered *to the system* in each of the three steps (A, B and C).

(c) What is the net heat transfer to or from the system over one complete cycle? What is the direction of the net heating?

(d) What is the net work done *by our system* over one complete cycle of this engine?

(e) What is the efficiency of this heat engine. Note that the efficiency is defined as the net work done by an engine divided by the amount of energy *added to the system* by heating.

(f) How does this compare with the Carnot efficiency? If the same, why is it the same? If different, why is it different?



Consider a heat engine operated using the three-step cycle shown above.

(a) Describe in words each of the three steps of the process, including whether they involve compression or expansion. You may assume that the working material is an ordinary material that expands when its temperature is increased. If you would need more information to describe any of the steps, please say so.

Solution:
A is an adiabatic compression, B is an isothermal expansion. C is a bit confusing. We can't tell if it is a compression or expansion without some additional information. All we know is that the material is cooling down and is heating its environment.

(b) Find the quantity of heat transfered *to the system* in each of the three steps (A, B and C).

Solution:
The heat is

$$Q = \int T dS \tag{41}$$

so we just need to find the area under each curve. For A, $Q_A = 0$. For B, it is $Q_B = T_H(S_b - S_a)$. C is only slightly harder, as we just need to add add a triangle to a rectangle: $Q_C = T_C(S_a - S_b) + \frac{1}{2}(T_H - T_C)(S_a - S_b)$. We should note that Q_C is negative, since the arrow is going to the left.

(c) What is the net heat transfer to or from the system over one complete cycle? What is the direction of the net heating?

Solution:
We could add together our previous answers, but it's a little easier to graphically recognize that we just need the area of the triangle (and the answer is positive), so it's just $Q_{\text{net}} = \frac{1}{2}(T_H - T_C)(S_b - S_a)$.

(d) What is the net work done *by our system* over one complete cycle of this engine?

Solution:
By the first law (and because internal energy is a state function), since this is a cyclical process and returns to the same state after one cycle, the net work done by our system must be equal to the net heat added to it. $W_{\text{net}} = \frac{1}{2}(T_H - T_C)(S_b - S_a)$

(e) What is the efficiency of this heat engine. Note that the efficiency is defined as the net work done by an engine divided by the amount of energy *added to the system* by heating.

Solution:
The heat *added* to the system is all added during step B, so our efficiency is

$$\eta = \frac{W_{\text{net}}}{Q_B} \tag{42}$$

$$= \frac{\frac{1}{2}(T_H - T_C)(S_b - S_a)}{T_H(S_b - S_a)} \tag{43}$$

$$= \frac{\frac{1}{2}(T_H - T_C)}{T_H} \tag{44}$$

$$= \frac{1}{2}\left(1 - \frac{T_C}{T_H}\right) \tag{45}$$

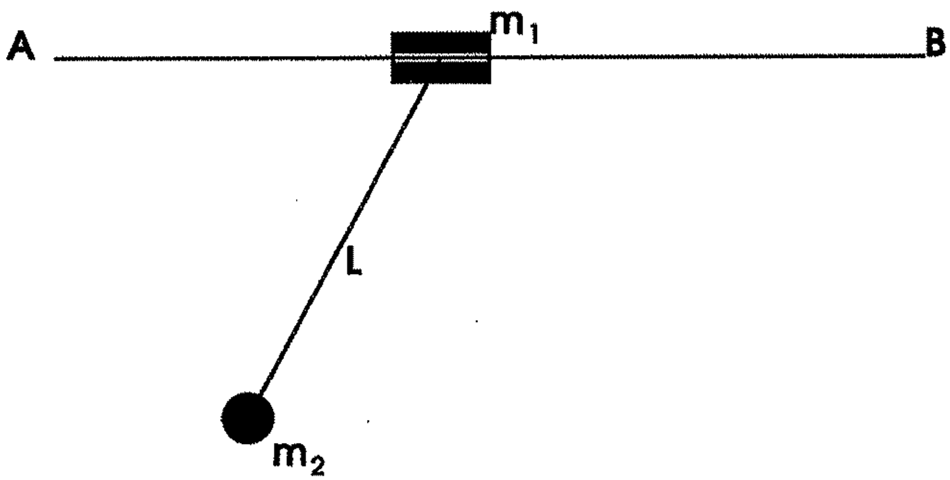
(f) How does this compare with the Carnot efficiency? If the same, why is it the same? If different, why is it different?

Solution:
The Carnot cycle (which consists of a rectangular cycle on our plot) has an efficiency of

$$\eta = \left(1 - \frac{T_C}{T_H}\right) \tag{46}$$

which is twice the efficiency of our engine. We find a lower efficiency because our system dumps heat to its enviroment while it is still hot, which is an irreversible process if the environment is cold. Or if the environment isn't all cold (consisting of different regions), we could have done more work by taking it to the cold environment (and adiabatically cooling it) before letting it dump heat.

A pendulum consists of a massless rod of length L connecting two masses m_1 and m_2. The pendulum is free to swing in a plane around m_1 and m_1, and therefore the hinge of the pendulum, can slide over a horizontal segment (see Figure).

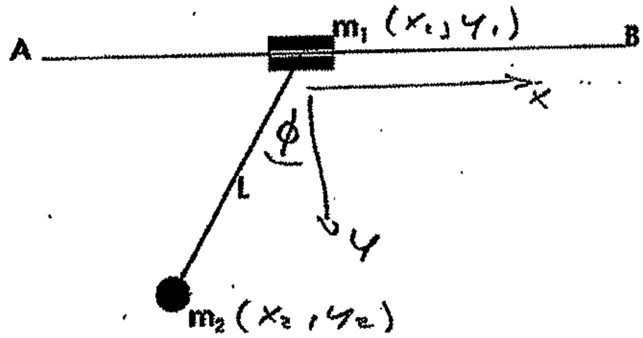


(a) Find the Lagrange equations of the pendulum.

(b) Starting from the Lagrange equations find the curve described by the point m_2 during (arbitrarily large) oscillations.

(c) Discuss the motion in the limiting cases $m_1 \gg m_2$ and $m_2 \gg m_1$. (This part can be tackled without first solving part b).

a)



generalized coordinates : x, ϕ

m_1: $x_1 = x$, $y_1 = 0$ m_2: $x_2 = x + l \sin\phi$, $y_2 = l\cos\phi$

and

$\dot{x}_1 = \dot{x}$ $\dot{x}_2 = \dot{x} + \dot{\phi} l \cos\phi$

$\dot{y}_1 = 0$ $\dot{y}_2 = -\dot{\phi} l \sin\phi$

kinetic energy:

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \left(\dot{x}^2 + 2 l \dot{x} \dot{\phi} \cos\phi + l^2 \dot{\phi}^2 \cos^2\phi + l^2 \dot{\phi}^2 \sin^2\phi \right)$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 \left(l^2 \dot{\phi}^2 + 2 l \dot{x} \dot{\phi} \cos\phi \right)$$

potential energy :

$$U = -m_2 g y_2 = -m_2 g l \cos\phi$$

Lagrangian:

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{m_2}{2}(l^2\dot{\phi}^2 + 2 l \dot{x}\dot{\phi}\cos\phi) + m_2 g l \cos\phi$$

a) cont

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$\Rightarrow$

$$\frac{\partial L}{\partial x} = 0 \, , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos\phi\right] \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial \phi} = -m_2 l \dot{x} \dot{\phi} \sin\phi - m_2 l g \sin\phi$$
$$= -m_2 l \sin\phi \, (\dot{x}\dot{\phi} - g)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(m_2 l^2 \dot{\phi} + m_2 l \dot{x} \cos\phi\right)$$
$$= m_2 l^2 \ddot{\phi} + m_2 l \ddot{x} \cos\phi - m_2 l \dot{x} \sin\phi \cdot \dot{\phi}$$

and

$$(m_1 + m_2)\dot{x} + m_2 l \dot{\phi} \cos\phi = \text{const} = p_x \quad ||$$

$$m_2 l^2 \ddot{\phi} + m_2 l \ddot{x} \cos\phi - m_2 l g \sin\phi = 0$$
$$\Rightarrow \quad l \ddot{\phi} + \ddot{x} \cos\phi - g \sin\phi = 0 \quad ||$$

b) because L does not depend on x

$$p_x = m_1 \dot{x}_1 + m_2 \dot{x}_2 = \text{const.}$$

$\Rightarrow$ without loss of generality we can assume $p_x = 0$ and integrate the first Lagrange equation: $\longrightarrow$

b) cont.

$$\int dt \left[(m_1+m_2)\dot{x} + m_2 l \dot{\phi} \cos\phi \right] = C \quad \leftarrow \text{integration const.}$$

$$\Rightarrow (m_1+m_2)x + m_2 l \sin\phi = C$$

or $\quad m_1 x_1 + m_2 x_2 = C$

interpretation: the x-component of the center of mass is stationary. For convenience choose coordinate system so that $c=0$.

$$m_1 x_1 + m_2 x_2 = 0$$

write in terms of x_2, y_2:

$$(m_1+m_2)(x_2 - l\sin\phi) + m_2 l \sin\phi = 0$$

$$(m_1+m_2)x_2 = m_1 l \sin\phi \quad | \text{ square}$$

$$(m_1+m_2)^2 x_2^2 = m_1^2 l^2 (1-\cos^2\phi)$$

$$= m_1^2 l^2 - m_1^2 y_2^2$$

$$\Rightarrow \frac{x_2^2}{\left(\frac{m_1}{m_1+m_2}\right)^2 l^2} + \frac{y_2^2}{l^2} = 1$$

→ equation of an ellipse with half axes: $a_x = \frac{m_1 l}{m_1+m_2}$; $a_y = l$, with $a_y > a_x$.

superimposed is an overall linear motion depending on the value of p_x.

c) limiting cases:

c1) $m_1 \gg m_2$ → regular pendulum

ellipse → circular motion.

c2) $m_2 \gg m_1$

$$\rightarrow a_x = \frac{m_1 l}{m_1 + m_2} \sim \frac{m_1}{m_2} l \rightarrow 0$$

⇒ m_1 moves back and forth in x,

while m_2 moves up and down.

Consider an infinitely long conducting cylindrical pipe of radius R, with linear charge density λ. The pipe is "cut in half" along two generatrices 180 degrees apart, into two coaxial half-cylinders (see Fig. 1) remaining in contact. Calculate the force of repulsion per unit length between the two half-cylinders.

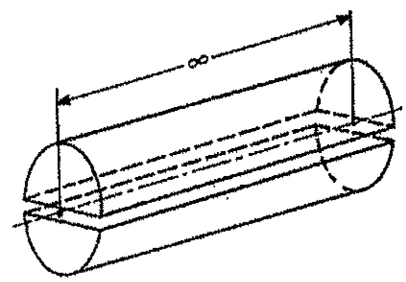
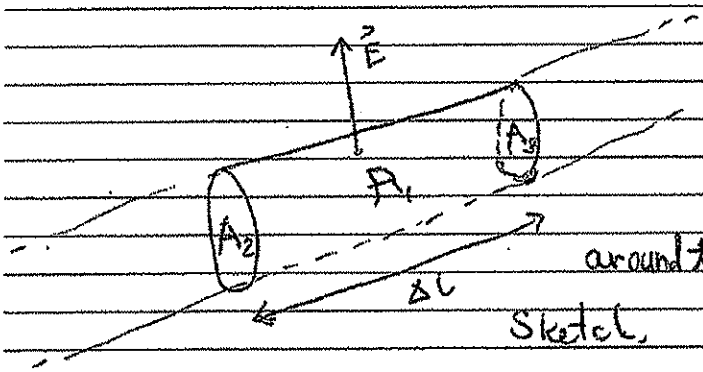


Figure 1. Two coaxial half-cylinders. For clarity, in this figure they are shifted apart in the vertical direction but in the problem, they remain in contact.

Problem 8 – Solution:

One can find the electric field at the cylinder surface using the Gauss Theorem.



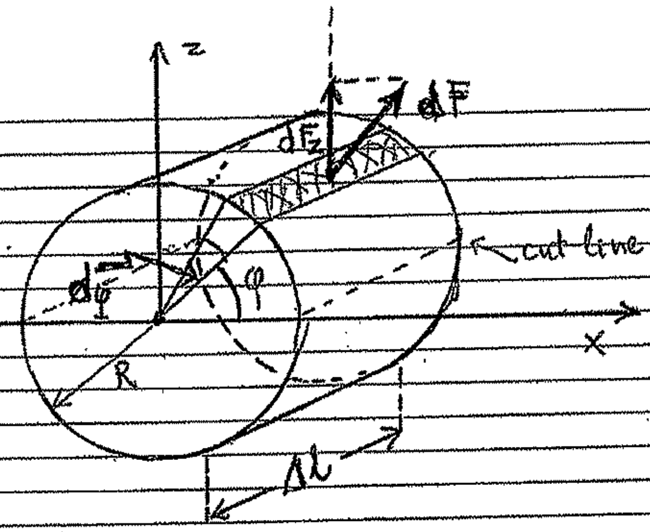
We wrap a Gaussian surface of length Δl around the pipe, as shown in the sketch.

Due to the symmetry, the field is normal to the cylinder surface, and pointing outwards (if the charge is +), or inward (for − charge) – but always exerting a force on the charges, "pushing them outwards."

The flux through the areas marked as A_2 and A_3 is zero, so that the net flux through the Gaussian surface is $\Phi = E \cdot A_1 = 2\pi R \Delta l \cdot E$

The charge contained within the Gaussian surface is $Q_{contained} = \lambda \cdot \Delta l$. So, from the Gauss Theorem we have $2\pi R \Delta l E = \lambda \cdot \Delta l / \varepsilon_0$

$$\Longrightarrow \quad E = \frac{\lambda}{2\pi R \varepsilon_0}$$



Consider a section of the pipe of Δl (length) & a "stripe"-shaped surface element "seen" from the axis at angle $d\varphi$.

The force exerted by Field E on the surface charge of a conductor is $dF = \frac{1}{2}\sigma \cdot E_n \cdot dA$ (σ – surface density of the charge; dA – surface element, E_n – normal E component). The reason why there is the $\frac{1}{2}$ coefficient is described in detail in the enclosed copies of pages 103-104 from "Introduction to Electrodynamics", D. J. Griffiths, Second Edition.

In the present case: $\sigma = \frac{\lambda}{2\pi R}$; $dA = \Delta L \cdot R \cdot d\varphi$; $dF_z = \sin\varphi \cdot dF$

Then, the total force in the z direction acting on the upper half is

$$F_z(\Delta l) = \int_0^{\pi} \frac{1}{2} \cdot \frac{\lambda}{2\pi R} \cdot \frac{\lambda}{2\pi R \varepsilon_0} \cdot \sin\varphi \cdot \Delta L \cdot R \cdot d\varphi$$
$$= \frac{\lambda^2 \Delta L}{8\pi^2 R \varepsilon_0} (-\sin\varphi)\Big|_0^{\pi} = \frac{\lambda^2 \Delta L}{4\pi^2 R \varepsilon_0}$$

Therefore, the force acting on a unit length of each half is $F_z(\Delta l)/\Delta L = \lambda^2/4\pi^2 R \varepsilon_0$

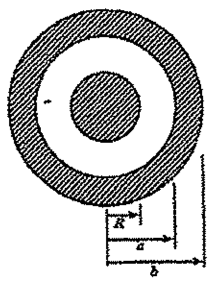


Figure 2.49

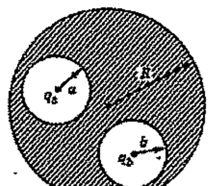


Figure 2.50

2.5.3 The Surface Charge on a Conductor; the Force on a Surface Charge

Because the field inside a conductor is zero, boundary condition (2.27) requires that the field immediately *outside* be

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{n} \qquad (2.41)$$

consistent with our earlier conclusion that the field is normal to the surface. In terms of potential, (2.30) yields

$$\boxed{\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon_0}} \qquad (2.42)$$

These equations enable you to calculate the surface charge on a conductor, if you can determine $\mathbf{E}$ or V; we shall use them frequently in the next chapter.

In the presence of an electric field, a surface charge will, naturally, experience a force. According to equation (2.2), the force per unit area, $\mathbf{f}$, is $\sigma\mathbf{E}$. But there's a problem here, for the electric field is *discontinuous* at a surface charge, so which value are we supposed to use: $\mathbf{E}_{above}$, $\mathbf{E}_{below}$, or something in between? The answer is that we should use the *average* of the two:

$$\mathbf{f} = \sigma\mathbf{E}_{average} = \tfrac{1}{2}\sigma(\mathbf{E}_{above} + \mathbf{E}_{below}) \qquad (2.43)$$

Why the average? The reason is very simple, though the telling makes it sound complicated. Let's focus our attention on a small patch of surface centered about the point in question (Fig. 2.51). Make it tiny enough so it is essentially flat, and the surface charge on it is essentially constant. The total field consists of two parts—that attributable to the patch itself, and that due to everything else (other regions of the surface, as well as any external sources that may be present):

$$\mathbf{E} = \mathbf{E}_{patch} + \mathbf{E}_{other}$$

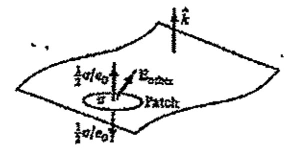


Figure 2.51

Now, the patch cannot exert a force on itself, any more than you can lift yourself by standing in a basket and pulling up on the handles. The force on the patch, then, is due exclusively to $\mathbf{E}_{other}$, and *this* suffers *no* discontinuity (if we removed the patch, the field in the "hole" would be perfectly smooth). The discontinuity is due entirely to the charge on the patch, which puts out a field $(\sigma/2\epsilon_0)$ on either side, pointing away from the surface (Fig. 2.51). Thus

$$\mathbf{E}_{above} = \mathbf{E}_{other} + \frac{\sigma}{2\epsilon_0}\hat{k}$$

$$\mathbf{E}_{below} = \mathbf{E}_{other} - \frac{\sigma}{2\epsilon_0}\hat{k}$$

and hence

$$\mathbf{E}_{other} = \tfrac{1}{2}(\mathbf{E}_{above} + \mathbf{E}_{below}) = \mathbf{E}_{average}$$

Averaging is really just a device for peeling off the contribution of the patch itself.

In the case of a conductor, the average field (equation 2.41) is $(1/2\epsilon_0)\sigma\hat{n}$, so the force per unit area is

$$\mathbf{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{n} \qquad (2.44)$$

This amounts to an outward electrostatic pressure on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field (2.41) just outside the surface,

$$P = \frac{\epsilon_0}{2}E^2 \qquad (2.45)$$

Problem 2.38 Equation (2.43) is not restricted to infinitesimally thin surfaces. Consider a slab of thickness a, and assume for the sake of argument that the charge density ρ and the electric field $\mathbf{E}$ depend only on x (see Fig. 2.52). The normal force per unit area on this slab is then

$$f_x = \int_0^a \rho(x)E_x(x)\,dx$$

Use Gauss's law (in differential form) to rewrite ρ, and show that

$$f_x = \frac{1}{2}\epsilon_0[E_x(a)^2 - E_x(0)^2] = \sigma E_{x_{ave}}$$