OSU PHYSICS DEPARTMENT
COMPREHENSIVE EXAMINATION #107

October 1, and 2, 2009

Comprehensive examination for Fall 2009

PART 1, Thursday October 1, 9:00 am

General Instructions

This Comprehensive Examination for Fall 2009 consists of eight problems of equal weight (20 points each). It has four parts. The first part (Problems 1-2) is handed out at 9:00 am on Thursday, October 1, and lasts three hours. The second part (Problems 3-4) will be handed out at 1:30 pm on the same day and will also last three hours. The third and fourth parts will be administered on Friday October 2, at 9:00 am and 1:30 pm.

Work carefully, indicate your reasoning, and display your work clearly. Even if you do not complete a problem, it might be possible to obtain partial credit – especially if your understanding is manifest. Use no scratch paper; do all work in the bluebooks, work each problem in its own numbered bluebook, and be certain that your chosen student letter (but not your name) is inside the back cover of every booklet. Be sure to make note of your student letter for use in the remaining parts of the examination.

If something is omitted from the statement of the problem or you feel there are ambiguities, please get up and ask your question quietly and privately, so as not to disturb the others. Put all materials, books, and papers on the floor, except the exam, bluebooks and the collection of formulas and data distributed with the exam. Calculators are not allowed. Please return all bluebooks and formula sheets at the end of the exam.

Use the last pages of your bluebooks for "scratch" work separated by at least one empty page from your solutions. "Scratch" work will not be graded.

Problem 1

A particle of mass m and velocity v is moving in one dimension and experiences a retarding force of magnitude $|F_r| = kme^{\alpha|v|}$, where k and α are positive constants. At time $t = 0$, the particle is located at $x_0 = 0$ and has initial velocity $v_0 > 0$.

a) Write Newton's equation of motion for this particle.

b) Solve the equation of motion to obtain the velocity as a function of time. Discuss the validity of your result.

c) How far from the starting point does the particle travel before stopping?

a) $m\ddot{x} = F$

$$m\ddot{x} = -kme^{\alpha|v|}$$ (− since reterding)

$$\boxed{\ddot{x} = -ke^{\alpha|v|}}$$ or $\dot{v} = -ke^{\alpha|v|}$

b) $$\frac{dv}{dt} = -ke^{\alpha|v|}$$

$$\int_{v_0}^{v} e^{-\alpha v}\,dv = -k\int_0^t dt$$ since $v>0$

$$-\frac{1}{\alpha}e^{-\alpha v}\Big|_{v_0}^{v} = -kt$$

$$e^{-\alpha v} - e^{-\alpha v_0} = \alpha kt$$

$$e^{-\alpha v} = e^{-\alpha v_0} + \alpha kt$$

$$-\alpha v = \ln\left(e^{-\alpha v_0} + \alpha kt\right)$$

$$\boxed{v(t) = -\frac{1}{\alpha}\ln\left(e^{-\alpha v_0} + \alpha kt\right)}$$ valid only for $v>0$

c) $$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = -ke^{\alpha v}$$

$$\int_{v_0}^{v} v e^{-\alpha v}\, dv = -k\int_0^x dx$$

$$\frac{e^{-\alpha v}}{\alpha^2}(-\alpha v - 1)\Big|_{v_0}^{v} = -kx$$

$$\frac{e^{-\alpha v}}{\alpha^2}(-\alpha v - 1) - \frac{e^{-\alpha v_0}}{\alpha^2}(-\alpha v_0 - 1) = -kx$$

at stopping $x = x_s$, $v = 0$

$$\Rightarrow \boxed{x_s = \frac{1}{\alpha^2 k}\left[1 - e^{-\alpha v_0}(1 + \alpha v_0)\right]}$$

Problem 2

In a simple model of a gas the possible positions of atoms are located on a lattice. The movement of an atom is then represented by an atom hopping from one lattice position to a neighboring one. We consider such a model in one dimension, where there are M possible positions on a linear chain with end-points connected (i.e. a linear ring). Therefore the possible positions are labelled by an index i, with $i = 1 \cdots M$. Atoms can hop from site i to sites $i \pm 1$, with $M+1$ being the same at 1 and 0 the same as M. Only one atom can be on a given site at a given time. The number of atoms N is very small, $N << M$. The density of the gas is given by $n = \frac{N}{M}$.

We apply a force that moves the atoms clock-wise. The conductivity of the ring is proportional to the number of atoms that can hop into an empty spot. Calculate how the conductivity depends on the density n, to second order.

Now assume that there is an additional energy J for a pair of neighboring atoms. Calculate how the conductivity depends on J, in the limit $1 << N << M$, again up to second order in n. Discuss your result for $J = 0$, $\beta J >> 1$, and $\beta J << -1$.

Solution Problem 2

The current is proportional to the number of atoms that can hop. The probability that an atom does not have a neighbor in the clockwise direction is $1 - \frac{N-1}{M}$ and hence we have for the conductivity σ:

$$\sigma \propto N\left(1 - \frac{N-1}{M}\right) \propto n(1 - \frac{1}{M} - n)$$

and for large M this is

$$\sigma \propto n(1-n)$$

When there is an interaction between atoms, the probability of having neighboring atoms changes. In the case $N << M$ we can ignore triplets, and only consider single atoms and pairs. Also, the probability of finding a pair is small. If we add one atom to the set, there are N ways it can end up next to an atom in the clock-wise direction, and $M - N \approx M$ ways it does not. But the probability of being next to an atom has an extra Boltzmann factor in it. Denote the probability of finding an atom without neighbors P_0 and of finding an atom with a neighbor in the clock-wise direction P_1, then we see

$$P_1 = \frac{Ne^{-\beta J}}{Ne^{-\beta J} + M}$$

Strictly speaking this gives us the probability for $N+1$ atoms, but since $1 << N$ that is the same as for N. Dividing by M yields

$$P_1 = \frac{ne^{-\beta J}}{ne^{-\beta J} + 1}$$

The conductivity is proportional to the number of atoms without a neighbor in the clock-wise direction, which is $N(1 - P_1)$, hence

$$\sigma \propto n(1 - \frac{ne^{-\beta J}}{ne^{-\beta J} + 1})$$

For $J = 0$ this gives, with $n << 1$,

$$\sigma \propto n(1 - \frac{n}{n+1}) \approx n(1-n)$$

as before. If $\beta J >> 1$, atoms do not want to be neighbors, we see that the coefficient of the second order term goes to zero. we have

$$n(1 - \frac{ne^{-\beta J}}{ne^{-\beta J}+1}) \approx n(1 - ne^{-\beta J}) \approx n$$

All atoms contribute to the current.

On the other hand, if $\beta J << -1$ the result reduces to

$$n(1 - \frac{ne^{-\beta J}}{ne^{-\beta J}+1}) = n\frac{1}{ne^{-\beta J}+1} \approx e^{\beta J} << 1$$

this also becomes very small. It is also independent of n. All atoms cluster together, and only the first of the series can move!

Problem 3

Consider a plane rigid rotator (constrained to move only within one plane) characterized by the electric dipole $\mathbf{d}$ and a moment of inertia I.

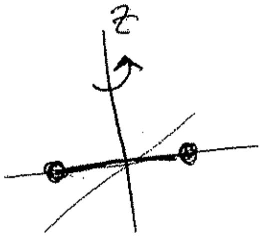
(a) Write down the Hamiltonian describing the rotational motion of this rotator.
(b) Find the energy levels of the rotator. What are its eigenstates?
(c) Now apply uniform electric field $\boldsymbol{\mathcal{E}}$ that lies in the plane of rotation. What is the Hamiltonian describing interaction of the rotator with the electric field?
(d) Let's say that the electric field is weak enough, so that the interaction found in (c) can be treated as a perturbation. Calculate the first nonvanishing corrections to the energy levels of the rotator due to this perturbation. *Hint*: in the second-order, use non-degenerate theory.

Problem #3

Solution:

plane rotator

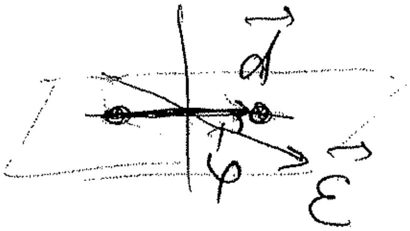
(a) $$H = \frac{\vec{L}^2}{2I} \overset{\swarrow}{=} \frac{L_z^2}{2I}$$



(b) $$\frac{L_z^2}{2I}|m\rangle = \underbrace{\frac{\hbar^2 m^2}{2I}}_{E_m^{(0)}}|m\rangle$$

$\Uparrow$ $\frac{1}{\sqrt{2\pi}} e^{im\varphi}$

(c) $$H' = -\vec{d}\cdot\vec{\mathcal{E}} =$$
$$= -d\mathcal{E}\cos\varphi$$



(d) Note that $E_m^{(0)}$ levels are doubly-degenerate

$\Downarrow$ ($|\pm m\rangle \Rightarrow$ same energy)

use degenerate perturbation theory

$\Downarrow$

need 2 × 2 matrix $\Rightarrow$ need to calculate

$\langle \pm m | d\mathcal{E}\cos\varphi | \pm m\rangle$; $\Leftarrow$

$-\langle \pm m | d\mathcal{E}\cos\varphi | \mp m\rangle$

$$\langle \pm m | \cos\varphi | \pm m \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{\mp i m\varphi} \cos\varphi \, e^{\pm i m\varphi} \, d\varphi$$

$$= \frac{1}{2\pi} \sin\varphi \Big|_0^{2\pi} = 0$$

$$\langle \pm m | \cos\varphi | \mp m \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{\mp 2 i m\varphi} \frac{e^{i\varphi}+e^{-i\varphi}}{2} \, d\varphi$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \left(e^{i\varphi(1\mp 2m)} + e^{-i\varphi(1\pm 2m)} \right) d\varphi = 0$$

$\delta(1\mp 2m) \Rightarrow$ Since m is integer $\Rightarrow$

$\neq 0$ if $m = \pm\frac{1}{2}$

So, $E_m^{(1)} = 0 \Rightarrow$ need 2nd-order perturbation

$$E_m^{(2)} = \sum_{k\neq m} \frac{|H'_{km}|^2}{E_m^{(0)} - E_k^{(0)}} = \sum_{k\neq m} \frac{|\langle k|\cos\varphi|m\rangle|^2 (d\mathcal{E})^2}{\frac{\hbar^2}{2I}(m^2-k^2)}$$

see Sakurai p. 302 on higher-order degenerate perturbation theory

$$= \frac{(d\mathcal{E})^2 \, 2I}{\hbar^2} \sum_{k\neq m} \frac{|\langle k|\cos\varphi|m\rangle|^2}{m^2-k^2}$$

$$\langle k|\cos\varphi|m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-k)\varphi}\cos\varphi\, d\varphi =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\Big(\underbrace{e^{i(m-k+1)\varphi}}_{\Downarrow \; \delta(m-k+1)} + \underbrace{e^{i(m-k-1)\varphi}}_{\Downarrow \; \delta(m-k-1)}\Big)\, d\varphi =$$

$$= \frac{1}{2}\left[\delta_{k,m+1} + \delta_{k,m-1}\right]$$

$$F_m^{(2)} = \frac{(d\mathcal{E})^2\, 2I}{\hbar^2}\sum_{k\neq m}\frac{1}{4}\,\frac{\delta_{k,m+1}+\delta_{k,m-1}}{m^2-k^2} =$$

$$= \frac{(d\mathcal{E})^2 I}{2\hbar^2}\Bigg[\underbrace{\frac{1}{m^2-(m+1)^2} + \frac{1}{m^2-(m-1)^2}}_{\overset{\shortparallel}{-\frac{1}{2m+1} + \frac{1}{2m-1} = \frac{2}{4m^2-1}}}\Bigg] = \underline{\frac{(d\mathcal{E})^2 I}{\hbar^2(4m^2-1)}}$$

Note: same result is obtained if $|-m\rangle$ state is used, i.e. if $|\langle k|\cos\varphi|-m\rangle|^2$ is calculated instead $\Rightarrow$

Also note that 2nd order does not remove the degeneracy with respect to ± m

Problem 4

Consider an idealized situation where the region of space between $0 < z < L$ has homogeneous fields $\vec{E} = \{E_x, 0,0\}, \vec{B} = \{0, B_y, 0\}$ (Fig.1).

a) A charged particle, charge q, mass m, is released from the origin $x = y = z = 0$ with no initial velocity. Derive a transcendental equation for its time of travel through the field region, and use the time of travel to find the point where the particle will exit the field region.

b) Find the correction to the above results due to small initial velocity of the particle v_z^0

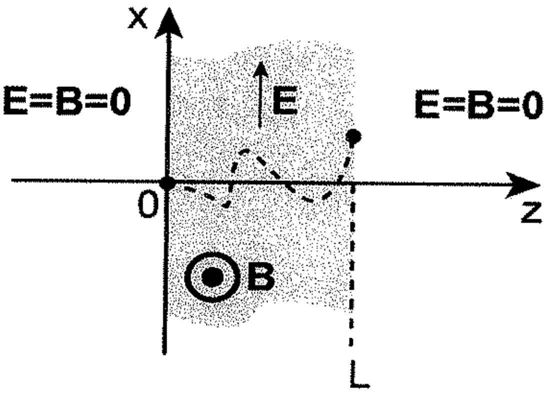
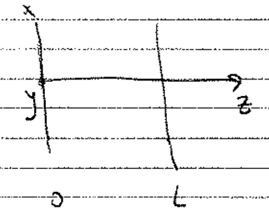


Fig. 1. Trajectory shown is for illustration only!

Solution

The force is given by:



$$\bar{F} = e\vec{E} + e[\bar{v}\times\bar{B}] =$$

$$= e\vec{E} + e\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & 0 & v_z \\ 0 & B_y & 0 \end{vmatrix} =$$

$$= \hat{i}\left(E_x - v_z B_y\right)e + \hat{k}\, v_x B_y e$$

$$\begin{cases} \dot{v}_x = \frac{e}{m}\left(E_x - v_z B_y\right) \\ \dot{v}_z = \frac{e}{m} v_x B_y \end{cases}$$

Multiply 2-nd eq by i, add two eqs together

$$\frac{d}{dt}\left(v_x - i v_z\right) = \frac{e}{m}\left(E_x - iB_y\left(v_x - i v_z\right)\right)$$

new variable: $\xi = v_x - i v_z$

$$\dot{\xi} = \frac{e}{m}\left(E_x - iB_y\cdot\xi\right) \qquad \dot{\xi} + i\frac{e}{m}B_y\cdot\xi = \frac{e}{m}E_x$$

$$\xi = Ce^{-iB_y\frac{e}{m}t} - \frac{E_x}{B_y}i$$

$$v_x = Re(\xi)\,;\ v_z = Im(\xi)$$

Initial conditions: $v_x = 0$; $v_z = 0$, $\xi(0) = C - \frac{E_x}{B_y}i = -v_z^0 i$

$$C = \frac{E_x}{B_y}i = C_1 i$$

$$\xi = \frac{E_x}{B_y}\left(i\cos\left(B_y\frac{e}{m}t\right) - \sin\left(B_y\frac{e}{m}t\right) - i\right)$$

$$\begin{cases} v_x = -\frac{E_x}{B_y}\sin\left(B_y\frac{e}{m}t\right) \\ v_z = \frac{E_x}{B_y}\left(1 - \cos\left(B_y\frac{e}{m}t\right)\right) \end{cases}$$

$$\begin{cases} x = \frac{E_x}{B_y}\frac{m}{eB_y}\cos\left(B_y\frac{e}{m}t\right) + \cancel{x_0}^{0} \\ z = \frac{E_x}{B_y}\left(t - \frac{m}{eB_y}\sin\left(B_y\frac{e}{m}t\right)\right) + \cancel{z_0}^{0} \end{cases}$$

time of flight:

$$\boxed{L = \frac{E_x}{B_y}\left[T - \frac{m}{eB_y}\sin\left(B_y\frac{e}{m}T\right)\right]}$$

Correction to the time of flight due to change in v:

$$\frac{E_x}{B_y}\left(T + \Delta T\right) - \left(\frac{E_x}{B_y}\Delta v\right)\frac{m}{eB_y}\sin\left(B_y\frac{e}{m}\left(T + \Delta T\right)\right) = L$$

$$\frac{E_x}{B_y}\left(\cancel{T} + \Delta T\right) - \cancel{\frac{E_x}{B_y}\frac{m}{eB_y}\sin\left(B_y\frac{e}{m}T\right)} - \Delta v\frac{m}{eB_y}\sin\left(B_y\frac{e}{m}T\right) -$$

$$- \frac{E_x}{B_y}\frac{\cancel{m}}{e\cancel{B_y}}\cos\left(B_y\frac{e}{m}T\right)\frac{\cancel{B_y}e}{\cancel{m}}\Delta T = \cancel{L}$$

$$\Delta T = \Delta v\frac{m}{eB_y}\cos\left(B_y\frac{e}{m}T\right)\frac{B_y}{E_x}\frac{1}{1 - \sin\left(B_y\frac{e}{m}T\right)}$$

$$\Delta x = \Delta v \cdot \frac{m}{eB_y} \cos\left(B_y \frac{e}{m} t\right) - \frac{E_x}{B_y} \sin\left(B_y \frac{e}{m} T\right) \Delta T =$$

$$= \Delta v \frac{m}{eB_y} \cos\left(B_y \frac{e}{m} t\right) \left[1 - \frac{\sin\left(B_y \frac{e}{m} T\right)}{1 - \sin\left(B_y \frac{e}{m} T\right)} \right]$$

Problem 5

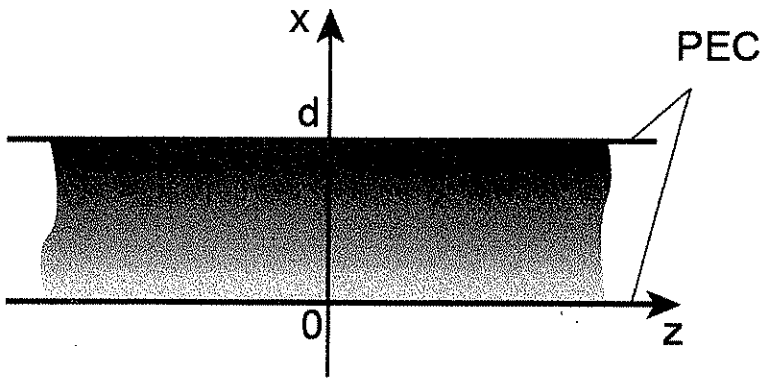
A parallel plate cavity of size d with perfect electric conductor (PEC) walls and vacuum core supports a set of standing-wave modes with frequencies $\omega_n = \pi n/d$.

Assume now that this cavity that is filled with spatially inhomogeneous dielectric permittivity $\epsilon(x) = \epsilon_1 \exp(\alpha x)$, with ϵ_1 being a constant. Starting from Maxwell's equations, derive the transcendental equation for frequencies of electromagnetic modes supported by the cavity. For simplicity, assume that the electric field of a mode is parallel to z axis.

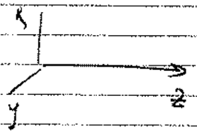
Hint: you may want to reduce the differential equation to Bessel equation

$$\xi^2 \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + (\xi^2 - m^2) f = 0$$

that has two linearly independent solutions $f = J_m(\xi)$ and $f = Y_m(\xi)$



$$\begin{cases} \text{curl}\,\bar{H} = \frac{1}{c}\frac{\partial \bar{D}}{\partial t} \\ \text{curl}\,\bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \end{cases}$$



$\mu_r = 1$; $\bar{B} = \bar{H}$;

$E, H \propto e^{-i\omega t + i k_z z}$; $\bar{H} = \{0, H_y, 0\}$; $\bar{E} = \{0, 0, E_z\}$

$\Rightarrow \text{curl}\,\bar{H} = -\frac{i\omega}{c}\epsilon E$

$\text{curl}\,E = i\frac{\omega}{c}\bar{H}$

$\text{curl curl}\,\bar{E} = +\frac{\omega^2}{c^2}\epsilon \cdot \bar{E}$

$\text{grad div}\,\bar{E} - \Delta \bar{E} = \frac{\omega^2}{c^2}\epsilon \bar{E}$

$\text{div}\,E = \text{div}\,\frac{\bar{D}}{\epsilon} = \frac{\partial}{\partial z}\frac{D_z}{\epsilon(x)} = 0$

wave equation:

$$\boxed{-\frac{\partial^2}{\partial x^2}E_z = \frac{\omega^2}{c^2}\epsilon(x)E_z}$$

Look for a solution in the form:

$E_z(x) = f(\gamma e^{\beta x})$

$\frac{\partial}{\partial x}E_z = f'(\gamma e^{\beta x})\gamma\beta e^{\beta x}$; $\frac{\partial^2}{\partial x^2}E_z = f''(\gamma e^{\beta x})\gamma^2\beta^2 e^{2\beta x} + f'\beta^2 e^{\beta x}\gamma$

$f''\gamma^2\beta^2 e^{2\beta x} + f'\beta^2\gamma e^{\beta x} + \frac{\omega^2}{c^2}\epsilon_0 e^{\alpha x} f(e^{\beta x}) = 0$

$\zeta = \gamma e^{\beta x}$;

$\beta^2\frac{d^2 f}{d\zeta^2}\zeta^2 + \beta^2\frac{df}{d\zeta}\zeta + \frac{\omega^2}{c^2}\epsilon_0 e^{\alpha x} f(\zeta) = 0$

$$\frac{d^2 f}{d\zeta^2}\zeta^2 + \frac{df}{d\zeta}\zeta + f\,\underbrace{\frac{\omega^2}{c^2}\frac{\epsilon_0}{\beta^2}e^{\alpha x}}_{\zeta^2} = 0$$

$\Rightarrow \beta = \alpha/2$; $\gamma = \frac{2}{\alpha}\frac{\omega}{c}\sqrt{\epsilon_0}$

$E_z(x) = c_1 J_0(\gamma e^{\beta x}) + c_2 Y_0(\gamma e^{\beta x})$

$E_z(0) = c_1 J(\gamma) + c_2 Y_0(\gamma) = 0$

$E_z(d) = c_1 J_0(\gamma e^{\beta d}) + c_2 Y_0(\gamma e^{\beta d}) = 0$

disp. eq: $\boxed{J_0(\gamma)Y_0(\gamma e^{\beta d}) - J_0(\gamma e^{\beta d})Y_0(\gamma) = 0}$

Problem 6

Consider a particle of spin ½. It is known that the particle is in an eigenstate of operator S_x.

(a) If S_z is measured, what are possible outcomes and their probabilities?
(b) At time $t = 0$, the particle is in the eigenstate of S_x that corresponds to the eigenvalue $-\hbar/2$. If we turn on the magnetic field B, so that the Hamiltonian describing the particle-magnetic field interaction is $H = (eB/mc)S_z$, what is the state of the particle at time $t > 0$? (e is the charge of the electron, m is its mass, and c is the speed of light)
(c) If S_x and S_z are measured at time $t = t_1$ after the magnetic field is turned on, what are the results of these measurements and their probabilities?
(d) What are expectation values of S_x and S_z at $t = t_1$? Is it what you expect based from the commutation relationships of these operators with the Hamiltonian? Explain.

QM Solutions

Problem # 6

(a) $|S_x = \pm \frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|S_z = \frac{\hbar}{2}\rangle \pm |S_z = -\frac{\hbar}{2}\rangle\right)$

So, possible outcomes are $S_z = \pm \frac{\hbar}{2}$ with probabilities $P\left(\pm\frac{\hbar}{2}\right) = \frac{1}{2}$

(b) At $t=0$: $|S_x = -\frac{\hbar}{2}\rangle = \frac{1}{\sqrt{2}}\left(|S_z = \frac{\hbar}{2}\rangle - |S_z = -\frac{\hbar}{2}\rangle\right)$;

Propagate in time $\Rightarrow$ $\psi(t) = e^{-\frac{i}{\hbar}\hat{H}t}\psi(0) \Rightarrow$

$$|S_x = -\frac{\hbar}{2}\rangle(t) \underset{\uparrow}{=} \frac{1}{\sqrt{2}}\left(e^{-i\frac{eB}{2mc}t}|S_z = \frac{\hbar}{2}\rangle - e^{i\frac{eB}{2mc}t}|S_z = -\frac{\hbar}{2}\rangle\right)$$

$\hat{H} = \frac{eB}{mc}\hat{S}_z$

$\hat{S}_z|S_z = \pm\frac{\hbar}{2}\rangle = \pm\frac{\hbar}{2}|S_z = \pm\frac{\hbar}{2}\rangle$

(c) At $t=t_1$: ②

results of measurements of S_x and S_z are $\underbrace{\pm\frac{\hbar}{2}}$

Probabilities:

$$\mathcal{P}\left(S_x=\frac{\hbar}{2}\right)=\left|\left\langle S_x=\frac{\hbar}{2}\middle|S_x=-\frac{\hbar}{2}\right\rangle(t_1)\right|^2=$$

$$=\left|\frac{1}{\sqrt{2}}\left(\left\langle S_z=\frac{\hbar}{2}\right|+\left\langle S_z=-\frac{\hbar}{2}\right|\right)\left(e^{-i\frac{eB}{2mc}t_1}\left|S_z=\frac{\hbar}{2}\right.-e^{i\frac{eB}{2mc}t_1}\left|S_z=-\frac{\hbar}{2}\right\rangle\right)\frac{1}{\sqrt{2}}\right|^2=\left|\frac{1}{2}\left(e^{-i\frac{eB}{2mc}t_1}-e^{i\frac{eB}{2mc}t_1}\right)\right|^2=\underbrace{\sin^2\left(\frac{eB}{2mc}t_1\right)}$$

$$\mathcal{P}\left(S_x=-\frac{\hbar}{2}\right)=\left|\left\langle S_x=-\frac{\hbar}{2}\middle|S_x=-\frac{\hbar}{2}\right\rangle(t_1)\right|^2=$$

$$=\underbrace{\cos^2\left(\frac{eB}{2mc}t_1\right)}$$

$$\mathcal{P}\left(S_z=\pm\frac{\hbar}{2}\right)=\left|\left\langle S_z=\pm\frac{\hbar}{2}\middle|S_x=-\frac{\hbar}{2}\right\rangle(t_1)\right|^2=$$

$$=\left|\pm\frac{1}{\sqrt{2}}e^{\mp i\frac{eB}{2mc}t_1}\right|^2=\underbrace{\frac{1}{2}}$$

(d) $\langle S_x \rangle = \frac{\hbar}{2}\,\mathcal{P}\left(S_x = \frac{\hbar}{2}\right) - \frac{\hbar}{2}\,\mathcal{P}\left(S_x = -\frac{\hbar}{2}\right) =$

$= \frac{\hbar}{2}\sin^2\left(\frac{eB}{2mc}t_1\right) - \frac{\hbar}{2}\cos^2\left(\frac{eB}{2mc}t_1\right) =$

$= -\frac{\hbar}{2}\cos\left(\frac{eB}{mc}t_1\right)$

$\langle S_z \rangle = \frac{\hbar}{2}\,\mathcal{P}\left(S_z = \frac{\hbar}{2}\right) - \frac{\hbar}{2}\,\mathcal{P}\left(S_z = -\frac{\hbar}{2}\right) = 0$

Time evolution of expectation values;

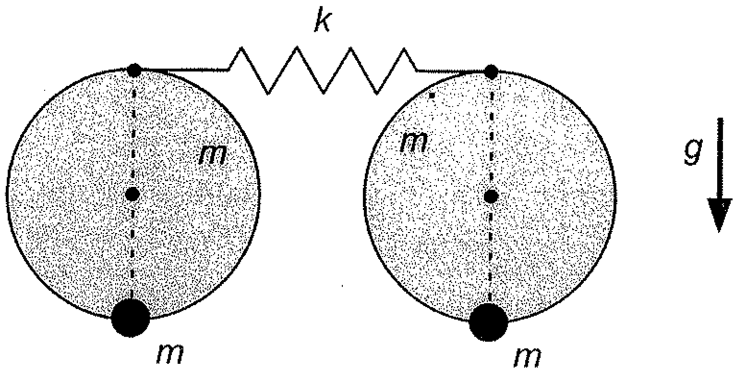
$$\frac{d\langle A \rangle}{dt} \overset{\langle \frac{\partial A}{\partial t} \rangle = 0 \; \swarrow}{=} \frac{1}{i\hbar}\langle [A, H] \rangle$$

In our case: $[S_x, H] \sim [S_x, S_z] \neq 0 \Rightarrow$ $\langle S_x \rangle$ is expected to evolve with time

$[S_z, H] \sim [S_z, S_z] = 0 \Rightarrow \langle S_z \rangle$ does **not** depend on time

Problem 7

An oscillator is made with a solid disk of mass m weighted by a small mass, also m, fixed on its rim. The disk is mounted on a frictionless horizontal axle through its center. Two such identical oscillators are connected by a spring attached to points on the rims opposite the small masses, as shown below. The spring has spring constant k and is at its relaxed length when the oscillators are in their equilibrium positions.



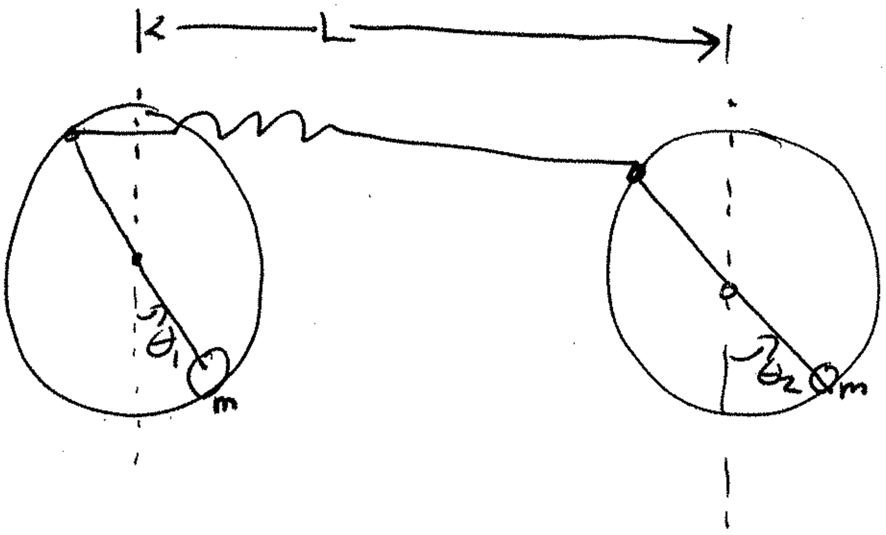
(a) Choose an appropriate set of generalized coordinates and write the kinetic and potential energies in the limit of small amplitude oscillations.

(b) Write the secular equation in determinant form.

(c) Find the eigenfrequencies (frequencies of the normal modes).

(d) Find the properly normalized eigenvectors.

(e) Sketch the normal modes of this system, discuss their symmetries, and indicate which normal mode is associated with which eigenfrequency.



$$T = \tfrac{1}{2} I \dot{\theta}_1^2 + \tfrac{1}{2} I \dot{\theta}_2^2 \qquad I = \underbrace{\tfrac{1}{2} mR^2}_{\text{disk}} + \underbrace{mR^2}_{\text{mass}} = \tfrac{3}{2} mR^2$$

$$T = \tfrac{1}{2}\left(\tfrac{3}{2} mR^2\right)\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$\Rightarrow \underline{m} = \begin{pmatrix} \tfrac{3}{2} mR^2 & 0 \\ 0 & \tfrac{3}{2} mR^2 \end{pmatrix}$$

$$U = mgy_1 + mgy_2 + \tfrac{1}{2} k \left[\Delta L\right]^2$$

$$= -mgR\cos\theta_1 - mgR\cos\theta_2 + \tfrac{1}{2} k \left[R\sin\theta_1 - R\sin\theta_2\right]^2$$

$\theta_1 = \theta_2 = 0$ is equilibrium

$\Rightarrow$ assume $\theta_1, \theta_2 \simeq 0$ to get harmonic approx

$$\Rightarrow U \simeq mgR \cdot \tfrac{1}{2}\left(\theta_1^2 + \theta_2^2\right) + \tfrac{1}{2} k R^2 (\theta_1 - \theta_2)^2$$

; changing U offset by $2mgR$

$$U = \tfrac{1}{2} mgR\left(\theta_1^2 + \theta_2^2\right) + \tfrac{1}{2} kR^2\left(\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2\right)$$

$$\Rightarrow \underline{A} = \begin{pmatrix} mgR + kR^2 & -kR^2 \\ -kR^2 & mgR + kR^2 \end{pmatrix}$$

b) $\det\left|\underline{A} - \underline{m}\,w^2\right| = 0$

$$\begin{vmatrix} mgR + kR^2 - \frac{3}{2}mR^2w^2 & -kR^2 \\ -kR^2 & mgR + kR^2 - \frac{3}{2}mR^2w^2 \end{vmatrix} = 0$$

c) $\Rightarrow \left(mgR + kR^2 - \frac{3}{2}mR^2w^2\right)^2 - \left(kR^2\right)^2 = 0$

$$mgR + kR^2 - \frac{3}{2}mR^2w^2 = \pm kR^2$$

$$w^2 = \frac{2}{3mR^2}\left[mgR + kR^2 \mp kR^2\right]$$

$$w_1^2 = \frac{2g}{3R}$$

$$w_2^2 = \frac{2g}{3R} + \frac{4k}{3m}$$

d) $(\underline{A} - \omega_1^2 \underline{m})\vec{a}_1 = 0$

$$\Rightarrow \left(mgR + kR^2 - \frac{3}{2}mR^2 \frac{2g}{3R}\right)a_{11} - kR^2 a_{21} = 0$$

$$kR^2(a_{11} - a_{21}) = 0 \quad \Rightarrow a_{11} = a_{21}$$

$$(\underline{A} - \omega_2^2 \underline{m})\vec{a}_2 = 0$$

$$\left(mgR + kR^2 - \frac{3}{2}mR^2\left(\frac{2g}{3R} + \frac{4k}{3m}\right)\right)a_{12} - kR^2 a_{22} = 0$$

$$-kR^2(a_{12} + a_{22}) = 0 \quad \Rightarrow a_{12} = -a_{22}$$

$$\Rightarrow \underline{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{11} & -a_{12} \end{pmatrix} \quad ; \quad \underline{a}^T = \begin{pmatrix} a_{11} & a_{11} \\ a_{12} & -a_{12} \end{pmatrix}$$

$$\underline{a}^T \underline{m}\, \underline{a} = \underline{1}$$

$$\begin{pmatrix} a_{11} & a_{11} \\ a_{12} & -a_{12} \end{pmatrix} \frac{3}{2}mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{11} & -a_{12} \end{pmatrix} = \underline{1}$$

$$\Rightarrow \frac{3}{2}mR^2 \cdot 2a_{11}^2 = 1 \qquad \Rightarrow a_{11} = a_{12} = \frac{1}{\sqrt{3mR^2}}$$

$$\frac{3}{2}mR^2 \cdot 2a_{12}^2 = 1$$

$$\Rightarrow \vec{a}_1 = \frac{1}{R\sqrt{3m}}(\hat{e}_1 + \hat{e}_2)$$

$$\vec{a}_2 = \frac{1}{R\sqrt{3m}}(\hat{e}_1 - \hat{e}_2)$$

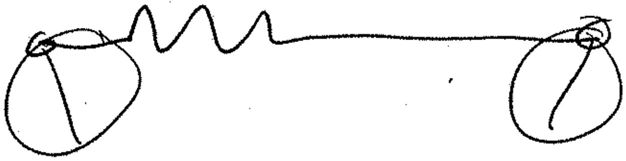
e) W_1: symmetric motion

$$w_1 = \sqrt{\frac{2g}{3R}}$$

spring doesn't stretch

⇒ freq same as single oscillator

W_2: antisymmetric motion



string stretches

more energy

⇒ higher frequency

$$W_2 = \sqrt{\frac{2g}{3R} + \frac{4k}{3m}} = W_1 \sqrt{1 + \frac{2kR}{mg}}$$

Problem 8

The pressure of a gas is increased in a reversible, adiabatic process. Originally the gas is at room temperature and atmospheric pressure. We know that for this gas at room temperature and under atmospheric pressure the coefficient of thermal expansion $\alpha = -\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,N}$ is equal to 0.05 K^{-1} and that the volume specific heat (heat capacity per unit volume) at constant pressure c_p is equal to 1500 J/m^3K. If the pressure is increases by 0.01 atmosphere, how much does the temperature increase? Use room temperature equal to 300K and atmospheric pressure equal to 10^5 Pascal. Note: you do not need a calculator for this problem.

Solution Problem 1

In order to find the change in temperature in a reversible adiabatic process when the pressure changes we have to use

$$\Delta T = \left(\frac{\partial T}{\partial p}\right)_{S,N} \Delta p$$

Because we have information about the specific heat, related to $\left(\frac{\partial S}{\partial T}\right)_{p,N}$ we should be able to use the following identity:

$$\left(\frac{\partial T}{\partial p}\right)_{S,N} = -\left(\frac{\partial T}{\partial S}\right)_{p,N}\left(\frac{\partial S}{\partial p}\right)_{T,N}$$

The first factor on the right hand side is directly related to c_p:

$$c_p = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{p,N}$$

Therefore, the second factor has to be related to α:

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,N}$$

Using

$$dU = TdS - pdV + \mu dN$$

and, with $G = U - TS + pV$,

$$dG = -SdT + Vdp + \mu dN$$

to get

$$\left(\frac{\partial V}{\partial T}\right)_{p,N} = \left(\frac{\partial^2 G}{\partial p \partial T}\right)_{N} = -\left(\frac{\partial S}{\partial p}\right)_{T,N}$$

or

$$\alpha = -\frac{1}{V}\left(\frac{\partial S}{\partial p}\right)_{T,N}$$

Put together, this yields:

$$\left(\frac{\partial T}{\partial p}\right)_{S,N} = \frac{T}{Vc_p}V\alpha = \frac{T\alpha}{c_p}$$

$$\left(\frac{\partial T}{\partial p}\right)_{S,N} = \frac{(300)(0.05)}{1500}\ m^3K/J = 0.01\ K/Pa$$

With $\Delta p = 0.01\ atm = 10^3\ Pa$ we get $\Delta T = (0.01)(1,000) = 10\ K$.