OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #109

September 27 and 28, 2010

Comprehensive examination for Fall 2010

PART 1, Monday, September 27, 9:00 am

General Instructions

This Comprehensive Examination for Fall 2010 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Monday, September 27, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:00 pm on the same day and will also last three hours. The third and fourth parts will be administered on Tuesday, September 28, at 9:00 am and 1:00 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam. Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

1 Physical constants

fine structure constant : $\alpha = \frac{e^2}{4\pi\epsilon_o \hbar c} = \frac{1}{137}$

Rydberg energy : $E_o = \frac{m_e e^4}{2\hbar^2 (4\pi\epsilon_o)^2} = \frac{m_e c^2 \alpha^2}{2}$

Bohr magneton : $\mu_B = \frac{e\hbar}{2m_e}$

Bohr radius : $a_o = \frac{4\pi\epsilon_o \hbar^2}{m_e e^2}$

2 Vector calculus relationships

Triple products:

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$

Product rules:

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \\
&\quad + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \\
\nabla \cdot (\phi \mathbf{A}) &= \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi \\
\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A} \nabla \cdot \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{A} + \\
&\quad + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}
\end{aligned}$$

Second derivatives:

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \\
\nabla \cdot (\nabla \times \mathbf{A}) &= 0
\end{aligned}$$

Green's theorem:

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) dV = \oint_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{S}$$

Spherical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) \\
&\quad + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right)\hat{\mathbf{r}} \\
&\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(r A_\phi)\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\phi} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) \\
&\quad + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} \\
&\quad + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} \\
&\quad + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)\hat{\mathbf{z}} \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

3 Quantum mechanics

Raising and lowering operators
for angular momentum:

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\, |j, m \pm 1\rangle$$

Perturbation theory for nondegenerate states:

$$E_n \approx E_n^0 + \langle n| V |n\rangle + \sum_{m \neq n} \frac{|\langle n| V |m\rangle|^2}{E_n^0 - E_m^0} + \cdots$$

Harmonic oscillator: $[a, a^\dagger] = 1$

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}} \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}} \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned}$$

4 Electromagnetism

Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Magnetic dipole field:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}$$

Energy density: $U = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$
Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

General solutions of Laplace's equation

in cylindrical coordinates (independent of z):

$$\begin{aligned} \Phi(\rho, \phi) = {} & a_o \log(\rho) \\ & + \sum_{n=1}^{\infty}\left(\frac{a_n}{\rho^n} + b_n\rho^n\right)(c_n \cos n\phi + d_n \sin n\phi) \end{aligned}$$

in spherical coordinates:

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta, \phi)$$

$$\Phi(r, \theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(with azimuthal symmetry)

5 Useful math formulas

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta)$$

$$\int_{-\infty}^{\infty} e^{ixy}dy = 2\pi\delta(x)$$

$$\int_0^{\infty} x^n e^{-x}dx = n!\text{, integer } n$$

$$(1+x)^n = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^k$$

$$\log(n!) \approx \frac{1}{2}\log(2\pi n) + n\log(n) - n$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta', \phi')Y_{lm}(\theta, \phi)$$

$$\frac{1}{|\mathbf{x} - r'\hat{\mathbf{z}}|} = \sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Spherical Bessel functions:

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$

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Legendre polynomials:

$$P_0(x) = 1 \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

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$$P_l^m(x) = \left(1 - x^2\right)^{m/2}\frac{d^m P_l}{dx^m}$$

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$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{i2\phi}$$

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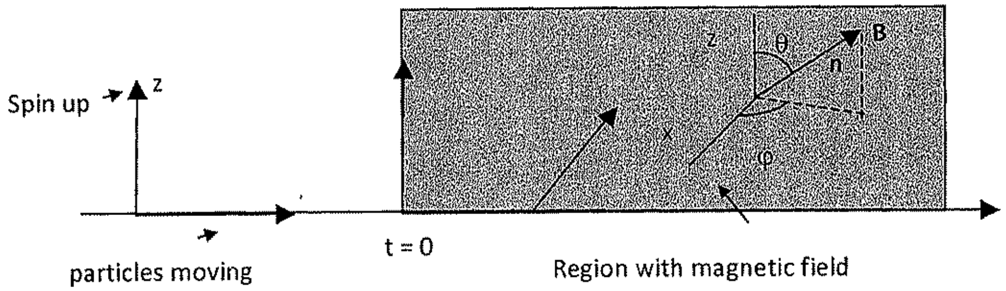
Problem #1 (QM)

A beam of neutrons with spin up along the positive z-axis enters (at time t = 0) region where there is a uniform magnetic field **B** (see Figure). The Hamiltonian of interaction between the particle and the magnetic field is $\mathbf{H} = -\boldsymbol{\mu}\cdot\mathbf{B} = 2\omega\mathbf{S}\cdot\mathbf{n} = 2\omega S_n$, where **S** is the particle's spin operator, **n** is the direction of the magnetic field $\mathbf{n} = \mathbf{i}\cos\varphi\sin\theta + \mathbf{j}\sin\varphi\sin\theta + \mathbf{k}\cos\theta$, and $\omega = \mu B/\hbar$. Ignore the spatial degrees of freedom.

1) Write down the state of the system at time t = 0 and time evolution operator.
2) Find matrix representation of the operator S_n in the S_z-basis.
3) Using the result of part 2), find matrix representation of the time evolution operator.
4) Find the state of the system at an arbitrary time t > 0.
5) For a particular case of $\boldsymbol{n} = \boldsymbol{i}$ (i.e. magnetic field along x-axis), find the expectation value of the spin operator (make sure to look at all components) in this state and discuss.

Pauli matrices are provided below.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



QM Problem #1

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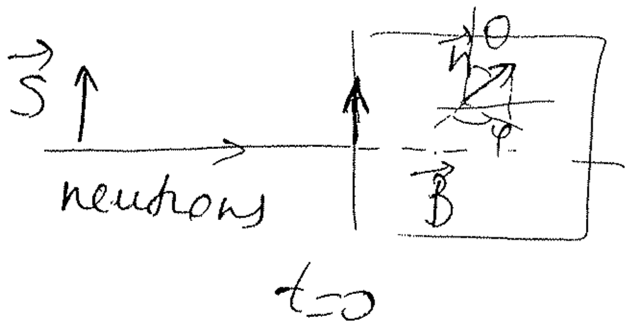


$H = 2\omega S_n$

1) $|\Psi(t=0)\rangle = |\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Time evolution operator: $e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{\hbar}2\omega S_n t}$

2) $S_n = S_x \cos\varphi \sin\theta + S_y \sin\varphi \sin\theta + S_z \cos\theta =$

$\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cos\varphi \sin\theta + \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\varphi \sin\theta +$

$+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$

3) $e^{-\frac{i}{\hbar}2\omega S_n t}$

$$\underset{\uparrow}{\doteq} \underset{\sim}{1} - \frac{i}{\hbar}\underline{\underline{2\omega S_n t}} - \underline{\frac{1}{\cancel{2}!}\left(\frac{2\omega}{\cancel{\hbar}}\right)^2 \frac{\cancel{\hbar}^2}{\cancel{4}} I t^2} +$$

$e^{iA} = 1 + iA - \frac{A^2}{2!} - i\frac{A^3}{3!} + \ldots = \cos A + i \sin A$

$S_n^2 = \frac{\hbar^2}{4}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} I$

$$+\underline{\frac{1}{3!}\, i \left(\frac{2\omega}{\hbar}\right)^3 \frac{\hbar^2}{4} I S_n t^3} \ldots =$$

$$= I \cos\omega t - i \frac{S_n}{\hbar/2} \sin\omega t =$$

$$= \cos\omega t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sin\omega t \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\omega t - i \sin\omega t \cos\theta & -i \sin\omega t \sin\theta\, e^{-i\varphi} \\ -i \sin\omega t \sin\theta\, e^{i\varphi} & \cos\omega t + i \sin\omega t \cos\theta \end{pmatrix}$$

4) $$|\psi(t)\rangle \doteq \begin{pmatrix} \cos\omega t - i \sin\omega t \cos\theta & -i \sin\omega t \sin\theta\, e^{-i\varphi} \\ -i \sin\omega t \sin\theta\, e^{i\varphi} & \cos\omega t + i \sin\omega t \cos\theta \end{pmatrix}$$

$$\cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|\psi(0)\rangle} = \underline{\begin{pmatrix} \cos\omega t - i \sin\omega t \cos\theta \\ -i \sin\omega t \sin\theta\, e^{i\varphi} \end{pmatrix}}$$

5) $\vec{n} \parallel Ox \Rightarrow \varphi = 0,\ \theta = \frac{\pi}{2} \Rightarrow |\psi(t)\rangle \doteq \begin{pmatrix} \cos\omega t \\ -i \sin\omega t \end{pmatrix}$

$$\langle S_x \rangle_t = \left(\cos\omega t \quad i \sin\omega t\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\omega t \\ -i \sin\omega t \end{pmatrix} =$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \begin{pmatrix} -i\sin\omega t \\ \cos\omega t \end{pmatrix} = 0$$

$$\langle S_y \rangle_t = \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} =$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \begin{pmatrix} -\sin\omega t \\ i\cos\omega t \end{pmatrix} = -\hbar \sin\omega t \cos\omega t =$$

$$= -\frac{\hbar}{2} \sin 2\omega t$$

$$\langle S_z \rangle_t = \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix} =$$

$$= \frac{\hbar}{2} \begin{pmatrix} \cos\omega t & i\sin\omega t \end{pmatrix} \begin{pmatrix} \cos\omega t \\ i\sin\omega t \end{pmatrix} = \frac{\hbar}{2} \cos 2\omega t$$

So, $\langle \vec{S} \rangle_t = \frac{\hbar}{2} \left(-\sin 2\omega t \, \vec{j} + \cos 2\omega t \, \vec{k} \right)$

$\nearrow$

rotation in the yz-plane

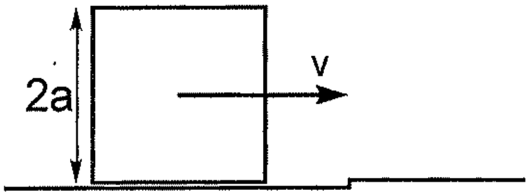
Problem 2

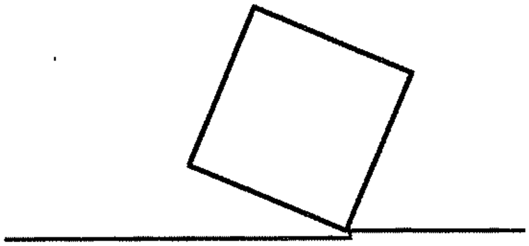
A cube of mass *m* and side 2*a* is sliding with velocity *v* along a frictionless, horizontal floor. The cube hits a very low step head-on and the leading edge comes abruptly to rest such that the cube then rotates about the step. See the figure below.

(a) Considering the quantities momentum, angular momentum, and energy, which are conserved and which are not conserved during the three processes: (i) sliding before the step collision, (ii) during the infinitesimally brief step collision, and (iii) after the collision (but before the cube topples over onto the floor)? Explain your answers.

(b) Find the angular velocity of the cube just after the collision.

(c) Find the minimum velocity required for the cube to roll over, in the same direction as the initial velocity, after hitting the step.





a) Conservation?

	P_x $\quad \Delta P_x = F_x \Delta t$	(about step) L_z $\quad \Delta L_z = \tau_z \Delta t = (r_x F_y - r_y F_x)\Delta t$	E
i	Yes $F_x = 0$ No friction	Yes $F_x = 0$ $\Sigma F_y = 0$	Yes No friction
ii	No $F_x \neq 0$ (step)	Yes $r_y = 0$ $\Sigma F_y = 0$	No Inelastic collision (deformation)
iii	No $F_x \neq 0$ (step)	No $\tau_{gravity} \neq 0$	Yes Step force does no work

b) $L_1 = L_2$ about step

$$m v a = I_{step}\, \omega_2$$

$$\omega_2 = \frac{m v a}{\frac{8}{3} m a^2} = \boxed{\frac{3v}{8a} = \omega_2}$$

$$I_{step} = I_{cm} + m(\sqrt{2}a)^2$$
$$= \frac{2}{3} m a^2 + 2 m a^2$$
$$= \frac{8}{3} m a^2$$

c) $E_2 = E_3$ $\qquad E_3$ at tipping point

want $KE_3 > 0$ to tip

$$\frac{1}{2} I \omega_2^2 + m g a = \frac{1}{2} I \omega_3^2 + m g \sqrt{2} a$$

$$\Rightarrow \frac{1}{2} I \omega_2^2 + m g a (1 - \sqrt{2}) > 0$$

$$\frac{1}{2}\left(\frac{8}{3} m a^2\right)\left(\frac{3V}{8a}\right)^2 + m g a (1 - \sqrt{2}) > 0$$

$$\Rightarrow V^2 > \frac{16}{3} g a (\sqrt{2} - 1)$$

$$\Rightarrow \boxed{V_{critical} = \sqrt{\frac{16}{3} g a (\sqrt{2} - 1)}}$$

3. Idealized gasoline engine

Consider an ideal gas consisting of N particles (molecules) with f degrees of freedom per molecule:

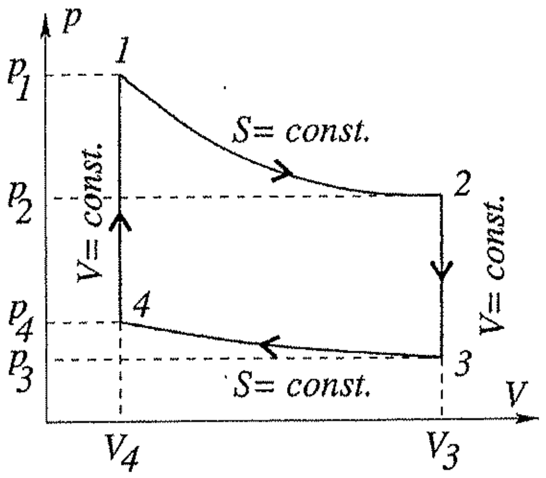
a) What are the equation of state and the internal energy U for this gas?

b) Consider an adiabatic process with constant particle number N: Starting from the results of a), show that

$$p^a V^b = const. \qquad \text{and} \qquad v^c T^d = const. \tag{1}$$

and determine the exponents a, b, c, d.

c) Consider an (idealized) four stroke engine (Otto motor) working with an ideal gas as a working medium. The (idealized) cycle is given in the figure below:



1 → 2 : expansion
2 → 3 : exhaust
3 : intake
3 → 4 : compression
4 → 1 : combustion

Using the results of part a) and the equations (1) determine the efficiency $\eta = \Delta W / \Delta Q$ of this cycle, where ΔW is the performed work, and ΔQ is the generated heat during combustion.

3. Idealized gasoline engine

Consider an ideal gas consisting of N particles (molecules) with f degrees of freedom per molecule:

a) What are the equation of state and the internal energy U for this gas?

$$\begin{aligned} pV &= NkT \\ U &= \frac{f}{2}NkT \end{aligned}$$

b) Consider an adiabatic process with constant particle number N: Starting from the results of a), show that

$$p^a V^b = const. \qquad \text{and} \qquad v^c T^d = const. \tag{1}$$

and determine the exponents a, b, c, d.

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N$$

$N = \text{const.} \rightarrow \mathrm{d}N = 0$.
Adiabatic process: $\mathrm{d}Q = T\mathrm{d}S = 0$.

$$\mathrm{d}U = -p\mathrm{d}V$$

From a)

$$\mathrm{d}U = \frac{f}{2}Nk\mathrm{d}T$$

$$p\mathrm{d}V + V\mathrm{d}p = Nk\mathrm{d}T$$

we find

$$\mathrm{d}U = p\mathrm{d}V + V\mathrm{d}p$$

and

$$\left(1 + \frac{f}{2}\right) p\mathrm{d}V = -\frac{f}{2}V\mathrm{d}p$$

$$\left(\frac{2}{f} + 1\right) \int_{V_0}^{V} \frac{\mathrm{d}\tilde{V}}{\tilde{V}} = -\int_{p_0}^{p} \frac{\mathrm{d}\tilde{p}}{\tilde{p}}$$

$$\left(\frac{2}{f} + 1\right) \ln\frac{V}{V_0} = -\ln\frac{p}{p_0}$$

$$pV^{1+\frac{2}{f}} = \text{const.}$$

Short answer for 2^{nd} equation: $pV = NkT = \text{const.} \times T$

$$\to V^{\frac{2}{f}} T = \text{const.}$$

Independent answer by integration: Using

$$\mathrm{d}U = -p\mathrm{d}V$$

and

$$\mathrm{d}U = \frac{f}{2} Nk\mathrm{d}T$$

together with

$$p = \frac{NkT}{V}$$

leads to

$$\left(\frac{2}{f}\right) \int_{V_0}^{V} \frac{\mathrm{d}\tilde{V}}{\tilde{V}} = -\int_{T_0}^{T} \frac{\mathrm{d}\tilde{T}}{\tilde{T}}$$

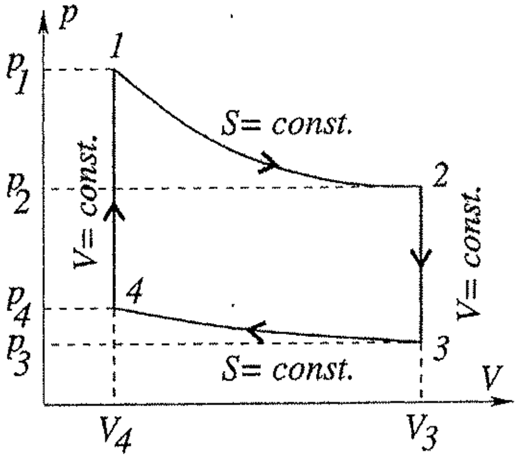
and

$$TV^{\frac{2}{f}} = \text{const.}$$

Finally

$$a = 1 \qquad b = 1 + \frac{2}{f} \qquad c = \frac{2}{f} \qquad d = 1\,.$$

c) Consider an (idealized) four stroke engine (Otto motor) working with an ideal gas as a working medium. The (idealized) cycle is given in the figure below:



1 → 2 : expansion
2 → 3 : exhaust
3 : intake
3 → 4 : compression
4 → 1 : combustion

Using the results of part a) and the equations (1) determine the efficiency $\eta = \Delta W / \Delta Q$ of this cycle, where ΔW is the performed work, and ΔQ is the generated heat during combustion.

$$\begin{array}{cccc} 1 \to 2: & \text{adiabatic} & \Delta Q = 0 & \Delta W_{12} > 0 \\ 2 \to 3: & \text{isochor} & \Delta W = 0 & \Delta Q_{23} < 0 \\ 3 \to 4: & \text{adiabatic} & \Delta Q = 0 & \Delta W_{34} < 0 \\ 4 \to 1: & \text{isochor} & \Delta W = 0 & \Delta Q_{41} > 0 \end{array}$$

The efficiency is (the exhaust heat Q_{23} does not matter)

$$\eta = \frac{\Delta W}{\Delta Q} = \frac{W_{12} + W_{34}}{Q_{41}}$$

To compute the work, we start from

$$W_1 = \int_{V_4}^{V_3} p(V) \mathrm{d}V$$

with

$$pV^{1+\frac{2}{f}} = \text{const.} = A$$

$$W_{12} = A \int_{V_4}^{V_3} V^{-1-2/f} \mathrm{d}V = -A\frac{f}{2}\left(V_3^{-2/f} - V_4^{-2/f}\right)$$

Similarly

$$W_{34} = -B \int_{V_4}^{V_3} V^{-1-2/f} \mathrm{d}V = B\frac{f}{2}\left(V_3^{-2/f} - V_4^{-2/f}\right)$$

Choose

$$\begin{aligned} A &= p_1 V_4^{1+2/f} \\ B &= p_4 V_4^{1+2/f} \end{aligned}$$

and

$$\Delta W = W_{12} + W_{34} = -\frac{f}{2}(p_1 - p_4) V_4^{1+2/f}\left(V_3^{-2/f} - V_4^{-2/f}\right) .$$

To compute the heat contribution we have $\Delta U = \Delta Q$ and

$$U = \frac{f}{2} NkT = \frac{f}{2} pV$$

which results in

$$Q_{41} = \frac{f}{2}(p_1 - p_4) V_4 .$$

Finally

$$\begin{aligned} \eta = \frac{\Delta W}{\Delta Q} &= -V_4^{2/f}\left(V_3^{-2/f} - V_4^{-2/f}\right) \\ &= 1 - \left(\frac{V_4}{V_3}\right)^{\frac{2}{f}} . \end{aligned}$$

#4

This problem deals with electric dipole radiation from a simple antenna in free space. A time-dependent current is applied to two wire segments oriented along the z axis, each being of length $d/2$. The segments are separated by a very small distance. The problem is to calculate $dP/d\Omega$ in the long wavelength limit far from the antenna.

a. Begin by writing the basic relation between $\vec{A}(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ using the retarded Green function $G(\vec{r},\vec{r}\,',t,t')$. Assume that $\vec{J}$ is time-harmonic.

b. Apply the far-field and long-wavelength approximations to this relation.

c. Show that $\vec{A}(\vec{r},t)$ depends upon $\vec{p}(t)$, the time-dependent dipole moment of the antenna. You will need the continuity relation and

$$\int \vec{J}(\vec{r}\,')dv' = -\int \vec{r}\,'(\vec{\nabla}\,'\cdot\vec{J}(\vec{r}\,'))dv' \ .$$

d. Determine $\vec{E}$ and $\vec{B}$ in terms of $\vec{p}$.

e. Derive the power radiated/steradian, $dP/d\Omega$, and the total power radiated, P.

f. A simple model for the current in the antenna is

$$I(z,t) = I_o(1-\frac{2|z|}{d})e^{-i\omega t} \ .$$

Construct $\vec{J}(\vec{r},t)$ and use it in your analysis to find $dP/d\Omega$ and P.

This problem describes electric dipole radiation from a simple antenna in free space. A time-dependent current is applied to two wire segments connected to a coaxial cable. The wires are oriented along the z axis, each being of length $d/2$ and separated by a very small distance. The problem is to calculate $dP/d\Omega$ in the long wavelength limit far from the antenna.

a. Begin by writing the basic relation between $\vec{A}(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ using the retarded Green function $G(\vec{r},\vec{r}',t,t')$. Assume that $\vec{J}$ is time-harmonic.

(4 points)

$$\vec{A}(\vec{r},t) = \frac{\mu_\circ}{4\pi}\int G(\vec{r},\vec{r}',t,t')\vec{J}(\vec{r}',t')dv'dt'$$

$$G(\vec{r},\vec{r}',t,t') = \delta(t'-(t-|\vec{r}-\vec{r}'|/c)/|\vec{r}-\vec{r}'| \text{ and } \vec{J}(\vec{r},t) = \vec{J}(\vec{r})e^{-i\omega t}$$

$$\vec{A}(\vec{r},t) \doteq \frac{\mu_\circ}{4\pi}e^{-i\omega t}\int \vec{J}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dv' \text{ with } k=\omega/c$$

b. Apply the far-field and long-wavelength approximations to this relation.

(4 points) Far field approximation: $r >> \lambda$.

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approxeq \frac{e^{ikr}}{r}e^{-ik\hat{r}\cdot\vec{r}'}$$

Long wavelength approximation: $\lambda >> d$ or $kd << 1$, so $e^{ik\hat{r}\cdot\vec{r}'} \approxeq 1$.

$$\vec{A}(\vec{r},t) = \frac{\mu_\circ}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \vec{J}(\vec{r}')dv'$$

c. Show that $\vec{A}(\vec{r},t)$ depends upon $\vec{p}(t)$, the time-dependent dipole moment of the antenna. You will need the continuity relation and

$$\int \vec{J}(\vec{r}')dv' = -\int \vec{r}'(\vec{\nabla}'\cdot\vec{J}(\vec{r}'))dv' \,.$$

(4 points)

$$\text{Continuity condition: } \nabla'\cdot\vec{J}(\vec{r}',t) = -\frac{\partial\rho(\vec{r}',t)}{\partial t} = i\omega\rho(\vec{r}')e^{-i\omega t}$$

$$\text{So, } \int \vec{J}(\vec{r}',t)dv' = -i\omega\int \vec{r}'\rho(\vec{r}')e^{-i\omega t}dv' = -i\omega\vec{p}(t) = -i\omega\vec{p}e^{-i\omega t}$$

d. Determine $\vec{E}$ and $\vec{B}$ in terms of $\vec{p}$.

(4 points)

$$\vec{A}(\vec{r},t) = -ik\vec{p}\frac{\mu_\circ}{4\pi}\frac{e^{ikr}}{r}e^{-i\omega t}$$

Supressing the time-dependence,

$$\vec{B} = \vec{\nabla} \times \vec{A} = -ik\frac{\mu_o}{4\pi}\left(\frac{ik}{r} - \frac{1}{r^2}\right)\hat{r} \times \vec{p}e^{ikr}$$

So, as $r \longrightarrow \infty$,

$$\vec{B} = k^2\frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\hat{r} \times \vec{p} \text{ , and}$$

$$\vec{E} = \frac{i}{k}\vec{\nabla} \times \vec{B} = -k^2\frac{\mu_o}{4\pi}\frac{e^{ikr}}{r}\hat{r} \times (\hat{r} \times \vec{p}) = -\hat{r} \times \vec{B}$$

e. Derive the time-averaged power radiated/steradian, $dP/d\Omega$, and the total power radiated, P. (2 points)

$$\frac{dP}{d\Omega} = \left(\vec{S}\cdot\hat{r}\right)r^2 = \frac{1}{2\mu_o}Re\left[\left(\vec{E} \times \vec{B}^*\right)\cdot\hat{r}\right]r^2 = r^2\frac{1}{2\mu_o}Re\left[\left(\vec{B} \times \hat{r}\right) \times \vec{B}^* \cdot \hat{r}\right]$$

$$\frac{dP}{d\Omega} = k^4\frac{\mu_o}{32\pi^2}|\,(\hat{r} \times \vec{p}) \times \hat{r}|^2 = k^4\frac{\mu_o}{32\pi^2}|\vec{p}|^2 sin^2\theta$$

$$P = \int\frac{dP}{d\Omega}d\Omega = k^4\frac{\mu_o}{32\pi^2}|\vec{p}|^2\int sin^3\theta d\theta d\phi = \frac{8\pi}{3}k^4\frac{\mu_o}{32\pi^2}|\vec{p}|^2 = k^4\frac{\mu_o}{12\pi}|\vec{p}|^2$$

f. A simple model for the current in the antenna is

$$I(z,t) = I_o(1 - \frac{2|z|}{d})e^{-i\omega t} \text{ .}$$

Construct $\vec{J}(\vec{r},t)$ and use it in your analysis to find $dP/d\Omega$ and P.
(2 points)

$$\vec{J}(z,t) = I(z,t)\hat{z}\delta(x)\delta(y) \Rightarrow \vec{\nabla}\cdot\vec{J} = \mp\frac{2I_o}{d}\delta(x)\delta(y)e^{-i\omega t} \text{ for } z \gtrless 0$$

$$\text{But } \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = i\omega\rho \text{ , so } \rho(z,t) = \pm\frac{2iI_o}{\omega d}\delta(x)\delta(y)e^{-i\omega t} \text{ for } z \gtrless 0$$

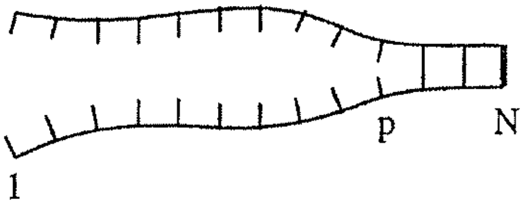
$$\vec{p}(t) = \int_{-d/2}^{d/2}\vec{r}\rho(z,t)dv = \hat{z}2e^{-i\omega t}\int_0^{d/2}\frac{2iI_o}{\omega d}zdz = i\frac{I_o d}{2\omega}\hat{z}e^{-i\omega t}$$

$$\frac{dP}{d\Omega} = \frac{I_o^2}{4}(kd)^2\frac{\mu_o}{32\pi^2}sin^2\theta \text{ and } P = I_o^2(kd)^2\frac{\mu_o}{48\pi} = I_o^2 R \text{ ,}$$

where R =radiation resistance.

5. Statistical mechanics of DNA melting

The micro states of a simple model for a double stranded polymer (i.e. DNA molecule) are given by:



(i) The two strands can form bonds at positions $1, 2, \ldots, N$.

(ii) A closed bond has energy $\varepsilon_0 = 0$. An open bond has energy $\varepsilon \neq 0$.

(iii) The p-th bond can only by open if all bonds at positions $1, 2, \ldots, p-1$ are already open.

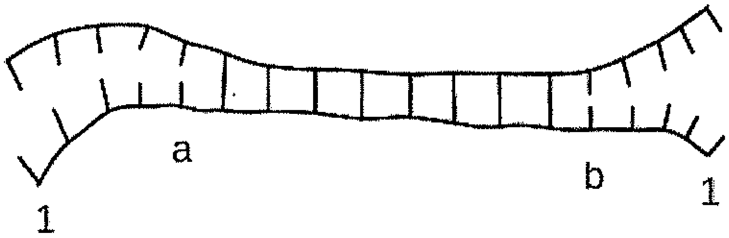
(iv) The N-th bond can't be opened.

a) The molecule is in contact with a heat bath with temperature T. Determine the partition function for the molecule.

b) What is the average number $\langle n \rangle$ of open bonds as function of temperature T and open bond energy ε ?

c) Determine the fraction $\langle n \rangle / N$ of open bonds in the limit $N \to \infty$ of an infinitely long molecule for both $\varepsilon > 0$ and $\varepsilon < 0$.

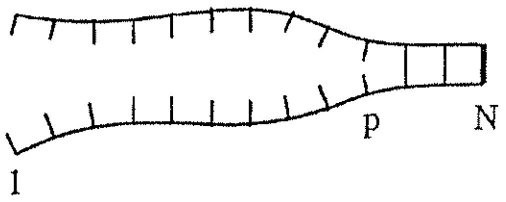
Now consider a double stranded polymer that can be unzipped (have open bonds) at both ends, i.e. (iv) is no longer valid and $a + b = p < N$. Assume $N =$ odd.



d) What is the partition function for this model? (You do not have to evaluate any sums, although closed forms exist.)

5. Statistical mechanics of DNA melting

The micro states of a simple model for a double stranded polymer (i.e. DNA molecule) are given by:



(i) The two strands can form bonds at positions $1, 2, \ldots, N$.

(ii) A closed bond has energy $\varepsilon_0 = 0$. An open bond has energy $\varepsilon \neq 0$.

(iii) The p-th bond can only by open if all bonds at positions $1, 2, \ldots, p-1$ are already open.

(iv) The N-th bond can't be opened.

a) The molecule is in contact with a heat bath with temperature T. Determine the partition function for the molecule.

Energy of microstate with p open bonds: $E_p = p\varepsilon$. No degeneracies. Partition function:

$$Z(T,N) = \sum_{p=0}^{N-1} \mathrm{e}^{-\beta\varepsilon p} = \sum_{p=0}^{N-1} x^p = \frac{1-x^N}{1-x}$$

with $x = \mathrm{e}^{-\beta\varepsilon}$.

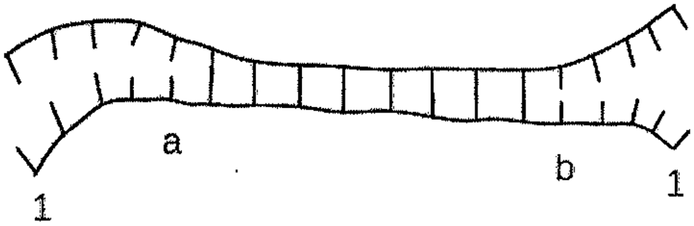
b) What is the average number $\langle n \rangle$ of open bonds as function of temperature T and open bond energy ε ?

$$\begin{aligned}
\langle n \rangle &= \langle p \rangle = \frac{1}{Z} \sum_{p=0}^{N-1} p\mathrm{e}^{-\beta\varepsilon p} \\
&= \frac{1}{Z}\left(-\frac{\partial Z}{\partial(\beta\varepsilon)}\right) = \frac{1}{Z}\left(-\frac{\partial Z}{\partial x}\frac{\partial x}{\partial(\beta\varepsilon})\right) = \frac{1}{Z} x \frac{\partial Z}{\partial x} \\
&= \frac{1}{Z} x \frac{1 - N x^{N-1} - (N-1) x^n}{(1-x)^2} = \cdots = \frac{x - N x^N - (1-N) x^{N+1}}{1 - x - x^N + X^{N+1}}
\end{aligned}$$

c) Determine the fraction $\langle n\rangle/N$ of open bonds in the limit $N \to \infty$ of an infinitely long molecule for both $\varepsilon > 0$ and $\varepsilon < 0$.

$$\begin{array}{lll} \varepsilon < 0 \;\to\; x > 1 & \lim\limits_{N\to\infty} \dfrac{\langle n\rangle}{N} = \dfrac{N-1}{N} = 1 \\ \varepsilon > 0 \;\to\; x < 1 & \lim\limits_{N\to\infty} \dfrac{\langle n\rangle}{N} = \lim\limits_{N\to\infty} \dfrac{1}{N} \times \dfrac{x}{1} = 0 \end{array}$$

Now consider a double stranded polymer that can be unzipped (have open bonds) at both ends, i.e. (iv) is no longer valid and $a + b = p < N$. Assume $N =$ odd.



d) What is the partition function for this model? (You do not have to evaluate any sums, although closed forms exist.)

Now a state with p open bonds can be formed in $g_p = p + 1$ ways.

$$\begin{aligned} Z &= \sum_{p=0}^{N-1} g_p \, e^{-\beta\varepsilon p} + Z_N \\ &= \sum_{p=0}^{N-1} (p+1)\, e^{-\beta\varepsilon p} + Z_N \end{aligned}$$

Problem #6 (QM)

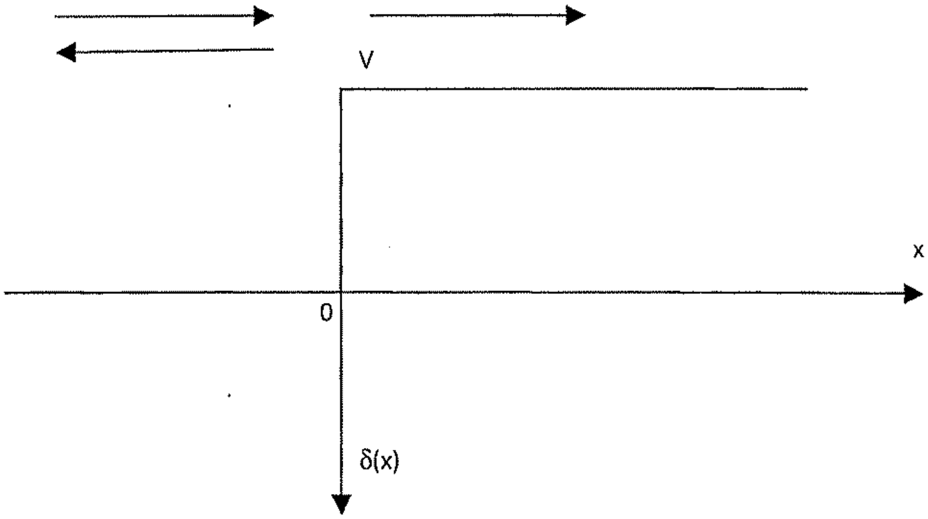
Consider a 1D potential with a step function component and an attractive delta function component just at the edge (see Figure), namely

$$V(x) = V\theta(x) - A\delta(x),$$

where θ and δ are theta- and delta-functions, respectively, and V and A are positive constants. The particle with energy E is incident from the left.

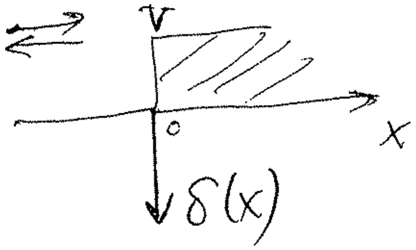
1) The particle with energy E is incident from the left, and $E > V$.
    (a) Write down the wave function describing the system.
    (b) Find the reflection coefficient, R, defined as the ratio of reflected and incident probability currents.
    (c) How does R behave as a function of energy E in the high-energy limit? Compare with the $R \sim 1/E^2$ behavior, obtained from a step-function barrier in the absence of the delta-function.

2) Consider the case $E < 0$.
    (a) Write down the wave function describing the system.
    (b) If there are any bound states, what are their energies?



QM Problem #6

$$V(x) = V\theta(x) - A\delta(x)$$



1) $E > V$

(a) $$\psi(x) = \begin{cases} e^{ikx} + Be^{-ikx}, & x<0 \\ Ce^{iqx}, & x>0 \end{cases},$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad q = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

(b) At $x=0$: $1+B = C$ ⇐ match ψ's

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x) \Rightarrow \text{integrate}, \ x \to 0$$

$$-\frac{\hbar^2}{2m}\left[\psi'(+0) - \psi'(-0)\right] - A\psi(0) = 0$$

$$\Downarrow$$

$$-\frac{\hbar^2}{2m}\left(iqC - ik(1-B)\right) - AC = 0 \Rightarrow$$

$$\underbrace{C}_{1+B}\left(\frac{\hbar^2}{2m}iq + A\right) = \frac{\hbar^2}{2m}ik(1-B) \Rightarrow$$

$$B = \frac{\frac{\hbar^2}{2m}i(k-q) - A}{\frac{\hbar^2}{2m}i(k+q) + A}$$

Then, $\underset{\uparrow}{R} = \frac{|J_r|}{J_{inc}} = |B|^2 =$

reflection coefficient

$$= \underbrace{\frac{\left(\frac{\hbar^2}{2m}\right)^2 (k-q)^2 + A^2}{\left(\frac{\hbar^2}{2m}\right)^2 (k+q)^2 + A^2}}$$

(c) High-energy limit; $E \to \infty \Rightarrow k, q$ are large

$$(k \pm q)^2 = \frac{2m}{\hbar^2}\left(E + E - V \pm 2\sqrt{E(E-V)}\right) \underset{E\to\infty}{\overset{}{\approx}} \frac{2m}{\hbar^2}\begin{cases} 4E \\ 0 \end{cases}$$

$$R \underset{E\to\infty}{\approx} \frac{A^2}{\frac{\hbar^2}{2m} \cdot 4E} = \underbrace{\frac{A^2 m}{2\hbar^2} \cdot \frac{1}{E}}$$

reflection drops off slower with E as compared to $R \sim \frac{1}{E^2}$ from a step-function without δ-function

2) $E<0$

In this case, $\Psi(x) = \begin{cases} De^{\kappa x}, & x<0 \\ De^{-qx}, & x>0 \end{cases}$

$$K = \sqrt{\frac{2m|E|}{\hbar^2}}, \quad q = \sqrt{\frac{2m(V+|E|)}{\hbar^2}}$$

At $x=0$: $-\frac{\hbar^2}{2m}\left[-\cancel{D}q - \cancel{D}\kappa\right] - A\cancel{D} = 0$

⇙

Constraint: $\underbrace{\frac{\hbar^2}{2m}(q+\kappa) = A}$

⇙

$$\frac{\hbar^2}{2m}\sqrt{\frac{2m}{\hbar^2}}\left(\sqrt{V+|E|} + \sqrt{|E|}\right) = A \Rightarrow$$

$$\frac{\hbar^2}{2m}\left(V + |E| + |E| + 2\sqrt{|E|(V+|E|)}\right) = A^2$$

$$4|E|(\cancel{V} + |E|) = \left(\frac{2m}{\hbar^2}A^2 - V - 2|E|\right)^2 =$$

$$= \left(\frac{2m}{\hbar^2}A^2 - V\right)^2 + \cancel{4|E|^2} - 4|E|\left(\frac{2m}{\hbar^2}A^2 \cancel{-V}\right) \Rightarrow$$

$$|E| = \frac{\left(\frac{2mA^2}{\hbar^2} - V\right)^2}{4\,\frac{2mA^2}{\hbar^2}} \Rightarrow \underbrace{E = -\frac{\hbar^2}{8mA^2}\left(\frac{2mA^2}{\hbar^2} - V\right)^2}$$

one bound state ↗

#7

Two semi-infinite slabs of dielectric materials are in contact, with the xy plane being the interface. The permittivities are ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$.

a. A single point charge q is placed in xy plane. Find Φ, $\vec{E}$, $\vec{D}$, and $\vec{P}$ everywhere and also any bound interfacial charge density.

b. Replace the single point charge with a conducting sphere of radius R centered at the interface between the two media. The total charge on the sphere is q. Again, find Φ, $\vec{E}$, $\vec{D}$, and $\vec{P}$ everywhere and the distribution of charge over the sphere

Two semi-infinite slabs of dielectric materials are in contact, with the xy plane being the interface. The permittivities are ϵ_1 for $z > 0$ and ϵ_2 for $z < 0$.

a. A single point charge q is placed in xy plane. Find Φ, $\vec{E}$, $\vec{D}$, and $\vec{P}$ everywhere and also any bound interfacial charge density.

(12 points) It is tempting to say

$$\vec{E}_1 = \frac{q}{4\pi\epsilon_1 r^2}\hat{r} \text{ for } z > 0 \text{ and } \vec{E}_2 = \frac{q}{4\pi\epsilon_2 r^2}\hat{r} \text{ for } z < 0 \ ,$$

but that would be wrong. The tangential components of these fields are not equal at the interface as required for the continuity of the tangential component of $\vec{E}$. The normal component of $\vec{E}$ is zero in the xy plane, so the continuity of the normal component of $\vec{D} = \epsilon\vec{E}$ is satisfied. The relation

$$\int_{sphere} \vec{D}\cdot\vec{da} = \int_{hs_1} \vec{D}_1\cdot\vec{da} + \int_{hs_2} \vec{D}_2\cdot\vec{da} = \frac{q}{4\pi r_2}2\pi r_2 + \frac{q}{4\pi r_2}2\pi r_2 = q$$

is satisfied, where hs_1 and hs_2 are hemispherical surfaces in regions 1 and 2. So, define q_1 and q_2 such that $\vec{E}_{tan}$ is continuous across the interface with the constraint that $q_1 + q_2 = 2q$.

$$\frac{q_1}{4\pi\epsilon_1 r^2} = \frac{q_2}{4\pi\epsilon_2 r^2} \Rightarrow \frac{q_1}{\epsilon_1} = \frac{q_2}{\epsilon_2} \Rightarrow q_2 = \frac{\epsilon_2}{\epsilon_1}q_1$$

$$q_1 + q_2 = 2q \Rightarrow q_1\left(1 + \frac{\epsilon_2}{\epsilon_1}\right) = 2q \Rightarrow q_1 = \frac{2q}{1 + \epsilon_2/\epsilon_1}$$

$$q_2 = \frac{\epsilon_2}{\epsilon_1}q_1 = \frac{2q}{1 + \epsilon_1/\epsilon_2}$$

Note that if $\epsilon_2 > \epsilon_1$ then $q_2 > q_1$. An alternative approach to this problem using the method of images for a charge q above the interface and taking the limit as this dispacement goes to zero yields the same result. The relevant quantities are:

$$\vec{E}_1(r) = \frac{2q}{1+\epsilon_2/\epsilon_1}\frac{1}{4\pi\epsilon_1}\frac{\hat{r}}{r^2} = \frac{1}{2\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{\hat{r}}{r^2} \text{ and } \vec{E}_2(r) = \frac{2q}{1+\epsilon_1/\epsilon_2}\frac{1}{4\pi\epsilon_2}\frac{\hat{r}}{r^2} = \frac{1}{2\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{\hat{r}}{r^2}$$

$$\Phi_1(r) = \frac{2q}{1+\epsilon_2/\epsilon_1}\frac{1}{4\pi\epsilon_1}\frac{1}{r} = \frac{1}{2\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{1}{r} \text{ and } \Phi_2(r) = \frac{2q}{1+\epsilon_1/\epsilon_2}\frac{1}{4\pi\epsilon_2}\frac{1}{r} = \frac{1}{2\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{1}{r}$$

$$\vec{D}_1 = \epsilon_1\vec{E} \ , \ \vec{D}_2 = \epsilon_2\vec{E} \text{ and } \vec{P}_{1,2} = \chi_{1,2}\vec{E}_{1,2} \ , \text{ where } \chi_{1,2} = \epsilon_{1,2} - \epsilon_o$$

There is no bound surface charge. At the interface $\vec{P}$ lies in the xy plane, and $\sigma_b = \vec{P}\cdot\hat{z} = 0$. However, the bound charge density is not zero:

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \Rightarrow \rho_{b1} = \frac{\chi_1}{\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{1}{r^3} \text{ and } \rho_{b2} = \frac{\chi_2}{\pi}\frac{q}{\epsilon_1+\epsilon_2}\frac{1}{r^3}$$

b. Replace the single point charge with a conducting sphere of radius R centered at the interface between the two media. The total charge on the sphere is q. Again, find Φ, $\vec{E}$, $\vec{D}$, and $\vec{P}$ everywhere and the distribution of charge over the sphere.

(8 points) The only difference between this situation and that of a point charge is that the charge is dispersed over the equipotential surface of the conducting sphere. The integral $\int \vec{D}\cdot\vec{da}$ over a spherical surface just outside the conducting sphere must yield the total charge q. Thus, $\vec{E}$, $\vec{D}$, $\vec{P}$, Φ and ρ_b are the same as in the case of the point charge. The field just above the surface of the conductor is $\vec{E}_{1,2} = (\sigma_{1,2}/\epsilon_{1,2})\,\hat{r}$, and, since $|\vec{E}_{1,2}|$ is independent of direction, the conclusion is that $\sigma_{1,2}$ is constant over the respective hemisphere. Hence, $\sigma_1 = q_1/2\pi R^2$ and $\sigma_2 = q_2/2\pi R^2$. Finally, there is now a bound surface charge density over the dielectric spherical cavity in contact with the conducting sphere:

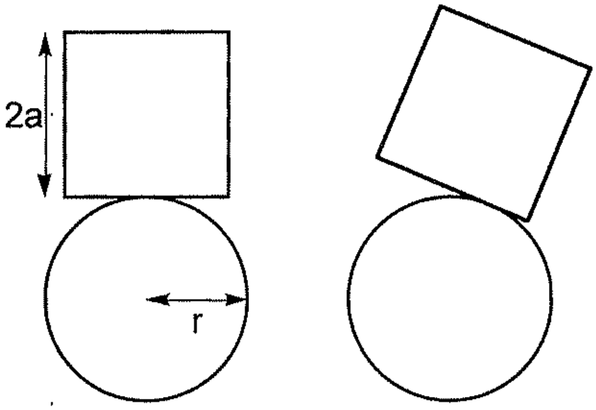
$$\sigma_{b1,2} = \vec{P}_{1,2}\cdot(-\hat{r}) = -\frac{\chi_{1,2}q_{1,2}}{2\pi\,(\epsilon_1+\epsilon_2)}\frac{1}{R^2}\,.$$

Problem 8

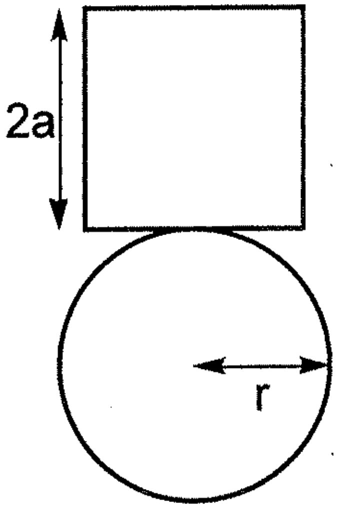
A long cylinder of radius *r* is held fixed with its axis horizontal. A cube of side 2*a* is balanced on top of the cylinder with its center vertically above the cylinder's axis and four of its sides parallel to the cylinder axis, as shown in Fig. (a). The static friction between the cube and cylinder is sufficient to prevent the cube from slipping, but it can tilt to the side, as shown in Fig. (b).

(a) The stability of the balance or equilibrium point in Fig. (a) depends on the relative sizes of the cube and cylinder. Determine the relationship that divides the stable and unstable regions.

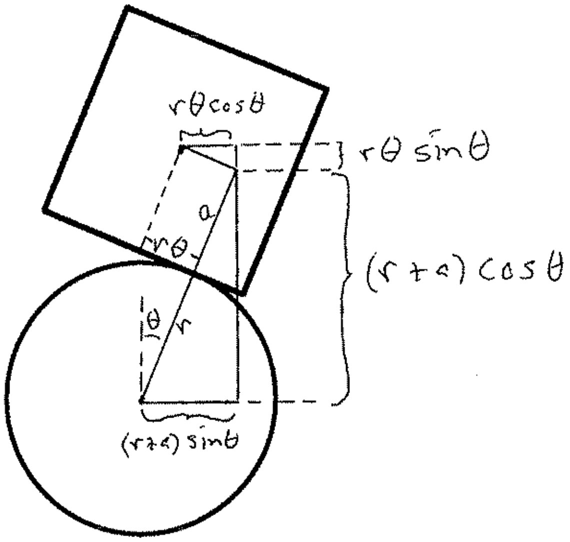
(b) Assuming that the equilibrium in Fig. (a) is stable, find the frequency of small oscillations of the cube about the balance point.



(a) (b)



(a)



(b)

a) Stable equilibrium: $\frac{dU}{d\theta}=0$, $\left.\frac{d^2U}{d\theta^2}\right|_{min} > 0$

$$U = mgh = mg\left[(r+a)\cos\theta + r\theta\sin\theta\right]$$

$$\frac{dU}{d\theta} = mg\left[r\theta\cos\theta - a\sin\theta\right] = 0 \quad @ \quad \theta = 0$$

$$\frac{d^2U}{d\theta^2} = mg\left[r\cos\theta - r\theta\sin\theta - a\cos\theta\right]$$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = mg\left[r-a\right]$$

$\Rightarrow$ $r > a$ Stable

$r < a$ Unstable

b) Lagrangian method for small oscillations

$$L = T - U = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - mgy$$

$$x = (r+a)\sin\theta - r\theta\cos\theta$$

$$y = (r+a)\cos\theta + r\theta\sin\theta \qquad I = \frac{2}{3}ma^2$$

$$\dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2 + r^2\theta^2\dot{\theta}^2$$

$$\Rightarrow L = \frac{m}{2}\left(a^2\dot{\theta}^2 + r^2\theta^2\dot{\theta}^2 + \frac{2}{3}a^2\dot{\theta}^2\right) - mg\left[(r+a)\cos\theta + r\theta\sin\theta\right]$$

make small angle approx now (same if later)

$$\Rightarrow L \simeq \frac{5}{6}ma^2\dot{\theta}^2 - mg\left[r\theta^2 + (r+a)\left(1-\frac{\theta^2}{2}\right)\right]$$

$$= \frac{5}{6}ma^2\dot{\theta}^2 - \frac{1}{2}mg(r-a)\theta^2 \qquad \text{ignoring constant}$$

$$\frac{\partial L}{\partial \theta} = -mg(r-a)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{5}{3}ma^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{5}{3}ma^2\ddot{\theta} + mg(r-a)\theta = 0$$

$$\ddot{\theta} = -\frac{3g(r-a)}{5a^2}\theta = -\omega^2\theta \quad \left[\begin{array}{l}\omega^2 > 0\\ \text{if } r > a\text{, stable}\\ \underline{\text{ok}}\end{array}\right]$$

$$\Rightarrow \omega_{s.o.} = \sqrt{\frac{3g(r-a)}{5a^2}}$$