OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #110

March 28 and 29, 2011

Comprehensive examination for Spring 2011

PART 1, Monday, March 28, 9:00 am

General Instructions

This Comprehensive Examination for Spring 2011 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, March 28, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, March 29, at 9:00 am and 1:00 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

1 Physical constants

fine structure constant : $\alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c} \approx \dfrac{1}{137}$

Rydberg energy : $E_o = \dfrac{m_e e^4}{2\hbar^2 (4\pi\epsilon_o)^2} = \dfrac{m_e c^2 \alpha^2}{2}$

Bohr magneton : $\mu_B = \dfrac{e\hbar}{2m_e}$

Bohr radius : $a_o = \dfrac{4\pi\epsilon_0 \hbar^2}{m_e e^2}$

2 Vector calculus relationships

Triple products:

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \\
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})
\end{aligned}$$

Product rules:

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \\
&\quad + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \\
\nabla\cdot(\phi\mathbf{A}) &= \phi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\phi \\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) + \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + \\
&\quad + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}
\end{aligned}$$

Second derivatives:

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \\
\nabla\cdot(\nabla\times\mathbf{A}) &= 0
\end{aligned}$$

Green's theorem:

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dV = \oint_S (\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S}$$

Spherical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi} \\
\nabla\cdot\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
\nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\mathbf{r}} \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) \\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}
\end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla\cdot\mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
\nabla\times\mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} \\
&\quad + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} \\
&\quad + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\hat{\mathbf{z}} \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

3 Quantum mechanics

Raising and lowering operators for ang. momentum:

$$\begin{aligned}
J_\pm &= J_x \pm iJ_y \\
J_\pm\left|j,m\right\rangle &= \hbar\sqrt{j(j+1) - m(m\pm 1)}\left|j,m\pm 1\right\rangle
\end{aligned}$$

Perturbation theory for nondegenerate states:

$$E_n \approx E_n^o + \left\langle n\right| V \left|n\right\rangle + \sum_{m\neq n}\frac{\left|\left\langle n\right| V \left|m\right\rangle\right|^2}{E_n - E_m} + \cdots$$

Harmonic oscillator: $[a, a^\dagger] = 1$

$$\begin{aligned}
a &= \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}} \\
a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}} \\
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\
a|n\rangle &= \sqrt{n}|n-1\rangle
\end{aligned}$$

4 Electromagnetism

Maxwell's equations:

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J}$$

Magnetic dipole field:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}$$

Energy density: $U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
Poynting vector: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

General solutions of Laplace's equation

in cylindrical coordinates (independent of z):

$$\Phi(\rho,\phi) = a_o\log(\rho) + \sum_{n=1}^{\infty}\left(\frac{a_n}{\rho^n} + b_n\rho^n\right)(c_n\cos n\phi + d_n\sin n\phi)$$

in spherical coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(with azimuthal symmetry)

5 Useful math formulas

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

$$\int_{-\infty}^{\infty} e^{ixy}dy = 2\pi\delta(x)$$

$$\int_0^{\infty} x^n e^{-x}dx = n!\,,\ \text{integer } n$$

$$\begin{aligned}
(1+x)^n &= \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^k \\
\log(n!) &\approx \frac{1}{2}\log(2\pi n) + n\log(n) - n \\
\sin(x\pm y) &= \sin x\cos y \pm \cos x\sin y \\
\cos(x\pm y) &= \cos x\cos y \mp \sin x\sin y
\end{aligned}$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$\frac{1}{|\mathbf{x}-r'\hat{\mathbf{z}}|} = \sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$

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Legendre polynomials:

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

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$$P_l^m(x) = \left(1 - x^2\right)^{m/2}\frac{d^m P_l}{dx^m}$$

Spherical harmonics:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\phi}$$

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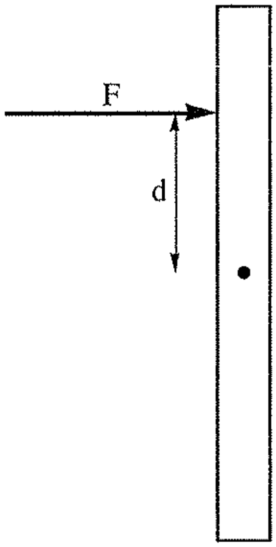
Problem 1

Many pieces of athletic equipment, such as tennis rackets, baseball bats, and golf clubs, have a "sweet spot." The simplest definition of the sweet spot is that point where a ball should be struck so as to cause the racket, bat or club to rotate about the hand that grips it and therefore impart no force or impulse to the hand. The sweet spot is also known as the center of percussion.

To model the sweet spot, consider a simple rod of mass *m*, length *L* and uniform density. The impact of the ball is modeled as a brief impulse $F\Delta t$ applied to the rod a distance *d* from the center of the rod, as shown below. For this impact point to be a sweet spot, there must be a conjugate point on the rod that does not move during the impulse—the rod rotates about an imaginary axis through that conjugate point (which is where you would grip the rod with your hand). In this simple model, there is no gravity, the rod is initially at rest, and we ignore any effects of the hand gripping the rod.

(a) For the impact point (*i.e.* sweet spot) shown below, find the location of the stationary conjugate point.

(b) Demonstrate that the work-kinetic energy theorem is obeyed for this process.



Problem 2

Consider a hydrogen atom in its ground state: $\psi_{1s}(r,\theta,\phi)=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$

(a) (3 pts) What is the expectation value of the kinetic energy in this state? Hint: the fastest way to answer this question is to use the virial theorem.

(b) What is the probability of finding the atom with a kinetic energy larger than the expectation value found in (a)?

To answer this question, consider the following:

(i) (8 pts) A measurement of kinetic energy is equivalent to a measurement of momentum. So, first find the probability density for finding the system with momentum p.

(ii) (2 pts) Find the momentum p_0 corresponding to the kinetic energy equal to its expectation value in part (a).

(iii) (7 pts) Using the results of parts (i) and (ii), determine the probability to find the system with momentum larger than p_0.

You might find this integral helpful:

$$\int_1^\infty \frac{x^2}{\left(1+x^2\right)^4}dx=\frac{3\pi-4}{192}$$

Problem 3

For a system, labeled i, which occupies energy eigenstates labeled α with arbitrary probabilities $p_{i,\alpha}$, the entropy is given by

$$S_i = -k_B \sum_\alpha p_{i,\alpha} \ln p_{i,\alpha} \,.$$

For two systems, $i = \{1, 2\}$ the probability of finding system 1 in state α and system 2 in state β is given by $p_{\alpha\beta} = p_{1,\alpha;2,\beta}$.

a) Show for two non-interacting systems, i.e. for which $p_{\alpha\beta} = p_{1,\alpha} \cdot p_{2,\beta}$, that entropy is additive:

$$S = S_1 + S_2 \,.$$

b) For two interacting systems, i.e. for which in general $p_{\alpha\beta} \neq p_{1,\alpha} \cdot p_{2,\beta}$, calculate $S - (S_1 + S_2)$ and show that the entropy obeys the inequality:

$$S \leq S_1 + S_2 \,.$$

c) Now consider one system with eigenstates α and an arbitrary probability distribution $p_{1,\alpha}$ with mean energy $\langle E \rangle = \sum_\alpha p_{1,\alpha} E_\alpha$ and compare it to a Boltzmann probability distribution

$$p_{0,\alpha} = \frac{1}{Z} \mathrm{e}^{-E_\alpha/k_B T} \,, \qquad Z = \sum_\alpha \mathrm{e}^{-E_\alpha/k_B T} \,,$$

with the same mean energy

$$\langle E \rangle = \sum_\alpha p_{0,\alpha} E_\alpha = \sum_\alpha p_{1,\alpha} E_\alpha \,.$$

Calculate $S_1 - S_0$ and show that

$$S_1 - S_0 = k_B \sum_\alpha p_{1,\alpha} \ln \left(\frac{p_{0,\alpha}}{p_{1,\alpha}} \right) \,.$$

From this result proof the following 2 statements:

(i) $S_0 \geq S_1$

(ii) $S_0 = S_1$ if and only if $p_{0,\alpha} = p_{1,\alpha} \; \forall \, \alpha$.

Hint for parts b) and c): $\ln x \leq x - 1 \quad ; \quad x \geq 0$

Problem 4

A beam of particles is subject to a simultaneous measurement of the angular momentum variables $\mathbf{L}^2$, L_z. The measurement gives pairs of values $\ell = m = 0$ and $\ell = 1,\ m = -1$ with probabilities ¾ and ¼ respectively.

(a) (2 pts) Reconstruct the state of the beam immediately before the measurement.

(b) (13 pts) The particles in the beam with $\ell = 1,\ m = -1$ are separated out and subjected to a measurement of L_x. What are the possible outcomes and their probabilities?

Hint: it may be useful to recall that the eigenstates $|1, m_x\rangle$ of the L_x observable can be expressed in the L_z-basis as:

$$|1, m_x\rangle = \sum_{m=-1}^{1} C_m(m_x)|1, m\rangle$$

where C_m are coefficients of expansion which depend on m_x.

(c) (5 pts) Construct the spatial wave functions of the states that could arise from the second measurement.

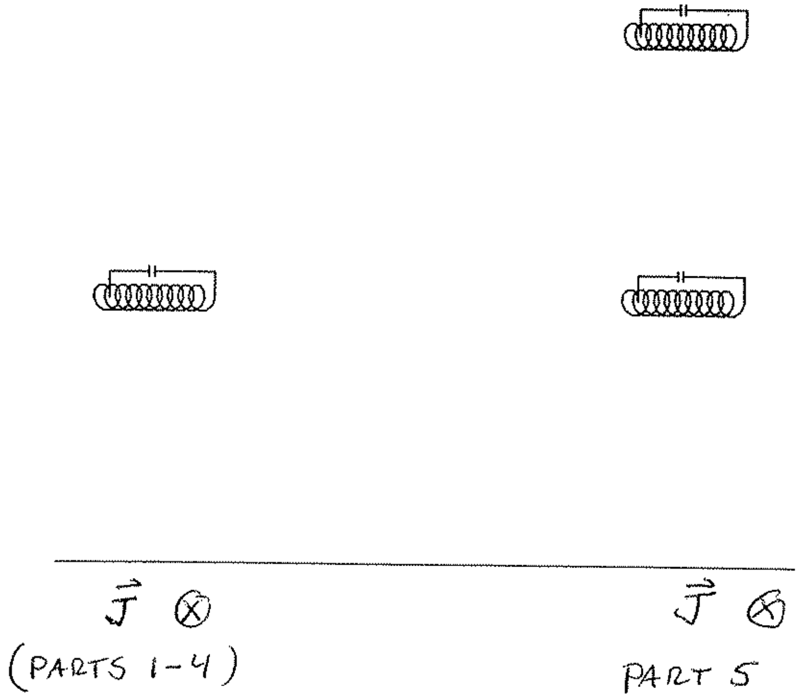
Problem 7

A small bead of mass m is free to slide on a frictionless, rigid rod. The rod pivots about one end and rotates in a vertical plane with a constant angular velocity ω. At $t = 0$, the rod is horizontal (with respect to the local uniform gravitational field) and the bead is a distance r_0 from the rotational axis and has zero radial velocity.

(a) Find the differential equation of motion of the bead.
(b) Solve the equation of motion to find the position $r(t)$ of the bead.

Problem 8

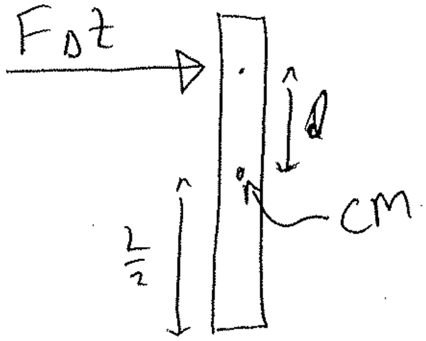
This problem explores the behavior of a small, straight coil positioned above a conducting plate with a spatially uniform, time-dependent current density parallel to the surface. The axis of the coil is parallel to the surface but perpendicular to the current density. Leads from the ends of the coil are brought together just outside the middle of the coil such that a small gap exists. When the current in the plate is varied sinusoidally, it is possible to observe a spark across this gap. In order to be able to make useful approximations, the length and width of the coil must be much smaller than the length and width of the surface of the plate. Also, the distance from the plate must be small as well.



The plate has a length L, width W and thickness T. The coil has length l and cross-sectional area a and lies a distance D above the metal surface. The coil is small compared to L and W. The coil is close to the surface so that $D << L$ and $D << W$. The number of turns N is large enough so that the usual solenoidal approximation can be used. The leads from the ends of the coil are brought together so that there is a very small gap between them, and potential at which the air in between experiences dielectric breakdown is V_o. The capacitance of the structure can be ignored. The time-dependent current in the slab is $I(t) = I_o e^{i\omega t}$.

1. What is the minimum frequency ω_s at which a spark will be created?
2. What is the self-inductance of the coil?
3. If the total resistance of the coil and the spark is R, what is the characteristic discharge time τ? What must R be such that $\tau \leq T_s$, where $T_s = 2\pi/\omega_s$?
4. The air in the interior of the coil is replaced with a material of permeability $\mu > \mu_o$. Do ω_s and τ change?
5. Suppose that a second identical coil is positioned parallel to and directly above the first coil. The distance between the two is d, and $d < D$. The coils are small enough so that $d >> l$. At the moment of dielectric breakdown, what are the magnetic dipole moments of the two coils? What is the force on each coil?

Sweet Spot

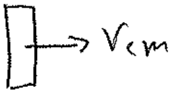


motion = motion of cm + motion about cm

$$F = \left(\frac{\Delta P}{\Delta t}\right)_{cm}$$

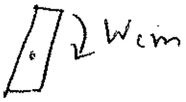
$$\Rightarrow F\Delta t = m\Delta v = m V_{cm}$$

$$\Rightarrow V_{cm} = \frac{F\Delta t}{m}$$



$$\tau = \left(\frac{\Delta L}{\Delta t}\right)_{cm} \Rightarrow dF\Delta t = I\Delta\omega = I\omega_{cm}$$

$$\Rightarrow \omega_{cm} = \frac{F\Delta t\, d}{I}$$



⇒ points below center have negative velocity component due to rotation

stationary point @ r : $v(r) = 0 = V_{cm} - r\omega_{cm}$

$$\Rightarrow r = \frac{V_{cm}}{\omega_{cm}} = \frac{F\Delta t}{m} \cdot \frac{I}{F\Delta t \cdot d} = \frac{I}{md} \text{ below CM}$$

$$I = \int r^2 dm = 2\int_0^{\frac{L}{2}} x^2 \rho\, dx = 2\,\frac{m}{L}\left[\frac{x^3}{3}\right]_0^{\frac{L}{2}} = \frac{1}{12} m L^2$$

$$\Rightarrow \boxed{r = \frac{L^2}{12d}}$$

$W = \Delta KE$ obeyed?

$$W = F(\Delta x)_{\text{point of force application}}$$

$$= F\ V_{ave} \cdot \Delta t = F\Delta t \left(\frac{V_{t=0} + V_{t=\Delta t}}{2} \right)$$

$$= \frac{F\Delta t}{2} \cdot \left(V_{cm} + d\,\omega_{cm} \right)$$

$$= \frac{F\Delta t}{2} \left(\frac{F\Delta t}{m} + d\ \frac{F\Delta t\, d}{I} \right)$$

$$= (F\Delta t)^2 \left(\frac{1}{2m} + \frac{d^2}{2I} \right)$$

$$\Delta KE = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I \omega_{cm}^2$$

$$= \frac{1}{2} m \left(\frac{F\Delta t}{m} \right)^2 + \frac{1}{2} I \left(\frac{F\Delta t\, d}{I} \right)^2$$

$$= (F\Delta t)^2 \left(\frac{1}{2m} + \frac{d^2}{2I} \right)$$

$\Rightarrow W = \Delta KE$ ok

Problem #2

Consider a hydrogen atom in its ground state

(a) Expectation value of the kinetic energy -?

Two approaches are possible:

- virial theorem $\Rightarrow \langle T \rangle = -\frac{1}{2}\langle r V' \rangle =$

$$\underset{\substack{\uparrow \\ \text{Coulomb} \\ \text{potential}}}{V \sim \frac{1}{r}} \Rightarrow \langle T \rangle = \underset{\substack{\uparrow \\ V = ar^n}}{\frac{n}{2}} \langle V \rangle \Rightarrow \quad = -\frac{1}{2}\langle V \rangle$$

$$\langle V \rangle = -2\langle T \rangle$$

$$\langle T \rangle + \langle V \rangle = \langle T \rangle - 2\langle T \rangle = -\langle T \rangle = \underbrace{E_1}_{\substack{\uparrow \\ \text{ground} \\ \text{state} \\ \text{energy}}}$$

$$\underset{\sim}{\langle T \rangle} = -E_1 = \underset{\sim}{\frac{e^2}{2a_0}}$$

- $\langle T \rangle = -\frac{\hbar^2}{2\mu}\int \psi_{1s}^* \vec{\nabla}^2 \psi_{1s}\, d^3r = \frac{e^2}{2a_0}$

(b) Probability of finding the atom with a kinetic energy $> \langle T \rangle$?

A measurement of kinetic energy is equivalent to a measurement of momentum.

Let's first find the probability density for finding the system with momentum $\vec{p}$ ⇒ ②

$$\varphi(p) = \int d^3r \, \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} \, \psi_{1s}(r) =$$

$$= \frac{2\pi}{(2\pi\hbar)^{3/2}(\pi a_0^3)^{1/2}} \int_0^\infty dr\, r^2 \int_{-1}^{1} d\cos\theta \, e^{ipr\cos\theta/\hbar} \, e^{-r/a_0} =$$

$$= \frac{1}{(2\pi)^{1/2}\hbar^{3/2}(\pi a_0^3)^{1/2}} \int_0^\infty dr\, r^2 \left[e^{i\frac{pr}{\hbar}-\frac{r}{a_0}} - e^{-i\frac{pr}{\hbar}-\frac{r}{a_0}} \right] \cdot$$

$$\cdot \frac{1}{\frac{ipr}{\hbar}} = \frac{1}{(2\pi\hbar)^{1/2}(\pi a_0^3)^{1/2}} \, \frac{-i}{p} \int_0^\infty dr\, r e^{-r/a_0} \cdot$$

$$\cdot \left[e^{i\frac{pr}{\hbar}} - e^{-i\frac{pr}{\hbar}} \right] \overset{}{\underset{\uparrow}{=}} \frac{+i}{p\,(2\pi\hbar)^{1/2}(\pi a_0^3)^{1/2}} \left[\frac{1}{\left(\frac{1}{a_0}+i\frac{p}{\hbar}\right)^2} - \frac{1}{\left(\frac{1}{a_0}-i\frac{p}{\hbar}\right)^2} \right]$$

$$\int_0^\infty r e^{-r\left(\frac{1}{a_0}\pm i\frac{p}{\hbar}\right)} dr = \frac{1}{\left(\frac{1}{a_0}\pm i\frac{p}{\hbar}\right)^2} \underbrace{\Gamma(2)}_{\shortparallel \atop 1}$$

$$= \frac{i\cdot 2}{\cancel{p}\,(2\pi\hbar)^{1/2}(\pi a_0^3)^{1/2}} \cdot \frac{-2i\frac{\cancel{p}}{\hbar}\frac{1}{a_0}}{\left(\frac{1}{a_0^2}+\frac{p^2}{\hbar^2}\right)^2} = \frac{4}{\pi\sqrt{2}}\left(\frac{a_0}{\hbar}\right)^{3/2} \frac{1}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^2}$$

The corresponding probability density is then $\mathcal{P}(p) = |\varphi(p)|^2 = \frac{8}{\pi^2}\left(\frac{a_0}{\hbar}\right)^3 \frac{1}{\left(1+\frac{a_0^2 p^2}{\hbar^2}\right)^4}$

Momentum corresponding to kinetic energy = expectation value of kinetic energy is $p_0 = \sqrt{2\mu\langle T\rangle} = \sqrt{2\mu\cdot\frac{e^2}{2a_0}} =$

$a_0 = \frac{\hbar^2}{\mu e^2} \nearrow = \frac{\hbar}{a_0}$

So, the probability to find the system with $p > p_0$ is $P = \int_{p_0}^{\infty} dp\, p^2\, \mathcal{P}(p)\cdot 4\pi =$

$$= 4\pi \int_{p_0 = \frac{\hbar}{a_0}}^{\infty} \frac{8}{\pi^2}\left(\frac{a_0}{\hbar}\right)^3 \frac{1}{\left(1+\frac{a_0^2 p^2}{\hbar^2}\right)^4}\, p^2\, dp \overset{\swarrow\ \xi = \frac{a_0 p}{\hbar}}{=}$$

$$= \frac{32}{\pi}\int_1^{\infty} \frac{\xi^2 d\xi}{(1+\xi^2)^4} = \frac{32}{\pi}\left[\frac{\pi}{32} - \frac{4+3\pi}{192}\right] = 1 - \frac{2}{3\pi} - \frac{1}{2} =$$

$$= \frac{1}{2} - \frac{2}{3\pi} \approx \underline{0.29}$$

problem 3

a) $\boxed{\text{requires ① } \sum_{\alpha} p_{\alpha} = 1}$

$$S = -k \sum_{\alpha\beta} p_{\alpha\beta} \ln p_{\alpha\beta}$$

$$= -k \sum_{\alpha\beta} p_{1\alpha} p_{2\beta} \ln(p_{1\alpha} \cdot p_{2\beta})$$

$$= -k \sum_{\alpha} \sum_{\beta} \left[p_{1\alpha} p_{2\beta} \ln p_{1\alpha} + p_{1\alpha} p_{2\beta} \ln p_{2\beta} \right]$$

$$= -k \left[\sum_{\alpha} (p_{1\alpha} \ln p_{1\alpha}) \underbrace{\sum_{\beta} p_{2\beta}}_{\overset{①}{=} 1} + \sum_{\beta} (p_{2\beta} \ln p_{2\beta}) \underbrace{\sum_{\alpha} p_{1\alpha}}_{\overset{①}{=} 1} \right.$$

$$= S_1 + S_2 \quad \square$$

b) $\boxed{\text{requires ② } p_{1\alpha} = \sum_{\beta} p_{\alpha\beta} \,,\; p_{2\beta} = \sum_{\alpha} p_{\alpha\beta}}$

$$\frac{1}{k}\left[S - (S_1 + S_2)\right] =$$

$$= -\sum_{\alpha\beta} p_{\alpha\beta} \ln p_{\alpha\beta} + \underbrace{\sum_{\alpha} p_{1\alpha} \ln p_{1\alpha}}_{\overset{②}{=} \sum_{\alpha} \sum_{\beta} p_{\alpha\beta} \ln p_{1\alpha}} + \underbrace{\sum_{\beta} p_{2\beta} \ln p_{2\beta}}_{\overset{②}{=} \sum_{\alpha\beta} p_{\alpha\beta} \ln p_{2\beta}}$$

$$= \sum_{\alpha\beta} p_{\alpha\beta} \ln\left(\frac{p_{1\alpha} p_{2\alpha}}{p_{\alpha\beta}}\right)$$

$\ln x \leq x - 1$

$$\leq \sum_{\alpha\beta} p_{\alpha\beta} \left(\frac{p_{1\alpha} p_{2\alpha}}{p_{\alpha\beta}} - 1\right) = \sum_{\alpha\beta} (p_{1\alpha} p_{2\beta} - p_{\alpha\beta})$$

$$\overset{②}{=} \sum_{\alpha} (p_{1\alpha} - p_{1\alpha}) = 0$$

$$\Rightarrow S \leq S_1 + S_2 \quad \square$$

c) $\boxed{\begin{array}{l}\text{requires } ③ \ \sum_\alpha (p_{1\alpha} - p_{0\alpha}) E_\alpha = 0 \quad (\text{given}) \\ \qquad ④ \ \ln p_{0\alpha} = -\beta E_\alpha - \ln Z \end{array}}$

$$\frac{S_0 - S_1}{k} = \sum_\alpha \left(p_{1\alpha} \ln p_{1\alpha} - p_{0\alpha} \ln p_{0\alpha} \right)$$

$$= \sum_\alpha \left[p_{1\alpha} \ln p_{1\alpha} - (p_{0\alpha} + p_{1\alpha} - p_{1\alpha}) \ln p_{0\alpha} \right]$$

$$= \sum_\alpha \left[p_{1\alpha} \ln \frac{p_{1\alpha}}{p_{0\alpha}} + \underbrace{(p_{1\alpha} - p_{0\alpha}) \underbrace{\ln p_{0\alpha}}_{\overset{④}{=} -\beta E_\alpha - \ln Z \ \hookrightarrow \text{const}}}_{= 0 \quad ①, ③} \right.$$

$$\Rightarrow S_1 - S_0 = k \sum_\alpha p_{1\alpha} \ln \frac{p_{0\alpha}}{p_{1\alpha}} \qquad (*)$$

$$\leq k \sum_\alpha (p_{0\alpha} - p_{1\alpha}) \overset{①}{=} 0$$

$$\Rightarrow \quad S_0 \geq S_1 \quad \square \qquad (i)$$

(ii) from (*) since $p_{1\alpha}$ is arbitrary

$S_0 = S_1$ iff $p_{0\alpha} = p_{1\alpha} \ \forall \alpha$

$\rightarrow S_1 - S_0 = k \sum_\alpha p_{1\alpha} \ln 1 = 0 \quad \square$

QM Problem #4

A beam of particles is subject to a simultaneous measurement of the angular momentum variables $\vec{L}^2$ & L_z. The measurement gives pairs of values $l = m = 0$ and $l = 1$, $m = -1$ with probabilities $3/4$ & $1/4$, respectively.

(a) State of the beam before the measurement

$$|\Psi\rangle = \frac{\sqrt{3}}{2}|0\ 0\rangle + \frac{1}{2}e^{i\underset{\substack{\uparrow\\ \text{arb. phase}}}{\alpha}}|1\ -1\rangle$$

(b) We are in the state $|1\ -1\rangle$. Now measure L_x

Possible outcomes: $|1, m_x = \pm 1\rangle$ & $|1, m_x = 0\rangle$

$$|1, m_x\rangle \underset{\substack{\uparrow\\ \text{expand}\\ \text{in the } L_z\text{-basis}}}{=} C_1^{(m_x)}|1\ 1\rangle + C_0^{(m_x)}|1\ 0\rangle + C_{-1}^{(m_x)}|1\ -1\rangle$$

Need to find relationships among C_1, C_0, & C_{-1} for each m_x

Act with $L_x := \frac{L_+ + L_-}{2}$ on $|1, m_x>$: ②

$$L_x |1, m_x> = \hbar m_x |1, m_x> = \frac{L_+ + L_-}{2}\Big[C_1(m_x)|1\,1>+ C_0(m_x)|1\,0> + C_{-1}(m_x)|1\,-1>\Big] =$$

$$= \frac{\hbar}{2}\sqrt{2}\,C_1(m_x)|1\,0> + \frac{\hbar}{2}\sqrt{2}\,C_0(m_x)\Big[|1\,1> + |1\,-1>\Big] + \frac{\hbar}{2}\sqrt{2}\,C_{-1}(m_x)|1\,0> =$$

$$= \hbar m_x\Big[C_1(m_x)|1\,1> + C_0(m_x)|1\,0> + C_{-1}(m_x)\cdot|1\,-1>\Big]$$

$$\Downarrow$$

$$\frac{\sqrt{2}}{2}C_0(m_x) = m_x C_{-1}(m_x) \Leftarrow |1\,-1>$$

$$\frac{\sqrt{2}}{2}C_0(m_x) = m_x C_1(m_x) \Leftarrow |1\,1>$$

$$\frac{\sqrt{2}}{2}\big(C_1(m_x) + C_{-1}(m_x)\big) = m_x C_0(m_x) \Leftarrow |1\,0>$$

So, $C_0(0) = 0$, $\frac{\sqrt{2}}{2}C_0(1) = \underbrace{C_{-1}(1)}_{\overset{\shortparallel}{C_1(1)}}$; $\frac{\sqrt{2}}{2}C_0(-1) = -C_{-1}(-1) = -C_1(-1)$

$C_1(0) = -C_{-1}(0)$

Then,

$$|1, m_x=1\rangle = C_1(1)\,|1\,1\rangle + \frac{2}{\sqrt{2}}C_1(1)\,|1\,0\rangle +$$

$$+ C_1(1)\,|1\,-1\rangle = \underbrace{C_1(1)}_{\overset{\parallel}{\frac{1}{2}}}\Big[|1\,1\rangle + \underbrace{\frac{2}{\sqrt{2}}}_{=\sqrt{2}}|1\,0\rangle + |1\,-1\rangle\Big]$$

$\leftarrow$ from normaliz.

$$|1, m_x=0\rangle = C_1(0)\,|1\,1\rangle + \underbrace{C_{-1}(0)}_{\overset{\parallel}{C_1(0)}}|1\,-1\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(|1\,1\rangle - |1\,-1\rangle\right)$$

$\uparrow$ from normalization

$$|1, m_x=-1\rangle \underset{\uparrow\ \text{similarly}}{=} \frac{1}{2}\left(|1\,1\rangle - \sqrt{2}\,|1\,0\rangle + |1\,-1\rangle\right)$$

Combine $|1, m_x\rangle$ to get $|1,-1\rangle$ in the L_z-bas.

$$|1, m_x=1\rangle + |1, m_x=-1\rangle = |1,1\rangle + |1,-1\rangle$$

$$|1, m_x=0\rangle = \frac{1}{\sqrt{2}}\left(|1, m_x=1\rangle + |1, m_x=-1\rangle - 2\,|1\,-1\rangle\right) \Rightarrow$$

$$\underline{|1,-1\rangle = \frac{1}{\sqrt{2}}\left(|1, m_x=1\rangle + |1, m_x=-1\rangle\right) - \frac{1}{\sqrt{2}}\,|1, m_x=0\rangle}$$

Now read off the probabilities of each outcome

$$\Swarrow$$

$$\mathcal{P}(m_x=\pm 1)=\frac{1}{4}\quad ;\quad \mathcal{P}(m_x=0)=\frac{1}{2}$$

(c) Construct spatial wave functions that could arise after the 2nd measurement

$$|1,\ m_x=\pm 1\rangle \underset{\substack{\uparrow\\ \text{from last page}}}{=} \frac{1}{2}\left[|1\,1\rangle \pm \sqrt{2}\,|1\,0\rangle + |1\,-1\rangle\right]$$

$$\Downarrow$$

$$Y_1^{m_x=\pm 1} = \frac{1}{2}\left[Y_1^1 + Y_1^{-1} \pm \sqrt{2}\,Y_1^0\right] = \frac{1}{2}\left[\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\cdot\left(e^{-i\varphi}-e^{i\varphi}\right) \pm \sqrt{2}\,\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta\right] = \sqrt{\frac{3}{8\pi}}\left(\pm\cos\theta - i\sin\varphi\sin\theta\right)$$

$$Y_1^{m_x=0} = \frac{1}{\sqrt{2}}\left(Y_1^1 - Y_1^{-1}\right) = -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\varphi$$

problem 5

a) $f \sim p$ but f increases if L increases → opposite sign of $-pdV$

$L \sim V$

$$\rightarrow dU = TdS + f\,dL$$

$$dS = \frac{1}{T}dU - \frac{f}{T}dL$$

b) $S(T,L) \longrightarrow \left.\frac{\partial S}{\partial L}\right|_T = ?$

Maxwell relation, T,L point to Helmholtz free energy:

$$F(T,L) = U - TS \longrightarrow dF = dU - TdS - SdT$$
$$= -SdT + f\,dL$$
$$= \left.\frac{\partial F}{\partial T}\right|_L dT + \left.\frac{\partial F}{\partial L}\right|_T dL$$

$$\Rightarrow \left.\frac{\partial F}{\partial T}\right|_L = -S \; , \; \left.\frac{\partial F}{\partial L}\right|_T = f$$

$$\Rightarrow -\left.\frac{\partial S}{\partial L}\right|_T = +\left.\frac{\partial f}{\partial T}\right|_L = 2aT(L-L_0)$$

$$\left.\frac{\partial S}{\partial L}\right|_T = -2aT(L-L_0)$$

c) $S(T,L)$

$$\longrightarrow dS = \left.\frac{\partial S}{\partial T}\right|_L dT + \left.\frac{\partial S}{\partial L}\right|_T dL$$
$$= \frac{C_L}{T}dT + \left.\frac{\partial S}{\partial L}\right|_T dL$$

$$① = \int_{S(T,L_0)}^{S(T,L)} dS = S(T,L) - S(T,L_0) \underset{\substack{\uparrow \\ \text{const}\,T}}{=} 0 + \int_{L_0}^{L} \frac{\partial S}{\partial L}\Big|_T dL = -aT(L-L_0)$$

$$② = \int_{S(T_0,L_0)}^{S(T,L_0)} dS = S(T,L_0) - S(T_0,L_0) \underset{\substack{\uparrow \\ \text{const}\,L=L_0}}{=} \int_{T_0}^{T} b\, d\tilde{T} + 0 = b(T-T_0)$$

$$① + ② \Rightarrow S(T,L) = S(T_0,L_0) + b(T-T_0) - aT(L-L_0)^2$$

d) $C_L = T \frac{\partial S}{\partial T}\Big|_L = bT - aT(L-L_0)^2$

verify $C_{L=L_0} = bT$ ✓

note: since $C_L > 0 \ \forall T$

$\rightarrow b - a(L-L_0)^2 > 0 \quad \forall T$

e) adiabatic : $\text{đ}Q = 0 \rightarrow dS = 0$

$\Rightarrow S = \text{const}$

$\Rightarrow S(T_i, L_i) - S(T_f, L_f) = 0$

$= b(T_i - T_f) - a\left(T_i(L_i-L_0)^2 - T_f(L_f-L_0)^2\right)$

$\Leftrightarrow T_i\left(b - a(L_i-L_0)^2\right) = T_f\left(b - a(L_f-L_0)^2\right)$

$$\rightarrow T_f = T_i \frac{b - a(L_i-L_0)^2}{b - a(L_f-L_0)^2}$$

$\rightarrow T_f > T_i$ temperature increases because work is performed and no heat can escape.

Problem 5

For a stretched plastic rod, length L, the tension force f is given by

$$f = aT^2(L - L_0)\,,$$

where a is a positive constant and L_0 is the length of the unstretched rod. For $L = L_0$ the heat capacity of this plastic rod is given by

$$C_L|_{L=L_0} = bT\,,$$

where b is a positive constant.

a) Write down the fundamental thermodynamic relation for this system:

$$\mathrm{d}U = \ldots$$

and express $\mathrm{d}S$ in terms of $\mathrm{d}U$ and $\mathrm{d}L$.

b) Consider the entropy as a function of temperature T and length L, $S(T, L)$. Calculate

$$\left.\frac{\partial S}{\partial L}\right|_T .$$

c) Assume that $S(T_0, L_0)$ is known, calculate $S(T, L)$ for general T and L.

d) Calculate the heat capacity C_L for general L.

e) Starting with a rod with initial length L_i and temperature T_i, the rod is adiabatically stretched to length $L_f > L_i$. What is the final temperature T_f? Is T_f larger or smaller than T_i ?

Problem 6

This problem is concerned with the electrostatic potential arising from sections of a surface over which the potential is specified. Each figure shows two thin conducting sections of a single spherical surface of radius R centered at the origin of the coordinate system. In terms of the angle θ measured from the z axis, the upper surface runs from 0 to θ and the lower surface from $180° - \theta$ to $180°$. The electrostatic potential of the upper shell is $+V$, and that of the lower shell is $-V$. In answering the following questions, clearly state and justify all approximations that you use.

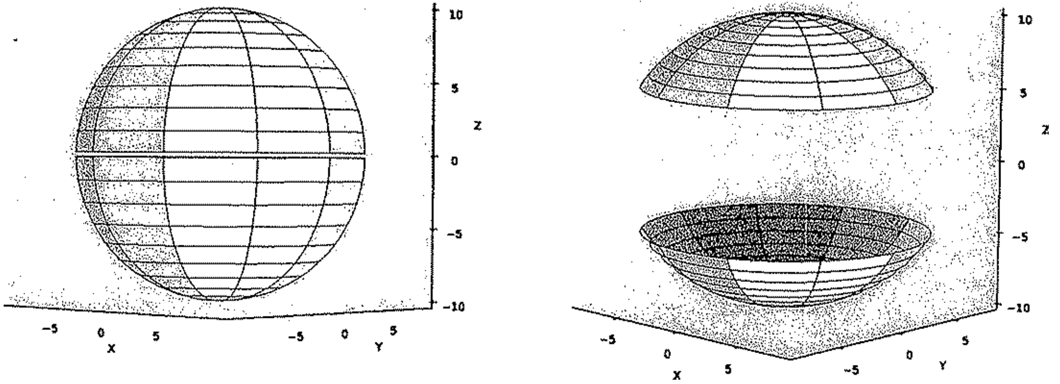


Figure 1: Sections of spherical shells for $\theta = 89°$ and $\theta = 60°$.

1. For $\theta = 89°$, find the potential everywhere within the sphere, that is, for $r < R$. What is the surface charge density distribution over the interior of the shells?
2. For $\theta = 89°$, find the potential everywhere outside the sphere, that is, for $r > R$. What is the surface charge density distribution over the exterior of the shell? At what distance from the origin does the leading term in the potential account for 90% of the total?
3. What is the capacitance of this structure?
4. Suppose that the lower shell is replace by a thick, conducting slab of infinite extent in the xy plane. Find the potential inside and outside the shell.
5. Returning to the original geometry, how does the potential for $r > R$ change if a thin layer, of thickness $d << R$, of a dielectric material is deposited on the outside of the shells?
6. Suppose that the angular range of each shell is reduced from 89° to 60°. What is the potential for $r > R$? Clearly state any ambiguities or approximations necessary.

This problem is concerned with the electrostatic potential arising from sections of a surface over which the potential is specified. Each figure shows two thin conducting sections of a single spherical surface of radius R centered at the origin of the coordinate system. In terms of the angle θ measured from the z axis, the upper surface runs from 0 to θ and the lower surface from $180^\circ - \theta$ to 180°. The electrostatic potential of the upper shell is $+V$, and that of the lower shell is $-V$. In answering the following questions, clearly state and justify all approximations that you use.

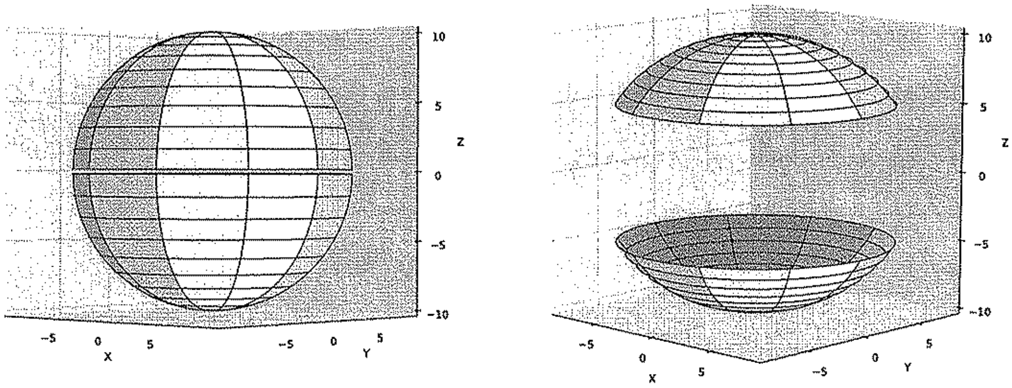


Figure 1: Sections of spherical shells for $\theta = 89^\circ$ and $\theta = 60^\circ$.

1. For $\theta = 89^\circ$, find the potential everywhere within the sphere, that is, for $r < R$. What is the surface charge density distribution over the interior of the shells?

Answer (6 points): The general solution is

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty} \left[A_l r^l P_l(\cos\theta) + \frac{B_l}{r^{l+1}} P_l(\cos\theta) \right] .$$

Begin by letting the gap between the hemispheres become infinitesimally small. This is a necessary first step in order to use the orthogonality relation for Legendre polynomials.

$$\int_{-1}^{1} P_l(x) P_k(x) dx = \frac{2}{2l+1}\delta_{lk} \Rightarrow \int_0^{\pi} \Phi(r,\theta) P_l(\cos\theta) \sin\theta d\theta = \frac{2}{2l+1}\left[A_l r^l + \frac{B_l}{r^{l+1}} \right]$$

$$r < R \Rightarrow \text{ all } B_l = 0$$

The potential over the shells, $\Phi(R,\theta) = V$ for $0 \le \theta < \pi/2$ and $-V$ for $\pi/2 < \theta \le \pi$, is antisymmetric over $[0,\pi]$, so only P_l functions which are antisymmetric (odd l values) will appear in Φ. Hence,

$$\int_0^{\pi} \Phi(R,\theta) P_l(\cos\theta) \sin\theta d\theta = V\left[\int_0^1 P_l(x) dx - \int_{-1}^{0} P_l(x) dx \right] = \frac{2}{2l+1} A_l R^l$$

Explicit integration for $l = 1, 3$ using $P_1(x) = x$ and $P_3(x) = (5x^3 - 3x)/2$ yields

$$A_1 = \frac{3V}{2R} \text{ and } A_3 = -\frac{7V}{8R^3} \Rightarrow \Phi(r,\theta) \approxeq V\left[\frac{3}{2}\frac{r}{R} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) \right] .$$

Notice that many more terms are needed to come close to satisfying the boundary condition $\Phi(R, 0 \leq \theta \leq \pi/2) = V$, just as a square wave of frequency ω is described by a slowly converging sum of $\sin(2n+1)\omega t$ functions. The charge density over the conducting interior surface of the sphere is

$$\sigma_{in} = -2\epsilon_\circ \frac{\partial \Phi}{\partial n} = 2\epsilon_\circ \frac{\partial}{\partial r}\Phi(r,\theta)|_{r=R} = 2\epsilon_\circ \left[A_1 P_1(\cos\theta) + 3A_3 R^2 P_3(\cos\theta)\right]$$

$$\sigma_{in} = \frac{3\epsilon_\circ V}{R}\left[P_1(x) - \frac{7}{4}P_3(x)\right] = \frac{3\epsilon_\circ V}{8R}\left[29x - 35x^3\right].$$

2. For $\theta = 89°$, find the potential everywhere outside the sphere, that is, for $r > R$. What is the surface charge density distribution over the exterior of the shell? At what distance from the origin does the leading term in the potential account for 90% of the total?

Answer (6 points): Outside the sphere $r > R$, and all $A_l = 0$ so that

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta).$$

$$\int_{-1}^{1} P_l(x)P_k(x)dx = \frac{2}{2l+1}\delta_{lk} \Rightarrow \int_0^\pi \Phi(r,\theta)P_l(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\frac{B_l}{r^{l+1}}$$

The potential over the shells, $\Phi(R,\theta) = V$ for $0 \leq \theta < \pi/2$ and $-V$ for $\pi/2 < \theta \leq \pi$, is antisymmetric over $[0,\pi]$, so only P_l functions which are antisymmetric (odd l values) will appear in Φ. Hence,

$$\int_0^\pi \Phi(R,\theta)P_l(\cos\theta)\sin\theta d\theta = V\left[\int_0^1 P_l(x)dx - \int_{-1}^0 P_l(x)dx\right] = \frac{2}{2l+1}\frac{B_l}{R^{l+1}}$$

Explicit integration for $l = 1, 3$ using $P_1(x) = x$ and $P_3(x) = (5x^3 - 3x)/2$ yields

$$B_1 = \frac{3VR^2}{2} \text{ and } B_3 = -\frac{7VR^4}{8} \Rightarrow \Phi(r,\theta) \approxeq V\left[\frac{3}{2}\frac{R^2}{r^2}P_1(\cos\theta) - \frac{7}{8}\frac{R^4}{r^4}P_3(\cos\theta)\right].$$

The charge density over the conducting exterior surface of the sphere is

$$\sigma_{ext} = -2\epsilon_\circ \frac{\partial \Phi}{\partial n} = -2\epsilon_\circ \frac{\partial \Phi(r,\theta)}{\partial r}|_{r=R} = 2\epsilon_\circ\left[\frac{2B_1}{R^3}P_1(\cos\theta) + \frac{4B_3}{R^5}P_3(\cos\theta)\right]$$

$$\sigma_{ext} = \frac{\epsilon_\circ V}{R}\left[2P_1(x) - 7P_3(x)\right] = \frac{5\epsilon_\circ V}{2R}\left[5x - 7x^3\right],$$

which is less than the interior charge density. For the leading term in Φ to be at least 9 times the second term

$$\frac{3R^2}{2r^2} \geq \frac{9 \cdot 7R^4}{2 \cdot 8r^4} \Rightarrow r \geq \sqrt{\frac{21}{8}}R.$$

3. What is the capacitance of this structure?

Answer (3 points): Finding the capacitance $C = Q/\Delta\Phi = (Q_{in} + Q_{ext})/2V$ requires integration of surface charge densities over the interior and exterior surfaces of only one hemisphere.

$$Q = \int\limits_{\substack{upper\\hemisphere}} [\sigma_{in}(\theta) + \sigma_{ext}(\theta)]\, R^2 d\Omega = 2\pi R^2 \int\limits_0^{\pi/2} [\sigma_{in}(\theta) + \sigma_{ext}(\theta)] \sin\theta d\theta\,.$$

$$Q_{in} = 2\pi R^2 \int\limits_0^1 \frac{3\epsilon_\text{o} V}{R}\left[29x - 35x^3\right] dx = \frac{69\pi\epsilon_\text{o} VR}{2} \text{ and } Q_{ext} = 2\pi R^2 \int\limits_0^1 \frac{5\epsilon_\text{o} V}{R}\left[5x - 7x^3\right] dx = \frac{15\pi\epsilon_\text{o} VR}{2}$$

$$C = 42\pi\epsilon_\text{o} R\,.$$

4. Suppose that the lower shell is replaced by a thick, conducting slab of infinite extent in the xy plane held at potential $\Phi = 0$. Find the potential inside and outside the upper shell.

Answer (1 points): The solutions for the potentials inside and outside the sphere are 0 in the xy plane, so the solutions are the same for this situation. The surface charge density is $-2\epsilon_\text{o}\frac{\partial\Phi}{\partial z}$.

5. Returning to the original geometry, how does the potential for $r > R$ change if a thin layer, of thickness $d << R$, of a dielectric material is deposited on the outside of the shells?

Answer (4 points): There are additional boundary conditions at the dielectric/air interface: the normal component of $\vec{D}$, the tangential component of $\vec{E}$ and Φ are continuous. Outside the dielectric, the simplest approximation for $\Phi(r > R, \theta)$ is

$$\Phi'(\vec{r}) = \sum_{l=0}^{\infty} \frac{B_l'}{r^{l+1}} P_l(\cos\theta)\,,$$

which, when used with the solution to question 2, cannot meet all the boundary conditions. A general approach for small d would be to base the solution on two unknown surface charge densities, one over the metal surface, σ_m, and the other over the outer and inner surfaces of the dielectric, σ_d. The surface charge density over the conducting surface will not be the same as σ_{ext} given previously. For $d << R$, consider a small patch of the dielectric/conductor structure over which the field is approximated as being purely radial. Then,

$$D_{air} = D_{dielectric} \Rightarrow E_{air} = (\epsilon/\epsilon_\text{o}) E_{dielectric} = (\epsilon/\epsilon_\text{o})\left[\frac{\sigma_m}{2\epsilon} - \frac{\sigma_d}{\epsilon}\right].$$

For $r >> R$, the field will be dipolar and proportional to E_{air}.

6. Suppose that the angular range of each shell is reduced from 89° to 60°. What is the potential for $r > R$? Clearly state any ambiguities or approximations.

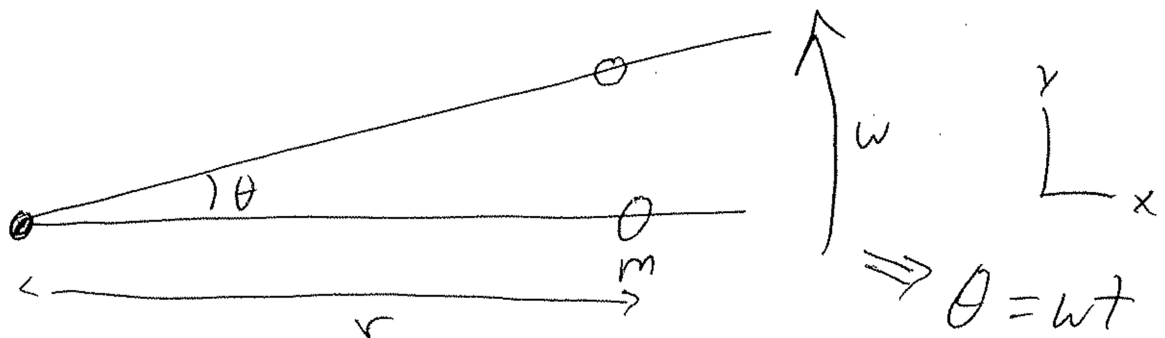
Answer (2 points): The greatest difficulty is that the orthogonality relation

$$\int_{-1}^{1} P_l(x)P_k(x)dx = \frac{2}{2l+1}\delta_{lk}$$

can no longer be used even in an approximate sense. The potential over the missing section of the spherical surface is unknown, but we do know that the potential and field are continuous over that surface. All we can state is that for large r, the solution will be of the form

$$\Phi(\vec{r}) = \frac{B_1}{r^2}P_1(\cos\theta)\,.$$

Rotating Rod w/ Bead



$\Rightarrow \underline{\theta = \omega t}$

$$x = r\cos\theta \;\rightarrow\; \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$y = r\sin\theta \;\rightarrow\; \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$L = T - U = \tfrac{1}{2}mv^2 - mgy = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta = \frac{m}{2}(\dot{r}^2 + r^2\omega^2) - mgr\sin(\omega t)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$\frac{\partial L}{\partial r} = mr\omega^2 - mg\sin(\omega t)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\Rightarrow mr\omega^2 - mg\sin(\omega t) - m\ddot{r} = 0$$

$$\Rightarrow \boxed{\ddot{r} = \omega^2 r - g\sin(\omega t)}$$

Homogeneous solution $(w=0)$: $\ddot{r} = w^2 r$

$$\Rightarrow r = a\cosh wt + b\sinh wt$$

Inhomogeneous solution try $r = A\sin wt$

$$\Rightarrow -w^2 A\sin wt = w^2 A\sin wt - g\sin wt$$

$$\Rightarrow A = \frac{g}{2w^2}$$

$$\Rightarrow r(t) = \frac{g}{2w^2}\sin wt + a\cosh wt + b\sinh wt$$

find a, b w/ initial conditions

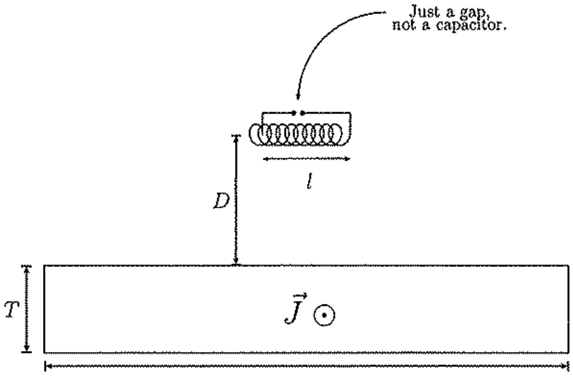
$r(0) = r_0$, $\dot{r}(0) = 0$

$$\Rightarrow r_0 = a \qquad \rightarrow a = r_0$$

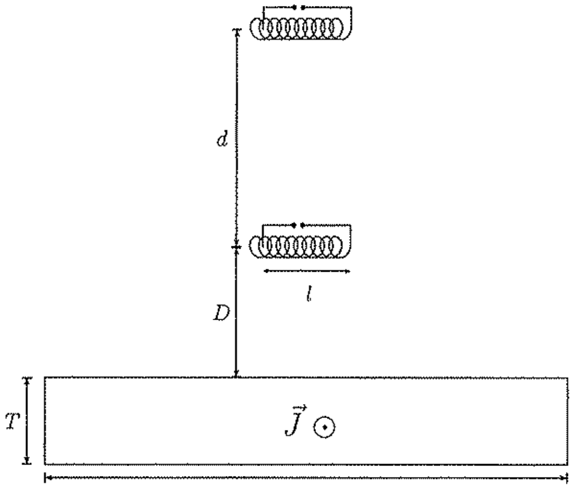
$$0 = \frac{g}{2w} + wb \qquad \rightarrow b = -\frac{g}{2w^2}$$

$$\Rightarrow \boxed{r(t) = r_0\cosh(wt) + \frac{g}{2w^2}\left[\sin(wt) - \sinh(wt)\right]}$$

This problem explores the behavior of a small, straight coil positioned above a conducting plate with a spatially uniform, time-dependent current density parallel to the surface. The axis of the coil is parallel to the surface but perpendicular to the current density. Leads from the ends of the coil are brought together just outside the middle of the coil such that a small gap exists. When the current in the plate is varied sinusoidally, it is possible to observe a spark across this gap. In order to be able to make useful approximations, the length and width of the coil must be much smaller than the length and width of the surface of the plate. The distance from the plate must be small as well.



(a) For parts 1 to 4.



(b) For part 5.

The plate has a length L, width W and thickness T. The coil has length l and cross-sectional area a and lies a distance D above the metal surface. The coil is small compared to L and W. The coil is close to the surface so that $D << L$ and $D << W$. The number of turns N is large enough so that the usual solenoidal approximation can be used. The leads from the ends of the coil are brought together so that there is a very small gap between them, and potential at which the air in between experiences dielectric breakdown is V_o. The capacitance of the structure can be ignored. The time-dependent current in the slab is $I(t) = I_o e^{i\omega t}$.

1. What is the minimum frequency ω_s at which a spark will be created?

Answer (6 points): Since D is small compared to the length and width of the plate, edge effects can be ignored and $\vec{B}$ assumed to be uniform over the length of the coil. Choosing, $\vec{J} = J\hat{z}$ and $\hat{y}$ to lie in the plane of the plate,

$$\vec{\nabla} \times \vec{B} = \mu_\circ J \text{ and } \vec{J} = \hat{z}I/TW \Rightarrow \vec{B} = \hat{y}\mu_\circ I/2W\,.$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \int \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} N \int \vec{B} \cdot \vec{da} = -aN\frac{\partial B}{\partial t}$$

$$\Delta\Phi(t) = \frac{a\mu_\circ N}{2W}\frac{\partial I}{\partial t} = \frac{i\omega a\mu_\circ N}{2W} I(t) \Rightarrow \omega_s = \frac{2WV_\circ}{a\mu_\circ I_\circ N}$$

2. What is the self-inductance of the coil?

Answer (5 points): Given a current I in the coil and assuming that the coil is close to an infinite solenoid, Ampere's Law yields $B(t) = \mu_\circ n I(t)$ in the interior, where $n = N/l$. Then, application of Faraday's Law yields

$$\Delta\Phi(t) = aN\frac{\partial B}{\partial t} = \frac{\mu_\circ N^2 a}{l}\frac{dI}{dt} \Rightarrow L = \frac{\mu_\circ N^2 a}{l}$$

3. If the total resistance of the coil and the spark is R, what is the characteristic discharge time τ? What must R be such that $\tau \leq t_s$, where $t_s = 2\pi/\omega_s$?

Answer (4 points):

$$L\frac{dI}{dt} = -IR \Rightarrow \frac{dI}{dt} = -\frac{R}{L}I \Rightarrow I(t) = I_\circ e^{-\frac{R}{L}t} \Rightarrow \tau = \frac{L}{R}$$

$$t_s = \frac{a\mu_\circ I_\circ N}{4\pi V_\circ W} \text{ and } \tau = \frac{\mu_\circ N^2 a}{Rl} \Rightarrow R \approx \frac{4\pi V_\circ W N}{I_\circ l}$$

4. The air in the interior of the coil is replaced with a rod of material of permeability $\mu > \mu_\circ$. Do ω_s and τ change?

Answer (2 points): The exact solution for $\vec{B}$ within the coil is not easy to determine, but approximately, B increases by the factor $\mu/\mu_\circ$. So, $\omega_s(\mu) = (\mu_\circ/\mu)\omega_s(\mu_\circ)$, and since $L = \mu N^2 a/l$, we find that $\tau(\mu) = (\mu/\mu_\circ)\tau(\mu_\circ)$.

5. Suppose that a second identical coil is positioned parallel to and directly above the first coil. The distance between the two is d, and $d < D$. The coils are small enough so that $d >> l$. At the moment of dielectric breakdown, what are the magnetic dipole moments of the two coils? What is the force on each coil?

Answer (3 points): To be precise, the self-consistent solution for $\vec{m}$ must include the effect of the magnetic dipole field of the other coil. However, assuming the coils are sufficiently far apart, at the moment of breakdown, $I = V_\circ/R$ and $\vec{m} = -Ia\hat{y}$ for each coil. For the magnetic dipole potential $\Phi_m(\vec{r}) = m\cos\theta/r^2$, where θ is measured relative to the direction of $\vec{m}$, the field at the position of the other coil is $\vec{B}_m(r = d, \theta = \pi/2) = -\vec{\nabla}\Phi_m(r = d, \theta = \pi/2) = \hat{y}m/d^3$. The potential energy is $U = -\vec{m}\cdot\vec{B}_m = m^2/d^3$, and the force is $F = 3m^2/d^4$ in the direction away from the other coil. Since the field from the plate is uniform and oriented along the axis of each coil, there is no force or torque on either dipole due to this field. Also, since the magnetic dipole moments are parallel, there is no torque on either dipole.